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Invariant Theory of a Class of Infinite-Dimensional Groups

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Abstract. The representation theory of a class of infinite-dimensional groups which are inductive limits of inductive systems of linear algebraic groups leads to a new invariant theory. In this article, we develop a coherent and comprehensive invariant theory of inductive limits of groups acting on inverse limits of modules, rings, or algebras. In this context, the *Fundamental Theorem of the Invariant Theory* is proved, a notion of *basis* of the rings of invariants is introduced, and a generalization of *Hilbert's Finiteness Theorem* is given. A generalization of some notions attached to the classical invariant theory such as *Hilbert's Nullstellensatz*, the primeness condition of the ideals of invariants are also discussed. Many examples of invariants of the infinite-dimensional classical groups are given.

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1. Introduction

In the preface to his book, *The Classical Groups: Their Invariants and Representations*, Hermann Weyl wrote *"The notion of an algebraic invariant of an abstract group γ cannot be formulated until we have before us the concept of a representation of γ by linear transformations, or the equivalent concept of a "quantity of type $\mathfrak{A}$." The problem of finding all representations or quantities of γ must therefore precede that of finding all algebraic invariants of γ."* His book has been and remains the most important work in the theory of representations of the classical groups and their invariants.

In recent years there is great interest, both in Physics and in Mathematics, in the theory of unitary representations of infinite-dimensional groups and their Lie algebras (see, e.g., [10], [9], [8] and the literature cited therein). One class of representations of infinite-dimensional groups is the class of *tame* representations of inductive limits of classical groups. They were studied thoroughly in the comprehensive and important work of Ol'shanskiĭ [13].

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As in Weyl's case with the classical groups, we also discovered a new type of invariants when we studied concrete realizations of irreducible tame representations of inductive limits of classical groups [22, 23]. One type of invariants that is extremely important in Physics is the Casimir invariants (see, e.g., [2]). Several of their generalizations to the case of infinite-dimensional groups may be found in [10], [14], [6], and [22]. However, to our knowledge, there is no systematic study of the *invariant theory of inductive limits of groups acting on inverse limits of modules, rings, or algebras.* In this article we develop a coherent and comprehensive theory of these invariants. To illustrate how they arise naturally from the representation theory of infinite-dimensional groups we shall consider the following typical examples.

Example 1.1. Set $V_k = \mathbb{C}^{1\times k}$ and let $A_k = P(V_k)$ denote the algebra of polynomial functions on V_k. Set $G_k = \mathrm{SO}_k(\mathbb{C})$ and $G_k^0 = \mathrm{SO}_k(\mathbb{R})$. Then G_k (resp. G_k^0) acts on V_k by right multiplication, and this induces an action of G_k (resp. G_k^0) on A_k. Then the ring of G_k (resp. G_k^0)-invariants is generated by the constants and $p_k^0 = \sum_{i=1}^k X_i^2$, where $(X_1, \dots, X_k) = X \in V_k$, and the G_k-invariant differential operators are generated by $\triangle_k = p_k^0(D) = \sum_{i=1}^k \partial_i^2$, where $\partial_i^2 = \frac{\partial^2}{\partial X_i^2}$. If $\mathcal{H}_k$ (resp. $\mathcal{H}_k^d$) denote the subspace of harmonic polynomials (resp. harmonic homogeneous of degree d), i.e., polynomials that are annihilated by $\triangle_k$, then for $k > 2$ we have the "separation of variables" theorem

$$P^{(m)}(V_k) = \sum_{i=0,\dots,[m/2]} \oplus (p_k^0)^{(i)} \mathcal{H}_k^{(m-2i)}, \tag{1.1}$$

where $P^{(m)}(V_k)$ denotes the subspace of all homogeneous polynomials of degree $m \geq 0$, and $[m/2]$ denotes the integral part of $m/2$. Moreover, each $(p_k^0)^{(i)}\mathcal{H}_k^{(m-2i)}$ is an irreducible G_k (resp. G_k^0)-module of signature $(\underbrace{m-2i, 0, \dots, 0}_{[k/2]})$. Now observe that a polynomial in k variables $(X_1, \dots, X_k)$ can be considered as a polynomial in l variables $(X_1, \dots, X_l)$ for $k \leq l$ in the obvious sense. It follows that A_k can be embedded in A_l so that the inductive limit A of A_k can be considered as the algebra of polynomials in infinitely many variables; in the sense that an element of A is a polynomial in n variables, where n ranges over $\mathbb{N}$. Let $G = \bigcup_{k=1}^\infty G_k$ (resp. $G^0 = \bigcup_{k=1}^\infty G_k^0$); then G acts on A in the following sense:

If $g \in G$ then $g \in G_k$, for some k; if $f \in A$ then $f \in A_l$, for some l. We may always assume that $k \leq l$ and $g \in G_l$, so that $g \cdot f$ is well-defined. Thus under the identification defined above, it makes sense to define $\mathcal{H}^d$ as the subspace of A which consists of all harmonic homogeneous polynomials of degree d. Then it was shown in [23] that $\mathcal{H}^d$ is an irreducible G-(resp. G^0)-module. But now what are the G-invariants? It is easy to see that no elements of A as well as no polynomial differential operators can be G-invariant. Now observe that if we let p^0 denote the formal sum $\sum_{i=1}^\infty X_i^2$ and $X = (X_1, \dots, X_k, \dots)$ denote the formal infinite row matrix, then Xg, $g \in G$ (i.e., $g \in G_k$ for some k), equals $((X_1, \dots, X_k)g, X_{k+1}, \dots)$, and it follows p^0 is formally G-invariant. Set $\triangle = \sum_{i=1}^\infty \partial_i^2$ and let $\triangle$ operate on A as follows:

If $f \in A$ then $f \in A_k$ for some $k \in \mathbb{N}$, and $\triangle f := \triangle_k f$.

Thus $f \in A$ is *harmonic* if $\triangle f = 0$, and $\mathcal{H}^d = \{f \in A^d \mid \triangle f = 0\}$.

This intuitive generalization of invariants can be rigorously formalized by defining p^0 as an element of the inverse (or projective) limit A_∞ of the algebras A_k. Then A_∞ is an algebra over $\mathbb{C}$ and one can define an action of G on A_∞. The subalgebra J of all elements of A_∞ which are pointwise fixed by this action is called the algebra of G-invariants, and $p^0 \in J$. It turns out that this can be done in a very general context.

It is also well-known that the ideal in A_k generated by p_k^0 is prime if $k > 2$. It will be shown that the ring of G-invariants in A_∞ is generated by the constants and by p^0, and the ideal in A_∞ generated by p_0 is prime.

Example 1.2. This example will be studied in great detail in Subsection 4.7., but since we want to use it to motivate the need to introduce a topology on A_∞ in Section 2., we shall give a brief description below.

Let $X_k = (x_{ij}) \in \mathbb{C}^{k\times k}$ and let A_k denote the algebra of polynomial functions in the variables x_{ij}. Let $G_k = \mathrm{GL}_k(\mathbb{C})$; then G_k operates on A_k via the co-adjoint representation. Set

$$T_k^n = \mathrm{Tr}(X_k^n), \qquad 1 \le n \le k;$$

then the subalgebra of all G_k-invariants is generated by the constants and by the algebraically independent polynomials $T_k^1, \dots, T_k^k$. Let G denote the inductive limit of the G_k's and let A_∞ denote the inverse limit of the A_k's. Let T^n denote the inverse limit of T_k^n; then it will be shown that $\{T^n; n \in \mathbb{N}\}$ is an *algebraically independent set of G-invariants.* However, if we let $\langle T^n; n \in \mathbb{N}\rangle$ denote the subalgebra of A_∞ generated by the T^n's, and J denote the subalgebra of G-invariants in A_∞, then we can only show that $\langle T^n; n \in \mathbb{N}\rangle$ is *dense* in J under the *topology of inverse limits* defined on A_∞. In general, we can give examples of ideals that are not closed in A_∞ (see Example 2.12). Thus in order to have a notion of *basis* for the rings of invariants it is necessary to introduce a topology on A_∞.

It turns out that, in general, the topology introduced in Section 2. is the most natural and the only nontrivial that one can define on inverse limits of algebraic structures.

In the spirit of Hilbert's Fourteenth Problem (see [12]) we shall also prove a sufficient condition for our rings of G-invariants to be finitely generated (Theorem 3.6). Some of the results in this article were presented in [20], [21] and [24].

2. Inverse limits of algebraic structures as topological spaces

Let $\mathbb{I}$ be an infinite subset of the set of natural numbers $\mathbb{N}$. Let $\mathcal{C}$ be a category. Suppose for each $i \in \mathbb{I}$ there is an object $A_i \in \mathcal{C}$ and whenever $i \le j$ there is a morphism $\mu_i^j : A_j \to A_i$ such that

(i) $\mu_i^i : A_i \to A_i$ is the identity for every $i \in \mathbb{I}$,

(ii) if $i \le j \le k$ then $\mu_i^k = \mu_i^j \circ \mu_j^k$.

Then the family $\{A_i; \mu_i^j\}$ is called an *inverse spectrum* over the index set $\mathbb{I}$ with *connecting morphisms* μ_i^j.

Form $\prod_{i\in\mathbb{I}} A_i$ and let p_i denote its projection onto A_i. The subset

$$\{a = (a_i) \in \prod_{i\in\mathbb{I}} A_i \mid a_i = p_i(a) = \mu_i^j \circ p_j(a) = \mu_i^j(a_j), \text{ whenever } i \leq j\}$$

is called the *inverse* (or *projective*) *limit* of the inverse spectrum $\{A_i; \mu_i^j\}$ and is denoted by A_∞ (or $\lim\limits_{\leftarrow} A_i$). The restriction $p_i|_{A_\infty} : A_\infty \to A_i$ is denoted by μ_i and is called the i^{th} *canonical map*. The elements of A_∞ are called *threads*.

In this article $\mathcal{C}$ can be either the category of modules, vector spaces, rings, or algebras over a field; then clearly if $A_\infty \neq \varnothing$ it belongs to the same category of the A_i. For example if each A_i is an algebra over the field $\mathbb{F}$ then A_∞ is an algebra over $\mathbb{F}$ with the operations defined as follows:

For $a = (a_i)$, $b = (b_i)$ in A_∞ and c in $\mathbb{F}$,

$$(a + b)_i := (a_i + b_i), \quad (ab)_i := a_i b_i, \quad (ca)_i := ca_i.$$

These operations are well-defined since the connecting morphisms μ_i^j are algebra homomorphisms. It follows that the canonical maps are also morphisms. In general the inverse limit A_∞ can be made into a topological space as follows:

Endow each A_i with the discrete topology. Then the Cartesian product $\prod_{i\in\mathbb{I}} A_i$ has a nontrivial product topology. Since each mapping μ_i^j is clearly continuous it follows that the projection maps p_i, and hence, the canonical maps μ_i are continuous. It follows from Theorem 2.3, p. 428, of [5] that the sets $\{\mu_i^{-1}(U) \mid$ all $i \in \mathbb{I}$, all subsets U of $A_i\}$ form a topological basis for A_∞. We have the following refinement.

Lemma 2.1. *The space A_∞ equipped with the topology defined above satisfies the first axiom of countability with the sets $\{\mu^{-1}(a_i) \mid$ all $i \in \mathbb{I}\}$ forming a countable topological basis at each point $a = (a_i)$ of A_∞. Moreover, $\mu_j^{-1}(a_j) \subset \mu_i^{-1}(a_i)$ whenever $i \leq j$.*

Proof. Let $a = (a_i) \in V$, where V is open in A_∞. Then there exist an $i \in \mathbb{I}$ and a subset U_i of A_i such that $a \in \mu_i^{-1}(U_i) \subset V$. This implies that $a_i = \mu_i(a) \in U_i$, and therefore, $\mu_i^{-1}(a_i) \subset \mu_i^{-1}(U_i) \subset V$. Since the set $\{a_i\}$ is open in A_i, $\mu_i^{-1}(a_i)$ is a basic open set in A_∞ containing a. This shows that A_∞ is first countable. Now let $i \leq j$ and let $b \in \mu_j^{-1}(a_j)$. Then $b_j = \mu_j(b) \in \mu_j(\mu_j^{-1}(a_j)) = \{a_j\}$, or $b_j = a_j$. This implies that $b_i = \mu_i^j(b_j) = \mu_i^j(a_j) = a_i$, and thus $b \in \mu_i^{-1}(a_i)$. This shows that $\mu_j^{-1}(a_j) \subset \mu_i^{-1}(a_i)$. ∎

Remark 2.2. In this article when we refer to the topological space A_∞ we mean that A_∞ is equipped with the topology defined by the topological basis $\{\mu_i^{-1}(a_i) \mid$ all $i \in \mathbb{I}$, and all $a = (a_i) \in A_\infty\}$, unless otherwise specified.

For S any subset of A_∞ let $\bar{S}$ denote the closure of S in A_∞. Lemma 2.1 implies the following

Lemma 2.3. *Let $S \subset A_\infty$. Then $x \in \bar{S}$ if and only if there is a sequence $\{x^n\}$ in S converging to x.*

Proof. See [5, Theorem 6.2, p. 218]. ■

Theorem 2.4. *Let $\{x^n\}$ be a sequence in A_∞. Then $x^n \to x$ if and only if for every $i \in \mathbb{I}$ there exists a positive integer N_i, depending on i, such that $x_i^n = x_i$ whenever $n \geq N_i$.*

Proof. By definition the sequence $\{x^n\}$ converges to x if: "for every neighborhood U of x $\exists N\ \forall n \geq N : x^n \in U$". By Lemma 2.1 it is sufficient to consider the neighborhoods of x of the form $\mu_i^{-1}(x_i)$, $\forall i \in \mathbb{I}$. This means that $x^n \to x$ if and only if

$$\text{“}\forall i \in \mathbb{I}\ \exists N_i\ \forall n \geq N_i : x_i^n = x_i\text{”}.$$ ■

Theorem 2.5. *If A_∞ belongs to the category $\mathcal{C}$ of modules, rings, etc., then the operations in A_∞ are continuous.*

Proof. For example, A_∞ is an algebra over a field and the operation is the multiplication in A_∞. Let $f : A_\infty \times A_\infty \to A_\infty$ be the map defined by

$$f(a, b) = a \cdot b, \qquad \forall\, a, b \in A_\infty.$$

Since A_∞ is first countable it follows that $A_\infty \times A_\infty$ is first countable. It follows from Theorem 6.3, p. 218, of [5] that f is continuous at (a, b) if and only if $f(a^n, b^n) \to f(a, b)$ for each sequence $(a^n, b^n) \to (a, b)$. By Theorem 2.4

$$\text{“}a^n \to a \text{ if and only if } \forall\, i \in \mathbb{I}\ \exists N_i^a\, \forall\, n \geq N_i^a : a_i^n = a_i\text{”},$$

$$\text{“}b^n \to b \text{ if and only if } \forall\, i \in \mathbb{I}\ \exists N_i^b\, \forall\, n \geq N_i^b : b_i^n = b_i\text{”}.$$

Thus $\forall\, i \in \mathbb{I}$ let $N_i = \max(N_i^a, N_i^b)$; then $\forall\, n \geq N_i$ we have $a_i^n = a_i$ and $b_i^n = b_i$. This implies that

$$\text{“}\forall\, i \in \mathbb{I}\ \exists N_i\ \forall\, n \geq N_i \qquad (a \cdot b)_i = a_i \cdot b_i = a_i^n \cdot b_i^n = (a^n \cdot b^n)_i\text{”}$$

which implies that $f(a^n, b^n) \to f(a, b)$, or f is continuous at (a, b). ■

For each $i \in \mathbb{I}$, let $S_i \subset A_i$ and assume that $\mu_i^j(S_j) \subset S_i$ whenever $i \leq j$. Then $\{S_i; \mu_i^j|_{S_j}\}$ is an inverse spectrum over $\mathbb{I}$. Theorem 2.8, p. 423, of [5] implies that the inverse limit S_∞ is homeomorphic to the subspace $A_\infty \cap \prod_{i \in \mathbb{I}} S_i$. In this article we shall identify S_∞ with this subspace.

Theorem 2.6. *Let S be any subset of A_∞, and let $S_i = \mu_i(S)$, all $i \in \mathbb{I}$; then $S_\infty = \bar{S}$.*

Proof. Let $s \in S$; then $s_i = \mu_i(s) \in S_i$, $\forall\, i \in \mathbb{I}$, and $\mu_i^j(s_j) = \mu_i^j \circ \mu^j(s) = \mu_i(s) = s_i$. Thus $\mu_i^j(S_j) \subset S_i$ and $S \subset S_\infty$. Let us show that S_∞ is closed in A_∞. Let $s^0 \in \bar{S}_\infty$; then Lemma 2.3 implies that there exists a sequence $\{s^n\}$ in S_∞ converging to s^0. By Theorem 2.4 it follows that for every $i \in \mathbb{I}$, there exists N_i such that $s_i^n = s_i^0$ whenever $n \geq N_i$. This implies that $s_i^0 \in S_i$ for every $i \in \mathbb{I}$, and hence, $s^0 \in S_\infty$. Thus S_∞ is closed, and it follows that $\bar{S} \subset S_\infty$. Now let $s \in S_\infty$; then by definition, for every $i \in \mathbb{I}$, there exists an element $s^i \in S$ such that $s_i = s_i^i$. Now the set $\{s^i \mid i \in \mathbb{I}\}$ is a sequence in S since $\mathbb{I}$ is an infinite subset of $\mathbb{N}$. For any $i, j \in \mathbb{I}$ such that $j \geq i$; then $s_i^j = \mu_i^j(s_j^j) = \mu_i^j(s_j) = s_i$. It follows that, for every $i \in \mathbb{I}$, there exists $N_i = i$ such that $s_i^j = s_i$ whenever $j \geq N_i = i$. Theorem 2.4 implies that $s^j \to s$, and thus $s \in \bar{S}$. Therefore, $S_\infty \subset \bar{S}$, and hence $\bar{S} = S_\infty$. ■

In the following theorems $\mathcal{C}$ is the category of (unital) rings but whenever it is appropriate the theorems remain valid if $\mathcal{C}$ is either the category of modules, vector spaces or algebras over a field $\mathbb{F}$. The proofs of Theorems 2.7 and 2.8 and Corollary 2.9 are straightforward.

Theorem 2.7. *Let $\{R_i; \mu_i^j \mid i \in \mathbb{I}\}$ be an inverse spectrum in the category $\mathcal{C}$ of unital rings. Then R_∞ is a unital ring and the following hold:*

(i) *If for all $i \in \mathbb{I}$, S_i are subrings of R_i such that $\mu_i^j(S_j) \subset S_i$ whenever $j \geq i$, then S_∞ is a subring of R_∞.*

(ii) *If S is a subring of R_∞ and $S_i = \mu_i(S)$, all $i \in \mathbb{I}$, then each S_i is a subring of R_i. Moreover, S_∞ is also a subring of R_∞ such that $S_\infty = \bar{S}$.*

Theorem 2.8. *Let $\{R_i; \mu_i^j \mid i \in \mathbb{I}\}$ be an inverse spectrum in the category $\mathcal{C}$ of commutative and unital rings. Then the following hold:*

(i) *If for all $i \in \mathbb{I}$, I_i are ideals of R_i such that $\mu_i^j(I_j) \subset I_i$ whenever $j \geq i$, then I_∞ is an ideal of R_∞.*

(ii) *If I is an ideal of R_∞, if $I_i = \mu_i(I)$, and if the canonical homomorphisms $\mu_i : R_\infty \to R_i$ are surjective, then each I_i is an ideal of R_i. Moreover, I_∞ is also an ideal of R_∞ such that $I_\infty = \bar{I}$.*

Let R be a unital commutative ring and let $S \neq \varnothing$ be any subset of R. Let $\langle S \rangle$ denote the *subring generated by* S ; i.e., the smallest subring containing S. Similarly if $S \neq \varnothing$ is a subset of R there exists a smallest ideal containing S. This ideal is called the *ideal generated by* S and is denoted by (S). The set S is then called a *system of generators* of this ideal. In fact an element of (S) can be written as $\sum_{\text{finite}} r_i s_i$ where $r_i \in R$, and $s_i \in S$.

Corollary 2.9. *Let S be any non-empty subset of R_∞ and set $\langle S \rangle_i = \mu_i(\langle S \rangle)$, $(S)_i = \mu_i((S))$, all $i \in \mathbb{I}$. Then the following hold:*

(i) *$\underleftarrow{\lim} \langle S \rangle_i$ is the smallest closed subring of R_∞ that contains S.*

(ii) *If the canonical homomorphisms μ_i are surjective, all $i \in \mathbb{I}$, then $\underleftarrow{\lim}(S)_i$ is the smallest closed ideal of R_∞ that contains S.*

A subset $\mathbb{L}$ of the index set $\mathbb{I}$ is called *cofinal* in $\mathbb{I}$ if $\forall\, i \in \mathbb{I}\ \exists l \in \mathbb{L} : i \leq l$. Since $\mathbb{I} \subset \mathbb{N}$ it is clear that $\mathbb{L} \subset \mathbb{I}$ is cofinal in $\mathbb{I}$ if and only if $\mathbb{L}$ is an infinite subset of $\mathbb{I}$.

Let $\{A_i; \mu_i^j\}$ be an inverse spectrum in a category $\mathcal{C}$ and let $\mathbb{L}$ be cofinal in $\mathbb{I}$. Then Theorem 2.7, p. 431, of [5] implies that $\underleftarrow{\lim} A_{i\in\mathbb{I}}$ is homomorphic to $\underleftarrow{\lim} A_{l\in\mathbb{L}}$. Clearly both limits are in the category $\mathcal{C}$ and they are also isomorphic. So we may without loss of generality assume that $\underleftarrow{\lim} A_{i\in\mathbb{I}} = \underleftarrow{\lim} A_{l\in\mathbb{L}}$.

Theorem 2.10. *If for every $i \in \mathbb{I}$, R_i is an integral domain, then R_∞ is an integral domain. If the connecting homomorphisms μ_i are surjective, all $i \in \mathbb{I}$, then every principal ideal I in the integral domain R_∞ is closed.*

Proof. If $a, b \in R_\infty$ are such that $a \cdot b = 0$ then $a_i \cdot b_i = (a \cdot b)_i = 0$ for all $i \in \mathbb{I}$. Since each R_i is an integral domain either $a_i = 0$ or $b_i = 0$. We may suppose without loss of generality that $a_i = 0$ for infinitely many indices $i \in \mathbb{I}$. Since this set of indices is cofinal in $\mathbb{I}$, Theorem 2.7 of [5] implies that $a = 0$. This implies that R_∞ is an integral domain. Now let I be a principal ideal of the integral domain R_∞ and let a be a generator of I. Since each μ_i is surjective, Theorem 2.8(ii) implies that each $I_i = \mu_i(I)$ is an ideal in R_i. For each $s_i \in I_i$ there exists an $s \in I$ such that $\mu_i(s) = s_i$. Since I is a principal ideal there exists $r \in R_\infty$ such that $s = ra$. This implies that $s_i = r_i a_i$, and thus each I_i is a principal ideal in R_i with a_i as a generator. Let $b \in I_\infty$; then $b_i \in I_i$, all $i \in \mathbb{I}$. Therefore, for each $i \in \mathbb{I}$ there exists $r_i \in R_i$ such that $b_i = r_i a_i$. We have, for all $j \geq i$,

$$\begin{aligned} r_i a_i &= b_i = \mu_i^j(b_j) = \mu_i^j(r_j a_j) = \mu_i^j(r_j)\mu_i^j(a_j), \text{ or} \\ r_i a_i &= \mu_i^j(r_j) a_i. \end{aligned} \tag{2.1}$$

If $I = \{0\}$ then obviously I is closed in R_∞. If $I \neq \{0\}$ then we may assume without loss of generality that $a_i \neq 0$ for sufficiently large i. For such an i, Eq. (2.1) implies that $\mu_i^j(r_j) = r_i$ since I_i is an integral domain. Set $r = (r_i)$; then since $\mu_i^j(r_j) = r_i$ whenever $j \geq i$ it follows that $r \in R_\infty$ and $b = ra \in I$. Thus $I = I_\infty$ and I is closed. ∎

Theorem 2.11. *For each $i \in \mathbb{I}$ let I_i be an ideal of R_i such that $\mu_i^j(I_j) \subset I_i$ whenever $j \geq i$. If I_i are prime for infinitely many $i \in \mathbb{I}$ then I_∞ is a prime ideal of R_∞.*

Proof. Since the set $\mathbb{L}$ of indices $l \in \mathbb{I}$ for which I_l are prime is infinite $\underleftarrow{\lim} I_{i \in \mathbb{I}} = \underleftarrow{\lim} I_{l \in \mathbb{L}}$ as remarked above. Thus we may assume without loss of generality that I_i are prime for all $i \in \mathbb{I}$. Suppose $a, b \in R_\infty$ such that $ab \in I_\infty$. Then by definition $a_i b_i = (ab)_i \in I_i$, $\forall\, i \in \mathbb{I}$. Since each I_i is prime either $a_i \in I_i$ or $b_i \in I_i$. Suppose that there are infinitely many $j \in \mathbb{I}$ such that $a_j \in I_j$. Then for each $i \in \mathbb{I}$ there exists $j \geq i$ such that $a_j \in I_j$. Since $\mu_i^j(I_j) \subset I_i$ it follows that $a_i = \mu_i^j(a_j) \in I_i$. Since i is arbitrary it follows that $a = (a_i) \in I_\infty$. If there is only a finite number of $j \in \mathbb{I}$ such that $a_j \in I_j$ there must be infinitely many $j \in \mathbb{I}$ such that $b_j \in J_j$, and the same argument as above shows that $b \in I_\infty$. Thus, $ab \in I_\infty$ implies either $a \in I_\infty$ or $b \in I_\infty$, and therefore I_∞ is prime. ∎

Example 2.12. We are giving below a class of examples which is typical of the category of objects that we will study in the remainder of this article.

Let R denote a commutative unital ring. Let k be a positive integer and let A_k denote the free commutative algebra $R[X_k] \equiv R[(X_{ij})]$ of polynomials with respect to the indeterminates X_{ij}, where i is any integer ≥ 1 and $1 \leq j \leq k$ (see [3], Chapter 4, for polynomial algebras in general). Let $(\alpha)_k = (\alpha_{11}, \ldots, \alpha_{1k}, \alpha_{21}, \ldots, \alpha_{2k}, \ldots)$ be a multi-index of integers ≥ 0 such that all but a finite number of the α_{ij} are nonzero. Set $X_k^{(\alpha)_k} = X_{11}^{\alpha_{11}} \dots X_{ij}^{\alpha_{ij}} \dots$. Then the set $\{X_k^{(\alpha)_k}\}$ is a basis for the R-module A_k when $(\alpha)_k$ ranges over all multi-indices defined above. Set $|(\alpha)_k| = \sum_{i,j} \alpha_{ij}$. Then every polynomial $p_k \in A_k$ can be written in exactly one way in the form

$$p_k = \sum_{|(\alpha)_k| \geq 0} c_{(\alpha)_k} X^{(\alpha)_k} \tag{2.2}$$

where $c_{(\alpha)_k} \in R$ and the $c_{(\alpha)_k}$ are zero except for a finite number; the $c_{(\alpha)_k}$ are called the *coefficients of* p_k; the $c_{(\alpha)_k} X^{(\alpha)_k}$ are called the *terms of* p_k. For $l \geq k$ every polynomial $p_l = \sum c_{(\alpha)_l} X^{(\alpha)_l}$ of A_l can be written uniquely in the form

$$p_l = \sum_{(\alpha')_l} c_{(\alpha')_l} X^{(\alpha')_l} + \sum_{(\alpha'')_l} c_{(\alpha'')_l} X^{(\alpha'')_l} \tag{2.3}$$

where in each $(\alpha')_l$ all the integers α'_{ij} are zero whenever $j > k$, and in each $(\alpha'')_l$ there must be an integer $\alpha''_{ij} > 0$ whenever $k < j \leq l$. Identify each $(\alpha')_l$ with an element $(\alpha)_k$ and define the map $\mu_k^l : A_l \to A_k$ by

$$p_k = \mu_k^l(p_l) = \sum_{(\alpha')_l} c_{(\alpha')_l} X^{(\alpha')_l} = \sum_{(\alpha)_k} c_{(\alpha)_k} X^{(\alpha)_k}. \tag{2.4}$$

Using Eqs. (2.3) and (2.4) we can easily deduce that μ_k^l is an algebra homomorphism and we have $\mu_k^m = \mu_k^l \circ \mu_l^m$ whenever $k \leq l \leq m$. In fact A_k can be considered as a subalgebra of A_l whenever $k \leq l$. Thus all the connecting homomorphisms μ_k^l are surjective. These connecting homomorphisms are called *truncation homomorphisms.*

Let A_∞ denote the inverse limit of the inverse spectrum $\{A_k; \mu_k^l\}$. Then since every element $p \in A_k$ can be considered as an element of A_l for $l \geq k$ we can identify p with a thread (p) in A_∞ by defining $p_l = p$ whenever $l \geq k$. Since the set of all integers $l \geq k$ is cofinal in $\mathbb{I} = \mathbb{N}$ the thread (p) is well-defined. It follows from Theorem 2.7 that A_∞ is nonempty and each A_k is a subalgebra of A_∞. If R is an integral domain then Théorème 1, p. 10, of [3] implies that each A_k is an integral domain, and hence by Theorem 2.10, A_∞ is an integral domain and every principal ideal I in A_∞ is closed.

For a fixed integer $n \geq 1$ let $A_{n,k}$ denote the algebra $R[(X_{ij})]$ for $1 \leq i \leq n$ and $1 \leq j \leq k$; then obviously $A_{n,k}$ is a subalgebra of A_k such that $\mu_k^l(A_{n,l}) = A_{n,k}$ whenever $l \geq k$. Set $A_{n,\infty} = \lim_{\leftarrow k} A_{n,k}$; then Theorem 2.7 implies that $A_{n,\infty}$ is a subalgebra of A_∞.

For each $i \geq 1$ define $p_k^i \in A_k$ by $p_k^i = \sum_{j=1}^k X_{ij}^2$. Consider the thread $f^i = (p_k^i)$ in A_∞. As remarked above for each k, p_k^i can be considered as an element of A_∞ so that if we set $f^{i,k} = p_k^i$ then the set $\{f^{i,k} \mid k \in \mathbb{N}\}$ is a sequence in A_∞. This sequence has a particular property in that its k^{th} term $f^{i,k}$ is a stationary thread at k. We claim that $\lim\limits_{k\to\infty} f^{i,k} = f^i$. Indeed, for every k there exists $N_k = k$ such that $f_k^{i,l} = \mu_k^l(p_l^i) = p_k^i = f_k^i$ whenever $l \geq k$. Theorem 2.4 implies that $\lim\limits_{k\to\infty} f^{i,k} = f^i$. In general, if $S^k = \sum_{i=1}^k g^i$, $k \in \mathbb{N}$, is a convergent sequence in A_∞, then we write its limit as $\sum_{i=1}^\infty g^i$. Similarly, if $P^k = \prod_{i=1}^k g^i$ is a convergent sequence in A_∞, then we write its limit as $\prod_{i=1}^{\infty} g^i$. Thus with this convention we have $f^i = \sum_{j=1}^{\infty} X_{ij}^2$, all $i \in \mathbb{N}$.

We claim that the ideal I generated by the set $\{f^i \mid i \in \mathbb{N}\}$ is not closed in A_∞. Indeed, let $S^n = \sum_{i=1}^{n} X_{ii} f^i$; then $\{S^n \mid n \in \mathbb{N}\}$ is a sequence in I such that

$$S_k^n = \mu_k(S^n) = \sum_{i=1}^{n} \mu_k(X_{ii})\mu_k(f^i) = \begin{cases} \sum_{i=1}^{n} X_{ii} f_k^i, & k > n, \\ \sum_{i=1}^{k} X_{ii} f_k^i, & k \leq n. \end{cases} \tag{2.5}$$

Set $S = \sum_{i=1}^{\infty} X_{ii} f^i$ and let us show that $S \in A_\infty$ and $\lim_{n\to\infty} S^n = S$. First consider $\{(\sum_{i=1}^{k} X_{ii} f_k^i)_k \mid k \in \mathbb{N}\}$. Then $\mu_k^l(\sum_{i=1}^{l} X_{ii} f_l^i) = \sum_{i=1}^{k} X_{ii} f_k^i$ whenever $l \geq k$. Thus $((\sum_{i=1}^{k} X_{ii} f_k^i)_k)$ is a thread in A_∞ and $S = ((\sum_{i=1}^{k} X_{ii} f_k^i)_k)$. Now we have from Eq. (2.5) "$\forall\, k \in \mathbb{N}\ \exists N_k = k\ \forall\, n \geq k : S_k^n = S_k$", which means that $\lim_{n\to\infty} S^n = S \in \bar{I}$.

Now let us show that $S \notin I$. A general element in I is of the form $g = \sum_{i=1}^{m} h^i f^i$ where $h^i \in A_\infty$, $1 \leq i \leq m$. Then $g_k = \mu_k(g) = \sum_{i=1}^{m} \mu_k(h^i)\mu_k(f^i) = \sum_{i=1}^{m} h_k^i f_k^i$ where the h_k^i belong to A_k, $1 \leq i \leq m$. Thus each h_k^i is a polynomial in the indeterminates X_{rs}, $1 \leq r \leq n_i$, $1 \leq s \leq k$. Let $n = \max\{n_i \mid 1 \leq i \leq m\}$; then clearly n is independent of k. Now choose $k > n$; then $S_k = \sum_{i=1}^{k} X_{ii} f_k^i$, and S cannot be an element g in I since the term $X_{kk} f_k^i$ of S_k does not occur in g_k.

3. Invariant theory of inductive limits of groups acting on inverse limits of rings

Let $\mathbb{I}$ be an infinite subset of the set of natural numbers $\mathbb{N}$. Let $\mathcal{C}$ be a category. Let $\{Y_i \mid i \in \mathbb{I}\}$ be a family of objects in the category $\mathcal{C}$. Suppose for each pair of indices i, j satisfying $i \leq j$ there is a morphism $\lambda_{ij} : Y_i \to Y_j$ such that

(i) $\lambda_{ii} : Y_i \to Y_i$ is the identity for every $i \in \mathbb{I}$,

(ii) if $i \leq j \leq k$ then $\lambda_{ik} = \lambda_{jk} \circ \lambda_{ij}$.

Then the family $\{Y_i; \lambda_{ij}\}$ is called a *direct* (or *inductive*) *system* with index set $\mathbb{I}$ and *connecting morphisms* λ_{ij}.

The image of $y_i \in Y_i$ under any connecting morphism is called a *successor* of y_i. Let $Y = \bigcup_{i\in\mathbb{I}} Y_i$ and call two elements $y_i \in Y_i$ and $y_j \in Y_j$ in Y *equivalent* whenever they have a common successor in the spectrum. This relation, R, is obviously an equivalence relation in Y. The quotient Y/R is called the *inductive* (or *direct*) *limit* of the spectrum, and is denoted by Y^∞ (or $\varinjlim Y_i$). Let $p : \bigcup_{i\in\mathbb{I}} Y_i \to Y^\infty$ be the projection; its restriction $p|Y_i$ is denoted by λ_i and is called the *canonical morphism* of Y_i into Y^∞. In general, Y^∞ may not have the same algebraic structure as the Y_i, but in many instances it does. For example, if $\{G_i; \lambda_{ij}\}$ is an inductive system of groups, the inductive limit of the operations on G_i defines on $\varinjlim G_i$ a group structure. Similar results hold for inductive limits of rings, modules, algebras, or Hilbert spaces; for details see [4, p. 139].

Now assume that for each $k \in \mathbb{I}$ we have a linear subgroup G_k of $\mathrm{GL}_k(\mathbb{C})$ such that G_k is naturally embedded (as a subgroup) in G_l, $k < l$; then we can define the inductive limit $G^\infty = \bigcup_{k\in\mathbb{I}} G_k$, and the connecting morphisms λ_{kl} are just the embedding isomorphisms of G_k into G_l.

Let A_∞ be the inverse limit of an inverse spectrum $\{A_i; \mu_i^j\}$ of a category of objects considered in Section 2.. Suppose that each A_k is acted on by the group G_k.

Lemma 3.1. *Assume that the homomorphisms μ_j^k and λ_{jk} satisfy the following condition*

$$g\cdot(\mu_j^k(a_k)) = \mu_j^k(\lambda_{jk}(g)\cdot a_k), \tag{3.1}$$

for all $g\in G_j$, $a_k\in A_k$ and $k\geq j$. Then there is a well-defined action of G^∞ on A_∞ given by

$$\begin{cases} (g\cdot a)_k := g\cdot a_k, & \\ (g\cdot a)_n := \lambda_{kn}(g)\cdot a_n, & \text{if } n\geq k,\\ (g\cdot a)_j := \mu_j^k(g\cdot a_k), & \text{if } j\leq k,\\ \forall\, g\in G_k,\ \forall\, a=(a_k)\in A_\infty. & \end{cases} \tag{3.2}$$

Proof. First, let us prove that $g\cdot a\in A_\infty$ whenever $g\in G_k$ and $a\in A_\infty$. For this we need to show that $(g\cdot a)_i = \mu_i^l((g\cdot a)_l)$ whenever $l\geq i$.

If $l\geq i\geq k$ then by Eq. (3.1) we have

$$\begin{aligned}\mu_i^l((g\cdot a)_l) &= \mu_i^l(\lambda_{kl}(g)\cdot a_l) = \mu_i^l(\lambda_{il}(\lambda_{ki}(g))\cdot a_l)\\ &= \lambda_{ki}(g)\cdot(\mu_i^l(a_l)) = \lambda_{ki}(g)\cdot a_i = (g\cdot a)_i.\end{aligned}$$

If $l\geq k>i$ then by definition we have

$$\begin{aligned}\mu_i^l((g\cdot a)_l) &= \mu_i^l(\lambda_{kl}(g)\cdot a_l) = \mu_i^k(\mu_k^l(\lambda_{kl}(g)\cdot a_l))\\ &= \mu_i^k(g\cdot\mu_k^l(a_l)) = \mu_i^k(g\cdot a_k) = (g\cdot a)_i.\end{aligned}$$

If $k>l\geq i$ then by definition we have

$$\mu_i^l((g\cdot a)_l) = \mu_i^l(\mu_l^k(g\cdot a_k)) = \mu_i^k(g\cdot a_k) = (g\cdot a)_i.$$

Now let us show that Eq. (3.2) defines an action of G^∞ on A_∞. Let $g_1\in G_i$ and $g_2\in G_k$. If $i<k$ we may identify g_1 with $\lambda_{ik}(g_1)$, if $k<i$ we may identify g_2 with $\lambda_{ki}(g_2)$. So we may assume without loss of generality that $g_1,g_2\in G_k$. We must show that

$$(g_1g_2)\cdot a = g_1\cdot(g_2\cdot a), \qquad \text{for all } a=(a_k)\in A_\infty.$$

For $n\geq k$ we have

$$\begin{aligned}((g_1g_2)\cdot a)_n &= \lambda_{kn}(g_1g_2)\cdot a_n = (\lambda_{kn}(g_1)\lambda_{kn}(g_2))\cdot a_n\\ &= \lambda_{kn}(g_1)\cdot(\lambda_{kn}(g_2)\cdot a_n) = \lambda_{kn}(g_1)\cdot(g_2\cdot a)_n = (g_1\cdot(g_2\cdot a))_n.\end{aligned}$$

For $j\leq k$ we have

$$\begin{aligned}((g_1g_2)\cdot a)_j &= \mu_j^k((g_1g_2)\cdot a_k) = \mu_j^k(g_1\cdot(g_2\cdot a_k))\\ &= \mu_j^k(g_1\cdot(g_2\cdot a)_k) = (g_1\cdot(g_2\cdot a))_j.\end{aligned}$$

Let e_i denote the identity element of G_i for all $i\in\mathbb{I}$. Then the unique element $e\in G^\infty$ such that $e=\lambda_i(e_i)$ for all $i\in\mathbb{I}$ is obviously the identity of G^∞, and we can easily verify that $e\cdot a = a$, $\forall\, a\in A_\infty$. ∎

Definition. An element $a_k \in A_k$ is said to be G_k*-invariant* if $g_k \cdot a_k = a_k$ for all $g_k \in G_k$. An element $a = (a_k) \in A_\infty$ is said to be G^∞*-invariant* if $g \cdot a = a$ for all $g \in G^\infty$.

The proofs of Lemmas 3.2 and 3.3 are straightforward.

Lemma 3.2. *An element $a = (a_k) \in A_\infty$ is G^∞-invariant if and only if each a_k is G_k-invariant.*

Lemma 3.3. *If $x \in A_k$ is G_k-invariant then $\mu_j^k(x)$ is G_j-invariant for all $j \leq k$.*

Now let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and consider the free commutative algebra $\mathbb{F}[X_k] = \mathbb{F}[(X_{ij})]$, $i \geq 1, 1 \leq j \leq k$, of polynomials as described in Example 2.12. For every $p \in \mathbb{F}[(X_{ij})]$ let $\tilde{p}$ denote the *polynomial function* obtained by substituting X_{ij} by $x_{ij} \in \mathbb{F}$. Since $\mathbb{F}$ is an infinite field the mapping $p \to \tilde{p}$ of $\mathbb{F}[X_k]$ onto $\mathbb{F}[x_k]$ is an algebra isomorphism (cf. [3, Proposition 9, p. 27]). Thus we can identify $\mathbb{F}[X_k]$ with $\mathbb{F}[x_k] = A_k$ and continue to call elements of A_k polynomials for the sake of brevity. Let μ_k^l be the *truncation homomorphisms* described in Example 2.12. Let A_∞ denote the inverse limit of the inverse spectrum $\{A_k; \mu_k^l\}$. Let $\{G_k; \lambda_{kl}\}$ be an inductive system of groups such that each G_k acts on A_k. Then it can be easily verified that condition (3.1) is satisfied, and thus the action of G^∞ on A_∞ given in Lemma 3.1 is well-defined. We have now the *Fundamental Theorem of the Invariant Theory of inductive limits of groups acting on inverse limits of polynomial algebras.* Since the action of each G_k on A_k is such that $g \cdot (p+q) = gp + g \cdot q$, $g \cdot (cp) = c(g \cdot p)$, and $g \cdot (pq) = (g \cdot p)(g \cdot q)$ for all $g \in G_k$, $p, q \in A_k$, and $c \in \mathbb{F}$, it follows that the action of G^∞ on A_∞ has the same algebraic structure (see [4, Section 6, p. 140]). This implies immediately that the subset of all G^∞-invariants in A_∞ is a subalgebra of A_∞.

Theorem 3.4. *For each $k \in \mathbb{I}$ let J_k denote the subalgebra of G_k-invariants in A_k. Let J denote the subalgebra of G^∞-invariants in A_∞. Then $J_\infty = \underset{\leftarrow}{\lim} J_k = J$, and hence, J is closed in A_∞.*

Proof. For each $k \in \mathbb{I}$, Theorem 2.7(ii) implies that $\mu_k(J)$ is a subalgebra of A_k. Lemma 3.2 implies that $\mu_k(J) \subset J_k$ for all $k \in \mathbb{I}$. Lemma 3.3 implies that $\mu_k^l(J_l) \subset J_k$ whenever $l \geq k$. Now Theorem 2.7(i) implies that J_∞ is a subalgebra of A_∞, and Theorem 2.7(ii) implies that $\underset{\leftarrow}{\lim} \mu_k(J)$ is also a subalgebra of A_∞. Obviously we have $\underset{\leftarrow}{\lim} \mu_k(J) \subset J_\infty$. Lemma 3.2 implies that $J_\infty \subset J$. Theorem 2.7(ii) implies that $\underset{\leftarrow}{\lim} \mu_k(J) = \bar{J}$. Thus, finally we have the chain of inclusions.

$$J \subset \bar{J} = \underset{\leftarrow}{\lim} \mu_k(J) \subset J_\infty \subset J. \tag{3.3}$$

Then the Theorem now follows immediately from Eq. (3.3). ■

In the Invariant Theory of the Classical Groups the subalgebra of invariants is generated by an algebraically independent set of polynomials. We shall generalize this result by introducing a notion of algebraic basis for an inverse limit of polynomial algebras.

Definition. 1. A family $\{f^\alpha\}_{\alpha\in\Lambda}$ of elements in A_∞ is said to be *algebraically independent* if the relation $p(\{f^\alpha\}) = 0$, where p is a polynomial in $\mathbb{F}[\{X^\alpha\}]_{\alpha\in\Lambda}$ where X^α is an indeterminate, implies $p = 0$. The family is said to be *algebraically dependent* if it is not algebraically independent.

It is clear from the definition of a polynomial that a family is algebraically independent if and only if every finite subfamily of this family is algebraically independent.

2. A family $\{f^\alpha\}_{\alpha\in\Lambda}$ of elements in A_∞ is said to *generate* A_∞ if $\overline{\langle\{f^\alpha\}_{\alpha\in\Lambda}\rangle} = A_\infty$, where $\langle\{f^\alpha\}_{\alpha\in\Lambda}\rangle$ denotes the subalgebra generated by the f^α, and the bar denotes the closure in the topology of inverse limits defined in Section 2.

3. An algebraically independent family of elements in A_∞ that generates A_∞ is called an *inverse limit basis* of A_∞.

4. For the standard definition of an algebraically independent family of polynomials see [3, p. 95].

Theorem 3.5. *Let $\{f^\alpha\}_{\alpha\in\Lambda}$ be a family of elements in A_∞. If for every finite subset of indices $\{\alpha_1, \dots, \alpha_n\} \subset \Lambda$ there exists an integer $k \in \mathbb{I}$, possibly depending on n, such that the subset of polynomials $\{f_k^{\alpha_1}, \dots, f_k^{\alpha_n}\}$ is algebraically independent in A_k, then $\{f^\alpha\}_{\alpha\in\Lambda}$ is algebraically independent in A_∞.*

Proof. Suppose $p(\{f^\alpha\}) = 0$, $p \in \mathbb{F}[\{X^\alpha\}]_{\alpha\in\Lambda}$; then $p(f^{\alpha_1}, \dots, f^{\alpha_n}) = 0$ for some finite subset of indices $\{\alpha_1, \dots, \alpha_n\}$. By hypothesis there exists an integer k such that $\{f_k^{\alpha_1}, \dots, f_k^{\alpha_n}\}$ is algebraically independent in A_k. Since the canonical map $\mu_k : A_\infty \to A_k$ is an algebra homomorphism it follows that $p(f_k^{\alpha_1}, \dots, f_k^{\alpha_n}) = 0$. Hence $p = 0$ and the theorem is proved. ∎

Theorem 3.6. *Let $\{f^\alpha\}_{\alpha\in\Lambda}$ be a family of elements in A_∞. If there exists $k_0 \in \mathbb{I}$ such that the family of polynomials $\{f_{k_0}^\alpha\}_{\alpha\in\Lambda}$ is algebraically independent in A_{k_0} then $\{f^\alpha\}_{\alpha\in\Lambda}$ is also algebraically independent in A_∞ and $\langle\{f^\alpha\}_{\alpha\in\Lambda}\rangle$ is closed in A_∞.*

Proof. The fact that $\{f^\alpha\}_{\alpha\in\Lambda}$ is algebraically independent follows immediately from Theorem 3.5. By Lemma 2.3 to prove that $\langle\{f^\alpha\}_{\alpha\in\Lambda}\rangle$ is closed we suppose that φ is the limit of a sequence $\{\varphi^n\}$ in $\langle\{f^\alpha\}_{\alpha\in\Lambda}\rangle$ and verify that $\varphi \in \langle\{f^\alpha\}_{\alpha\in\Lambda}\rangle$.

By Theorem 2.4, $\varphi^n \to \varphi$ if and only if for every $i \in \mathbb{I}$ there exists a positive integer N_i, depending on i, such that $\varphi_i^n = \varphi_i$ whenever $n \geq N_i$. In particular, for $i = k_0$ there exists N_{k_0} such that $\varphi_{k_0}^n = \varphi_{k_0}$ whenever $n \geq N_{k_0}$. Thus for $i \geq k_0$ we can choose $N_i \geq N_{k_0}$. Therefore for $n \geq N_i$ we have

$$\varphi_i = \varphi_i^n = p_n(\{f_i^\alpha\}) = (p_n(\{f^\alpha\}))_i,$$

where p_n is a polynomial depending on n. Since $\mu_{k_0}^i$ is an algebraic homomorphism, $\varphi_{k_0} = \varphi_{k_0}^n = \mu_{k_0}^i(\varphi_i^n) = p_n(\{f_{k_0}^\alpha\})$. The fact that $\{f_{k_0}^\alpha\}_{\alpha\in\Lambda}$ is algebraically independent implies that all the polynomials p_n are the same for sufficiently large n. Let p denote such a polynomial. Then we have $\varphi_i = (p(\{f^\alpha\}))_i$ for all $i \in \mathbb{I}$. This means that $\varphi \in \langle\{f^\alpha\}_{\alpha\in\Lambda}\rangle$, and this achieves the proof of the theorem. ∎

Remark 3.7. Suppose $\{f^\alpha\}_{\alpha\in\Lambda}$ is a family of elements in A_∞ such that $\{f_i^\alpha\}_{\alpha\in\Lambda}$ is an *algebraic basis* for the polynomial algebra A_i for all $i \geq k_0$ for some $k_0 \in \mathbb{I}$, i.e., the family $\{f_i^\alpha\}_{\alpha\in\Lambda}$ is algebraically independent in A_i and $\langle\{f_i^\alpha\}_{\alpha\in\Lambda}\rangle = A_i$. Then Theorem 3.6 implies that $\{f^\alpha\}_{\alpha\in\Lambda}$ is also an *algebraic basis* for A_∞. Thus in this case the notion of algebraic basis and inverse limit basis for A_∞ coincide, and the notion of (inverse limit) basis does indeed generalize the notion of algebraic basis.

Corollary 3.8. *We preserve the notations of Theorem* 3.4*. Suppose* $\{j^\alpha\}_{\alpha\in\Lambda}$ *is a family of elements in* J *such that* $\{j_k^\alpha\}_{\alpha\in\Lambda}$ *is an algebraic basis for* J_k *for all* $k \geq k_0$*. Then* $\{j^\alpha\}_{\alpha\in\Lambda}$ *is an algebraic basis for* J*.*

Proof. By Theorem 3.4, $J = J_\infty$ and Theorem 3.6 implies that $\{j^\alpha\}_{\alpha\in\Lambda}$ is an algebraic basis for J_∞. Thus the corollary is proved. ∎

Example 3.9. Let A_k be the algebra of polynomials in k variables $x_1, \dots, x_k$ in $\mathbb{F}$. Let $\{A_k; \mu_k^l\}$ denote the inverse spectrum with connecting homomorphisms $\mu_k^l : A_l \to A_k$, $l \geq k$, $l, k \in \mathbb{N}$. The μ_k^l are truncation homomorphisms, which in this case can be defined simply by setting $\mu_k^l(x_j) = x_j$ for $1 \leq j \leq k$ and $\mu_k^l(x_j) = 0$ for $k < j \leq l$, and by extending algebraically to all polynomials in A_l. Let A_∞ denote the inverse limit of the inverse spectrum $\{A_k; \mu_k^l\}$. Then the set $\{x^{(\alpha)}\}_{(\alpha)\in\Lambda}$, where $x^{(\alpha)} = x^{\alpha_1} \cdots x^{\alpha_k}$, and $(\alpha) = (\alpha_1, \dots, \alpha_k)$ is a multi-index, forms an inverse limit basis for A_∞ when (α) ranges over all multi-indices, and $k = 1, 2, \dots$, etc.

Now let $G_k \subset \mathrm{GL}_k(\mathbb{C})$ be a reductive algebraic group, let V_k be a complex vector space of dimension k on which G_k acts linearly. Let $\mathbb{C}[V_k]$ denote the ring of all polynomial functions on V_k. Let $\mathbb{C}[V_k]^{G_k}$ denote the subring of all G_k-invariant polynomial functions. Then we have the following Hilbert's Finiteness Theorem: *There exist* s *algebraically independent* G*-invariants* $p_1, \dots, p_s$ *such that* $\mathbb{C}[V_k]^{G_k} = \mathbb{C}[p_1, \dots, p_s]$. (See [7] and [16]). Set $A_k = \mathbb{C}[V_k]$ and $J_k = \mathbb{C}[V_k]^{G_k}$. We preserve the notation of Theorem 3.4 and assume in addition that each $G_k, k \in \mathbb{I}$, is a reductive linear algebraic group. Then by Hilbert's Finiteness Theorem there exists a set of algebraically independent polynomials $\{f_k^\alpha\}_{\alpha\in\Lambda_k}$ that generates J_k, where the index set Λ_k is a finite subset of $\mathbb{N}$. It follows that, for all pairs (l, k) such that $l \geq k$, we may assume that $\Lambda_k \subset \Lambda_l$. Set $\Lambda = \bigcup_{k\in\mathbb{I}} \Lambda_k$. In general if the V_k are infinite-dimensional then the J_k may not be finitely generated but we still have $\Lambda_k \subset \Lambda_l$ for $l \geq k$.

Theorem 3.10. *For each* $k \in \mathbb{I}$ *let* $\{f_k^\alpha\}_{\alpha\in\Lambda_k}$ *be a set of generators for* J_k*. If* $f^\alpha = \varprojlim f_k^\alpha$ *then the family* $\{f^\alpha\}_{\alpha\in\Lambda}$ *generates* J*. In particular, if* $\{f_k^\alpha\}_{\alpha\in\Lambda_k}$ *is an algebraic basis for* J_k *then* $\{f^\alpha\}_{\alpha\in\Lambda}$ *is an inverse limit basis for* J*.*

Proof. Let $J' = \langle\{f^\alpha\}; \alpha \in \Lambda\rangle$; then by assumption $\mu_k(J') = J_k$, for all $k \in \mathbb{I}$. By Theorem 2.6, $\bar{J'} = J_\infty$. By Theorem 3.4, $J_\infty = J$. Therefore $\bar{J'} = J$. Now if in addition the sets $\{f_k^\alpha\}_{\alpha\in\Lambda_k}$ are algebraically independent, then by assumption every finite subset of indices $\{\alpha_1, \dots, \alpha_m\}$ of Λ is contained in Λ_k for some $k \in \mathbb{I}$; therefore, the set $\{f_k^{\alpha_1}, \dots, f_k^{\alpha_m}\}$ is algebraically independent. Theorem 3.5 implies that the set $\{f^\alpha; \alpha \in \Lambda\}$ is algebraically independent. Thus by definition $\{f^\alpha; \alpha \in \Lambda\}$ is an inverse limit basis for J. ∎

Remark 3.11. In some examples in Section 4., for each $k \in \mathbb{I}$ the set of generators $\{f_k^\alpha\}_{\alpha\in\Lambda_k}$ are not algebraically independent yet the set $\{f^\alpha\}_{\alpha\in\Lambda}$ can be shown to be algebraically independent.

We conclude this section by generalizing to the case of inverse limits of polynomial algebras one of the most fundamental theorems in the theory of polynomial algebras, namely, *Hilbert's Nullstellensatz.* This theorem plays an important role in the *theory of algebraic invariants.*

Let I be a proper ideal in the polynomial ring in k complex variables $\mathbb{C}[Z] = \mathbb{C}[z_1, \ldots, z_k]$. Then the algebraic variety of I is defined as the set

$$V(I) = \{Z \in \mathbb{C}^k \mid p(Z) = 0, \forall\, p \in I\}.$$

If V is a subset of $\mathbb{C}^k$ then the ideal of V is the ideal in $\mathbb{C}[Z]$ defined by

$$\mathcal{I}(V) = \{p \in \mathbb{C}[Z] \mid p(Z) = 0, \forall\, Z \in V\}.$$

The nilradical of I is defined as the set

$$\sqrt{I} = \{p \in \mathbb{C}[Z] \mid p^n \in I \text{ for some integer } n \geq 1\}.$$

Then it can be easily shown that $\sqrt{I}$ is the intersection of all prime ideals in $\mathbb{C}[Z]$ which contain I. Then Hilbert's Nullstellensatz can be simply stated as $\mathcal{I}(V(I)) = \sqrt{I}$ (See [7, p. 142]).

For each $k \in \mathbb{N}$ let $A_k = \mathbb{C}[Z]_k$. For $l \geq k$ let $\mu_k^l : A_l \to A_k$ denote the *truncation homomorphism.* Let $A_\infty = \underleftarrow{\lim} A_k$. Let $i_{kl} : \mathbb{C}^k \to \mathbb{C}^l$ denote the embedding defined by $i_{kl}(Z) = i_{kl}(z_1, \ldots, z_k) = (\underbrace{z_1, \ldots, z_k, 0, \ldots, 0}_{l})$, $\forall\, Z \in \mathbb{C}^k$.

Let $\mathbb{C}^\infty = \bigcup_{k\in\mathbb{N}} \mathbb{C}^k$ denote the inductive limit of the spectrum $\{\mathbb{C}^k; i_{kl}\}$. Then it is easy to show that

$$p(i_{kl}(Z)) = [\mu_k^l(p)](Z),\ \forall\, p \in A_l,\ \forall\, Z \in \mathbb{C}^k,\ \forall\, k, l \in \mathbb{N},\ l \geq k. \tag{3.4}$$

Then for every $f = (f_k) \in A_\infty$ and every $Z \in \mathbb{C}^\infty$, i.e., $Z \in \mathbb{C}^k$ for some $k \in \mathbb{N}$, we define

$$f(Z) := f_k(Z). \tag{3.5}$$

Eq. (3.4) implies that Eq. (3.5) is well-defined, i.e., the complex number $f(Z)$ is independent of k. This allows us to define the following concept: For any subset V of $\mathbb{C}^\infty$ let $\mathcal{I}(V) = \{f \in A_\infty \mid f(Z) = 0,\ \forall\, Z \in V\}$.

Lemma 3.12. *For any $V \subset \mathbb{C}^\infty$ the set $\mathcal{I}(V)$ is an ideal in A_∞. Moreover $\mathcal{I}(V)$ is closed.*

Proof. If $f, f' \in \mathcal{I}(V)$, $h \in A_\infty$ and $c \in \mathbb{C}$, then $(f + cf')(Z) = f(Z) + cf'(Z) = 0$, $\forall\, Z \in \mathbb{C}^\infty$, and $(hf)(Z) = h(Z)f(Z) = 0$. Thus $\mathcal{I}(V)$ is an ideal in A_∞.

Suppose a sequence $\{f^n\} \subset \mathcal{I}(V)$ converges for $f \in A_\infty$. Then by Theorem 2.4 we have

$$\text{“}\forall\, k \in \mathbb{N},\ \exists\, N_k\ \forall\, n \geq N_k : f_k^n = f_k\text{”}.$$

This implies that for every k, $f_k(Z) = f_k^n(Z) = 0$ for all $Z \in V^k$, where $V^k = V \cap \mathbb{C}^k$. This means that $f(Z) = 0$, $\forall\, Z \in V$, and thus $f \in \mathcal{I}(V)$. ∎

Now let I be an ideal in A_∞ and set $V(I) = \{Z \in \mathbb{C}^\infty \mid f(Z) = 0, \forall f \in I\}$. If $f = (f_l) \in I$ then $Z \in V(I)$ if $Z \in \mathbb{C}^k$ for some $k \in \mathbb{N}$, and $f(Z) = f_k(Z) = 0$. Choose k to be the smallest integer such that $Z \in \mathbb{C}^k \backslash \mathbb{C}^{k-1}$ and define $V^k(I) = \{Z \in \mathbb{C}^k \mid f_k(Z) = 0, \forall f \in I\}$. Then $V^k(I)$ may be empty but we always have $V^k(I) \subset V^l(I)$, $\forall l \geq k$, and $V(I) = \bigcup_{k\in\mathbb{N}} V^k(I)$. Indeed, $Z \in V(I)$ if and only $Z \in V^k(I)$ for some $k \in \mathbb{N}$. And if $l \geq k$ then Eq. (3.3) implies that $Z \in V^l(I)$. Thus if $V(I)$ is not empty then there exists a smallest integer k_0 such that $V^k(I) = \varnothing$ for $k < k_0$ and $V^k(I) \supset V^{k_0}(I) \neq \varnothing$ for $k \geq k_0$. Thus in any case we can write $V(I) = \bigcup_{k\in\mathbb{N}} V^k(I)$.

Theorem 3.13. (*The Nullstellensatz for inverse limits of polynomial rings*) *Let I be an ideal in A_∞ and let $I_k = \mu_k(I)$, $\forall k \in \mathbb{N}$. Then $\mathcal{I}(V(I)) = \varprojlim(\sqrt{I_k})$.*

Proof. The proof of the theorem consists of the following logically equivalent statements: Let $f = (f_k)$; then

$$\begin{aligned}
&\text{“}f \in \mathcal{I}(V(I))\text{”} \iff \text{“}f(Z) = 0, \quad \forall Z \in V(I)\text{”} \\
&\quad\iff \text{“}f(Z) = 0, \quad \forall Z \in V^k(I),\ \forall k \in \mathbb{N}\text{”} \\
&\quad\iff \text{“}f_k(Z) = 0, \quad \forall Z \in V^k(I_k),\ \forall k \in \mathbb{N}\text{”} \\
&\quad\iff \text{“}f_k \in \sqrt{I_k}, \quad \forall k \in \mathbb{N}\text{”} \quad \text{(by the classical form of the Nullstellensatz)} \\
&\quad\iff f \in \varprojlim(\sqrt{I_k}).
\end{aligned}$$

■

Corollary 3.14. *In Theorem* 3.13 *suppose in addition that I is closed and that each I_k is radical. Then $\mathcal{I}(V(I)) = I$.*

Proof. We have $\mathcal{I}(V(I)) = \varprojlim(\sqrt{I_k}) = \varprojlim(I_k) = I_\infty = \bar{I} = I$. ■

4. Invariant theory of the infinite-dimensional classical groups

In this section we apply the results of Sections 2. and 3. to the invariant theory of inductive limits of the classical groups and the symmetric groups as they act on inverse limits of polynomials in many variables. Our basic reference is [26]. As remarked by H. Weyl, the results are valid in any field of characteristic zero, but since in all our examples the underlying field is $\mathbb{C}$, we shall restrict ourselves to this case. Also, as noted by H. Weyl, *“nothing of algebraic import is lost by unitary restriction”* (for Weyl's “unitarian trick” see [25, Lemma 4.11.13, p. 349]), and by the *principle of permanence of the identities* we shall consider only the case of the complex classical groups acting on polynomial algebras over $\mathbb{C}$. Also, all theorems in this section are consequences of results in previous sections, especially of Corollary 3.8 and Theorem 3.10. Thus we shall only give a detailed proof for the case $\mathrm{O}^\infty(\mathbb{C})$.

4.1. Invariant theory of the orthogonal group $\mathrm{O}^{\infty}(\mathbb{C})$.

For $k \in \mathbb{N}$ let $\mathrm{O}_k(\mathbb{C}) = \{g \in \mathrm{GL}_k(\mathbb{C}) \mid g^{-1} = g^t\}$ and set $G_k = \mathrm{O}_k(\mathbb{C})$. For $k \leq l$ define the connecting isomorphism λ_{kl} of G_k into G_l by

$$\lambda_{kl}(g) = \begin{array}{r} \\ {\scriptstyle k}\{ \\ {\scriptstyle l-k}\{ \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{}^{k} & \overbrace{}^{l-k} \end{array} \\ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \qquad \text{for all } g \in G_k. \tag{4.1}$$

Let $\mathrm{O}^{\infty}(\mathbb{C}) = G^{\infty} = \bigcup_{k \in \mathbb{N}} G_k$ denote the inductive limit of the inductive system $\{G_k\,;\lambda_{kl}\}$. Let X_k denote the matrix of indeterminates X_{ij}, $i \geq 1$, $1 \leq j \leq k$. Let X_k^i denote the i^{th} row of X_k. We shall consider both cases when $1 \leq i \leq n$ and when the index set $\{i\}$ is unbounded. As in Example 2.12 we let $A_k = \mathbb{C}[X_k]$ and $\mu_k^l \colon A_l \to A_k$, $k, l \in \mathbb{N}$, $l \geq k$, denote the truncation homomorphisms. Let A_{∞} denote the inverse limit of the inverse spectrum $\{A_k\,;\mu_k^l\}$.

For a fixed k, G_k acts on A_k by right translation, i.e.,

$$(g, p) \longrightarrow g \cdot p, \qquad \text{where } (g \cdot p)(X_k) := p(X_k g),\ g \in G_k,\ p \in A_k. \tag{4.2}$$

Then according to [26, Theorem (2.9A), p. 53], the subalgebra J_k of G_k-invariants in A_k is generated by 1 and the homogeneous quadratic polynomials

$$f_k^{i_1 i_2} = \left(X_k^{i_1}, X_k^{i_2}\right) = \sum_{j=1}^{k} X_{i_1 j} X_{i_2 j}, \qquad \forall\, i_1, i_2 \geq 1. \tag{4.3}$$

If $1 \leq i_1, i_2 \leq n \leq k$, then the $f_k^{i_1 i_2}$ are algebraically independent. Otherwise, all relations between them are algebraic consequences of relations of the following type (see [26, Theorem (2.17.A), p. 75]):

$$\begin{vmatrix} \left(X_k^{i_1}, X_k^{j_1}\right) & \ldots & \left(X_k^{i_1}, X_k^{j_{k+1}}\right) \\ \vdots & & \vdots \\ \left(X_k^{i_{k+1}}, X_k^{j_1}\right) & \ldots & \left(X_k^{i_{k+1}}, X_k^{j_{k+1}}\right) \end{vmatrix} = 0. \tag{4.4}$$

Example 4.1. Suppose that $1 \leq i \leq 3$, $k = 2$, and $X_3 = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$, where X, Y, Z are two-dimensional row vectors. Then J_k is generated by 1 and

$$(X, X), \quad (X, Y), \quad (X, Z), \quad (Y, Y), \quad (Y, Z), \quad (Z, Z)$$

and the relations between these generators are algebraic consequences of the relation

$$\begin{vmatrix} (X, X) & (X, Y) & (X, Z) \\ (Y, X) & (Y, Y) & (Y, Z) \\ (Z, X) & (Z, Y) & (Z, Z) \end{vmatrix} = 0.$$

Let us verify that there is a well-defined action of G^∞ on A_∞ given by Eq. (3.2) of Lemma 3.1; i.e., we must verify Eq. (3.1) for $g \in G_k$, $p \in A_l$, and $l \geq k$. We have

$$\left[g \cdot \left(\mu_k^l (p)\right)\right] (X_k) = \left[\mu_k^l (p)\right] (X_k g) .$$

Since $p \in A_l$, p is a polynomial in the indeterminates X_{ij}, for $1 \leq i \leq m$ and $1 \leq j \leq l$. By identifying the indeterminates X_{ij} with the variables $X_{ij} \in \mathbb{C}$ we can write

$$p (X_l) = p (X_k \,\vdots\, X_{l-k}) ,$$

where X_l is an $m \times l$ matrix, X_k is an $m \times k$ matrix, and X_{l-k} is an $m \times (l-k)$ matrix. And $\mu_k^l (p) (X_k)$ is just $p (X_k \,\vdots\, 0)$. Therefore,

$$\mu_k^l (p) (X_k g) = p (X_k g \,\vdots\, 0) .$$

On the other hand,

$$\begin{aligned} \mu_k^l (\lambda_{kl} (g) \cdot p) (X_k) &= [\lambda_{kl} (g) \cdot p] (X_k \,\vdots\, 0) \\ &= p \left([X_k \,\vdots\, 0] \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) = p (X_k g \,\vdots\, 0) . \end{aligned}$$

Since these relations hold for all $X_k \in \mathbb{C}^{m \times k}$ and $g \in G_k$ we have

$$\mu_k^l (\lambda_{kl} (g) \cdot p) = g \cdot \left(\mu_k^l (p)\right) , \qquad \forall\, g \in G_k,\ p \in A_l,\ l \geq k,$$

which is exactly the relation (3.1).

Let

$$f^{i_1 i_2} = \sum_{j=1}^{\infty} X_{i_1 j} X_{i_2 j}, \qquad \forall\, i_1, i_2 \geq 1. \tag{4.5}$$

Then we have the following Fundamental Theorem of Invariants for $\mathrm{O}^\infty (\mathbb{C})$ acting on A_∞.

Theorem 4.2. (i) *The set $\{1, f^{i_1 i_2} \mid i_1, i_2 \geq 1\}$ forms an inverse limit basis for the subalgebra J of all $\mathrm{O}^\infty (\mathbb{C})$-invariants in A_∞.*

(ii) *If $1 \leq i_1, i_2 \leq n$, then 1 and the $\frac{1}{2} n (n+1)$ formal sums $f^{i_1 i_2}$ form an algebraic basis for the subalgebra J of all O^∞-invariants in A_∞.*

Proof. Part (i) follows from Theorem 3.10. Part (ii) follows from Corollary 3.8. ∎

4.2. Invariant theory of the special orthogonal group $\mathrm{SO}^\infty (\mathbb{C})$.

For $k \in \mathbb{N}$ let $G_k = \mathrm{SO}_k(\mathbb{C}) = \{g \in \mathrm{O}_k(\mathbb{C}) \mid \det(g) = 1\}$. Then in general the subalgebra J_k of G_k-invariants is generated by 1, the homogeneous quadratic polynomials $f^k_{i_1 i_2}$ as defined by Eq. 4.3, and the determinants

$$[(X_k^{i_1}, \ldots , X_k^{i_k})] = \det \begin{bmatrix} X_k^{i_1} \\ \vdots \\ X_k^{i_k} \end{bmatrix} , \tag{4.6}$$

where we again identify the indeterminate X_{ij} with the variable $X_{ij} \in \mathbb{C}$ (see [26, pp. 41–53]). But since for $l > k$, $\mu_k^l([(X_l^{i_1}, \ldots, X_l^{i_l})]) = 0$, it follows from Theorem 3.10 and Corollary 3.8 that the Fundamental Theorem of Invariants for $\mathrm{SO}^\infty(\mathbb{C})$ acting on A_∞ has exactly the same form as Theorem 4.2.

Theorem 4.3. (*The Nullstellensatz for the homogeneous ideal of invariants of* $\mathrm{O}^\infty(\mathbb{C})$ *and* $\mathrm{SO}^\infty(\mathbb{C})$). *In Theorem* 4.2(ii) *let* $I = (f^{i_1 i_2})$ *denote the homogeneous ideal in* A_∞ *generated by the* $\frac{1}{2}n(n+1)$ *elements* $f^{i_1 i_2}$, $1 \le i_1,\ i_2 \le n$. *If* I_k *denotes the ideal in* A_k *generated by the polynomials* $f_k^{i_1 i_2}$, $1 \le i_1,\ i_2 \le n$, *and* $I_\infty = \varprojlim I_k$ *then we have* $\mathcal{I}(V(I)) = I_\infty$.

Proof. Since obviously the canonical homomorphisms $\mu_k : A_\infty \to A_k$ are surjective Theorem 2.8(ii) implies that $\mu_k(I)$ is an ideal in A_k. Clearly, $\mu_k(I) \subset I_k$, and since $\mu_k(f^{i_1 i_2}) = f_k^{i_1 i_2}$ it follows that $\mu_k(I) = I_k$. By [17, Theorem 2.5, p. 11] the ideals I_k are prime for $k > 2n$. Hence by Theorem 2.11, I_∞ is a prime ideal of A_∞, and $\sqrt{I_\infty} = I_\infty$. Therefore, by Theorem 3.13, $\mathcal{I}(V(I)) = I_\infty$. ∎

4.3. Invariant theory of the symplectic group $\mathrm{Sp}^\infty(\mathbb{C})$.

The symplectic group $G_k = \mathrm{Sp}_{2k}(\mathbb{C})$ is defined as the group of all linear transformations which leave a skew-symmetric bilinear form invariant. For our purpose we choose the skew-symmetric bilinear form

$$[x, y] = (x_1 y_2 - x_2 y_1) + (x_3 y_4 - x_4 y_3) + \cdots + (x_{2k-1} y_{2k} - x_{2k} y_{2k-1}) \tag{4.7}$$

for all $x, y \in \mathbb{C}^{1\times 2k}$.

If we let S_k denote the $k \times k$ block diagonal matrix

$$\begin{bmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{bmatrix} \tag{4.8}$$

then Eq. (4.7) can be written as $[x, y] = x S_k y^t$. Then G_k can be defined as

$$G_k = \{g \in \mathbb{C}^{2k\times 2k} \mid g S_k g^t = S_k\}. \tag{4.9}$$

It can be easily verified that if $g \in G_k$ then automatically $\det(g) = 1$.

For $k \le l$ define the connecting isomorphism λ_{kl} of G_k into G_l by

$$\lambda_{kl}(g) = \begin{array}{r} \\ 2k\{ \\ 2(l-k)\{ \end{array} \!\!\begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{2k} & \overbrace{\quad}^{2(l-k)} \end{array} \\ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \qquad \text{for all } g \in G_k. \tag{4.10}$$

Let $\mathrm{Sp}^\infty(\mathbb{C}) = G^\infty = \bigcup_{k\in\mathbb{N}} G_k$ denote the inductive limit of the inductive system $\{G_k; \lambda_{kl}\}$. Let X_k denote the matrix of indeterminates X_{ij}, $i \ge 1$, $1 \le j \le 2k$.

Let X_k^i denote the i^{th} row of X_k. Let Y_k denote the matrix of indeterminates Y_{ij}, $i \geq 1$, $1 \leq j \leq 2k$. Let Y_k^i denote the i^{th} row of Y_k. Let $A_k = \mathbb{C}[X_k, Y_k]$ and $\mu_k^l : A_l \to A_k$, $l \geq k$, denote the truncation homomorphisms. Let A_∞ denote the inverse limit of the inverse spectrum $\{A_k; \mu_k^l\}$. For a fixed k, G_k acts on A_k via

$$\begin{cases} (g,p) \longrightarrow g \cdot p, \text{ where} \\ (g \cdot p)(X_k, Y_k) := p(X_k g, Y_k g^{\checkmark}), \\ g \in G_k,\ p \in A_k,\ g^{\checkmark} = (g^{-1})^t. \end{cases} \tag{4.11}$$

Then according to [26, Theorem (6.1.A), p. 167] the subalgebra J_k of G_k-invariants in A_k is generated by 1 and by the homogeneous quadratic polynomials

$$\begin{cases} f_k^{i_1 i_2} = [X_k^{i_1}, X_k^{i_2}] = X_k^{i_1} S_k (X_k^{i_2})^t, & i_2 > i_1 \geq 1, \\ \varphi_k^{j_1 j_2} = [Y_k^{j_1}, Y_k^{j_2}] = Y_k^{j_1} S_k (Y_k^{j_2})^t, & j_2 > j_1 \geq 1, \\ h_k^{i_1 j_1} = (X_k^{i_1}, Y_k^{j_1}) = X_k^{i_1} (Y_k^{j_1})^t, & i_1, j_1 \geq 1. \end{cases} \tag{4.12}$$

These generators are not algebraically independent. All relations between them are algebraic consequences of relations of the type (4.4) for the h_k^{ij}'s and of the types $\mathcal{J}_1 \equiv 0, \dots, \mathcal{J}_k \equiv 0$ for the $f_k^{i_s j_s}$'s and $\varphi_k^{i_s j_s}$'s (see [26, Theorem (6.1.B), p. 168] for the definition of $\mathcal{J}_1 \equiv 0, \dots, \mathcal{J}_k \equiv 0$).

Similarly to the case $G^\infty = \mathrm{O}^\infty(\mathbb{C})$ we can easily verify that there is a well-defined action of $G^\infty = \mathrm{Sp}^\infty(\mathbb{C})$ on A_∞.

Let $X = \lim\limits_{k \to \infty} X_k$, $Y = \lim\limits_{k \to \infty} Y_k$, and $S = \lim\limits_{k \to \infty} S_k$. Set

$$\begin{cases} f^{i_1 i_2} = X^{i_1} S (X^{i_2})^t, & i_2 > i_1 \geq 1, \\ \varphi^{j_1 j_2} = Y^{j_1} S (Y^{j_2})^t, & j_2 > j_1 \geq 1, \\ h^{i_1 j_1} = X^{i_1} (Y^{j_1})^t, & i_1, j_1 \geq 1. \end{cases} \tag{4.13}$$

Then we have the following Fundamental Theorem of Invariants of $\mathrm{Sp}^\infty(\mathbb{C})$ acting on A_∞.

Theorem 4.4. (i) *The set* $\{1, f^{i_1 i_2}, \varphi^{j_1 j_2}, h^{i_1 j_1}\}$ *forms an inverse limit basis for the subalgebra J of all* $\mathrm{Sp}^\infty(\mathbb{C})$*-invariants in* A_∞.

(ii) *If both X and Y have only a finite number of rows then the set* $\{1, f^{i_1 i_2}, \varphi^{j_1 j_2}, h^{i_1 j_1}\}$ *forms an algebraic basis for the subalgebra J of all* $\mathrm{Sp}^\infty(\mathbb{C})$*-invariants in* A_∞.

Proof. Part (i) follows from Theorems 3.5 and 3.10. Part (ii) follows from Corollary 3.8. ■

If X_k is an $n \times 2k$ matrix for all $k \in \mathbb{N}$, and $A_k = \mathbb{C}[X_k]$, then the subalgebra J_k of G_k-invariant polynomials is generated by the set $\{1, f_k^{i_1 i_2} \mid 1 \leq i_1 < i_2 \leq n\}$. Let $I_k = (f_k^{i_1 i_2})$ denote the homogeneous ideal in A_k generated by the $\frac{1}{2}n(n-1)$ homogeneous quadratic polynomials $f_k^{i_1 i_2}$, $1 \leq i_1 < i_2 \leq n$. For $k > n$, the ideals I_k are shown to be prime in [18, Theorem 1.5, p. 269]. Hence by Theorem 2.11, $I_\infty = \varprojlim I_k$ is a prime ideal in A_∞, and we have

Theorem 4.5. *(The Nullstellensatz for the homogeneous ideal of invariants of* $\mathrm{Sp}^\infty(\mathbb{C})$*). Let* X_k *denote an* $n \times 2k$ *matrix of indeterminates, let* $A_k = \mathbb{C}[X_k]$*, and* $A_\infty = \varprojlim A_k$*. Let* $I_k = (f_k^{i_1 i_2})$*, and* $I_\infty = \varprojlim I_k$*. Let* I *denote the ideal in* A_∞ *generated by the* $\frac{1}{2}n(n-1)$ *invariants* $f^{i_1 i_2}$*,* $1 \le i_1 < i_2 \le n$*. Then we have* $\mathcal{I}(V(I)) = I_\infty$*.*

4.4. Invariant theory of the general linear group $\mathrm{GL}^\infty(\mathbb{C})$**.**

Let X_k (resp. Y_k) denote the matrix of indeterminates X_{ij} (resp. Y_{ik}), $1 \le i$, $1 \le j \le k$. Let X_k^i (resp. Y_k^i) denote the i^{th} row of X_k (resp. Y_k). Let $A_k = \mathbb{C}[X_k, Y_k]$. Set $G_k = \mathrm{GL}_k(\mathbb{C})$; then the action of G_k on A_k is also given by Eq. (4.11). Then according to [26, Theorem (2.6.A), p. 45] the subalgebra of G_k-invariants in A_k is generated by 1 and by the homogeneous quadratic polynomials

$$\begin{cases} h_k^{i_1 i_2} & = (X_k^{i_1}, Y_k^{i_2}) = X_k^{i_1}(Y_k^{i_2})^t, \\ & = \sum_{j=1}^k X_{i_1 j} Y_{i_2 j}, \qquad i_1, i_2 \ge 1. \end{cases} \tag{4.14}$$

If $1 \le i_1, i_2 \le n \le k$ then they are algebraically independent. Otherwise, all relations between them are algebraic consequences of relations of the following type:

$$\begin{vmatrix} (X_k^{i_1}, Y_k^{j_1}) & \cdots & (X_k^{i_1}, Y_k^{j_{k+1}}) \\ \vdots & & \vdots \\ (X_k^{i_{k+1}}, Y_k^{j_1}) & \cdots & (X_k^{i_{k+1}}, Y_k^{j_{k+1}}) \end{vmatrix}. \tag{4.15}$$

As in the case of $\mathrm{Sp}^\infty(\mathbb{C})$, we let $\mathrm{GL}^\infty(\mathbb{C}) = G^\infty = \bigcup_{k \in \mathbb{N}} G_k$ denote the inductive limit of the inductive system $\{G_k; \lambda_{kl}\}$. Then there is a well-defined action of G^∞ on A_∞. Set

$$h^{i_1 i_2} = (X^{i_1}, Y^{i_2}) = X^{i_1}(Y^{i_2})^t, \qquad i_1, i_2 \ge 1. \tag{4.16}$$

Then we have the Fundamental Theorem of Invariants of $\mathrm{GL}^\infty(\mathbb{C})$ acting on A_∞.

Theorem 4.6. (i) *The set* $\{1, h^{i_1 i_2}\}$ *forms an inverse limit basis for the subalgebra* J *of all* $\mathrm{GL}^\infty(\mathbb{C})$*-invariants in* A_∞*.*

(ii) *If both* X *and* Y *have only a finite number of rows then the set* $\{1, h^{i_1 i_2}\}$ *forms an algebraic basis for* J*.*

Proof. Same as in Theorem 4.4. ∎

If X_k is a $p \times k$ matrix and Y_k is a $q \times k$ matrix for all $k \in \mathbb{N}$, let $I_k = (h_k^{i_1 i_2})$ (resp. $I = (h^{i_1 i_2})$) denote the ideal in A_k (resp. A_∞) generated by the pq invariants $h_k^{i_1 i_2}$ (resp. $h^{i_1 i_2}$), $1 \le i_1 \le p$, $1 \le i_2 \le q$. For $k > \max(p, q)$ the ideals I_k are shown to be prime in [19, Theorem 5.1, p. 213]. Hence by Theorem 2.11, $I_\infty = \varprojlim I_k$ is prime and we have the Nullstellensatz for $\mathrm{GL}^\infty(\mathbb{C})$, $\mathcal{I}(V(I)) = I_\infty$.

4.5. Invariant theory of the special linear group $\mathrm{SL}^\infty(\mathbb{C})$.

The setup is exactly the same as the general linear group $\mathrm{GL}^\infty(\mathbb{C})$ except that, in addition to the $h_k^{i_1 i_2}$'s in Eq. (4.14), the generators of the G_k-invariants are

$$[(X_k^{i_1}, \dots, X_k^{i_k})] = \det \begin{bmatrix} X_k^{i_1} \\ \vdots \\ X_k^{i_k} \end{bmatrix}, \text{ and } [(Y_k^{j_1}, \dots, Y_k^{j_k})] = \det \begin{bmatrix} Y_k^{j_1} \\ \vdots \\ Y_k^{j_k} \end{bmatrix}. \tag{4.17}$$

However, the images of the determinants under the truncation homomorphisms μ_k^l, $l > k$, are all zero. The Fundamental Theorem of Invariants and the Nullstellensatz of $\mathrm{SL}^\infty(\mathbb{C})$ acting on A_∞ are exactly the same as those of $\mathrm{GL}^\infty(\mathbb{C})$.

Before studying the co-adjoint action of $\mathrm{GL}^\infty(\mathbb{C})$ we shall discuss the invariant theory of the infinite symmetric group since there is an intimate relation between the invariant theory of $\mathcal{S}_k$ and that of the co-adjoint action of $\mathrm{GL}_k(\mathbb{C})$ (see, e.g., [16, Theorem 1.5.7, p. 10]).

4.6. Invariant theory of the infinite symmetric group $\mathcal{S}^\infty$.

Let R be any commutative ring with unit. Let $A_k = R[X_k]$ denote the polynomial ring in k variables $(x_1, \dots, x_k) = X_k$. Let $\mathcal{S}_k$ denote the symmetric group of all permutations of the set $\{1, \dots, k\}$. Then $\mathcal{S}_k$ acts on A_k via

$$\begin{cases} (\sigma, p) \longrightarrow \sigma \cdot p, \ \sigma \in \mathcal{S}_k, \ p \in A_k, \text{ where} \\ (\sigma \cdot p)(x_1, \dots, x_k) = p(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)}). \end{cases} \tag{4.18}$$

Then the subring J_k of $\mathcal{S}_k$-invariant polynomials has an algebraic basis of the form

$$\begin{aligned} S_k^0 &= 1, \\ S_k^1 &= \textstyle\sum_{1 \le i \le k} x_i, \\ S_k^2 &= \textstyle\sum_{1 \le i < j \le k} x_i x_j, \\ &\vdots \\ S_k^3 &= \textstyle\sum_{1 \le i_1 < i_2 < i_3 \le k} x_{i_1} x_{i_2} x_{i_3}, \\ &\vdots \\ S_k^k &= x_1 x_2 \cdots x_k. \end{aligned} \tag{4.19}$$

The polynomials $S_k^1, \dots, S_k^k$ are called the *elementary symmetric functions* (see [26, p. 37] or [1, Theorem 3.4, p. 548]). Set

$$T_k^n = \sum_{1 \le i \le k} x_i^n, \qquad 1 \le n \le k; \tag{4.20}$$

then it can be shown [26, pp. 38–39] that the set $\{T_k^n; 0 \le n \le k\}$ also forms an algebraic basis of J_k. In fact, there is a recursive formula expressing $\{T_k^n\}$ in terms of $\{S_k^n\}$, and vice versa [26, p. 39].

We can embed $\mathcal{S}_k$ into $\mathcal{S}_l$ for $k < l$ by defining $\lambda_{kl} : \mathcal{S}_k \to \mathcal{S}_l$ as follows:

$$\lambda_{kl}(\sigma)(i) = \begin{cases} \sigma(i), & 1 \le i \le k, \\ i, & k < i \le l, \end{cases} \qquad \forall\, \sigma \in \mathcal{S}_k. \tag{4.21}$$

Let $\mathcal{S}^\infty = \bigcup_{k=1}^\infty \mathcal{S}_k$ be the inductive limit of the inductive system $\{\mathcal{S}_k; \lambda_{kl}\}$.

Let A_∞ denote the inverse limit of the inverse spectrum $\{A_k; \mu_k^l\}$, where the connecting homomorphisms $\mu_k^l : A_l \to A_k$, $l > k$, are the truncation homomorphisms. In fact, μ_k^l can be simply defined by setting

$$\mu_k^l(x_i) = \begin{cases} x_i, & 1 \le i \le k, \\ 0, & k < i \le l. \end{cases}$$

Then it is straightforward to verify that

$$\sigma \cdot \mu_k^l(p) = \mu_k^l(\lambda_{kl}(\sigma) \cdot p).$$

Set

$$S^n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \qquad T^n = \sum_{i \in \mathbb{N}} x_i^n, \qquad n = 1, 2, \dots ; \tag{4.22}$$

then Theorem 3.10 implies the following

Theorem 4.7. *Each of the sets $\{1, S^n; n \in \mathbb{N}\}$ and $\{1, T^n; n \in \mathbb{N}\}$ forms an inverse limit basis for the subring J of all $\mathcal{S}^\infty$-invariants in the ring A_∞.*

See [11, Theorem (3.2), p. 3] for another proof that $\{S^n\}$ is algebraically independent.

4.7. Invariant theory of the co-adjoint action of $\mathrm{GL}^\infty(\mathbb{C})$**.**

Let $X_k = (x_{ij}) \in \mathbb{C}^{k \times k}$, and let $A_k = \mathbb{C}[X_k]$ denote the algebra of all polynomial functions in the variables x_{ij}, $1 \le i,\ j \le k$. Set $G_k = \mathrm{GL}_k(\mathbb{C})$; then the adjoint representation of G_k on $\mathbb{C}^{k \times k}$ is defined by

$$g \cdot X_k = g X_k g^{-1}, \qquad \forall\, g \in G_k, X \in \mathbb{C}^{k \times k}. \tag{4.23}$$

The co-adjoint representation of G_k on A_k is defined by

$$\begin{cases} (g, p) \longrightarrow g \cdot p, \text{ where } (g \cdot p)(X_k) = p(g^{-1} \cdot X_k) = p(g^{-1} X_k g), \\ \forall\, g \in G_k,\ p \in A_k. \end{cases} \tag{4.24}$$

If $X_k \in \mathbb{C}^{k \times k}$ let $\chi_{X_k}(t) = \det(I - tX_k)$ denote the characteristic polynomial of X_k in the indeterminate t. Then we have

$$\begin{aligned} \chi_{X_k}(t) &= \det(I - tX_k) \\ &= 1 - S^1(X_k)t + S^2(X_k)t^2 + \cdots + (-1)^k S_k^k(X_k) t^k, \end{aligned} \tag{4.25}$$

and it can be shown that

$$S_k^n(X_k) = \sum \det \begin{bmatrix} X_{i_1 i_1} & \cdots & X_{i_1 i_n} \\ \vdots & & \vdots \\ X_{i_n i_1} & \cdots & X_{i_n i_n} \end{bmatrix} \equiv \sum \triangle_{i_1 \cdots i_n}^{i_1 \cdots i_n}(X_k), \tag{4.26}$$

where the sum is over all n-shuffles $(i_1, \dots, i_n)$, $i_1 < i_2 < \cdots < i_n$, $1 \le n \le k$.

If $Y_k \in \mathbb{C}^{k \times k}$ let $\mathrm{Tr}(Y) = \sum_{i=1}^k Y_{ii}$ denote the trace of Y and define

$$T_k^n = \mathrm{Tr}(X_k^n), \qquad 1 \le n \le k. \tag{4.27}$$

Let J_k denote the subalgebra of all G_k-invariant polynomials. Then we have the following fundamental theorem for the theory of G_k-invariant polynomials.

Theorem 4.8. *The algebra J_k is generated by the constants and the algebraically independent polynomials $S_k^1, \dots, S_k^k$. The same statement holds for $T_k^1, \dots, T_k^k$. Moreover, the following recursive formula also holds:*

$$(-1)^n (n+1) S_k^{n+1} = \sum_{i+j=n} (-1)^i S_k^i T_k^{j+1}, \qquad n = 0, 1, \dots, k-1. \tag{4.28}$$

Proof. See [16, Theorem 1.5.7, p. 10] and [26, p. 39]. ■

As in Subsection 4.4., it is easy to verify that there is a well-defined action of $G^\infty = \mathrm{GL}^\infty(\mathbb{C})$ on A_∞.

Let $S^n = \underleftarrow{\lim} S_k^n$ and $T^n = \underleftarrow{\lim} T^n$. Let $X = \lim X_k$ denote the inductive limit of X_k. Then S^n and T^n can be symbolically represented as

$$S^n = \sum_{i_1 < i_2 < \cdots < i_n} \triangle_{i_1 \cdots i_n}^{i_1 \cdots i_n}(X), \qquad T^n = \mathrm{Tr}(X^n), \qquad n \in \mathbb{N}. \tag{4.29}$$

Then Theorem 3.10 implies the following

Theorem 4.9. *Each of the sets $\{1, S^n; n \in \mathbb{N}\}$ and $\{1, T^n; n \in \mathbb{N}\}$ forms an inverse limit basis for the subalgebra J of all G^∞-invariants in the algebra A_∞.*

Remark 4.10. Although they are both bases for J, for practical applications the basis $\{1, T^n; n \in \mathbb{N}\}$ is more suitable because of its simpler form (see [22, Remark 3.12]).

5. Conclusion

In this article we have developed a coherent and comprehensive invariant theory of inductive limits of groups acting on inverse limits of the categories of modules, rings, or algebras. On one hand, we have succeeded in generalizing Hilbert's finiteness theorem to many cases in our context. On the other hand, there are many cases in which the bases of the rings of invariants are not finitely generated, and in these cases the notion of an *algebraic basis is not adequate.* This led us to introduce the notion of *inverse limit basis* which naturally involves a topology on inverse limits. This is illustrated by examples in Subsections 4.6. and 4.7.. Also, the example in Subsection 4.7. can be generalized to the case of the inductive limit G^∞ of the chain $\{G_k\}$, where each G_k is a semisimple connected complex analytic group acting on the algebra of polynomial functions on its Lie algebra via the co-adjoint representation. Then the *Chevalley restriction theorem* (see, e.g., [25, Theorem 4.9.2, p. 335]) gives us a procedure to find a basis for the ring of G_k-invariants. This was achieved in a more general context by Procesi in [15]. We hope to generalize these invariants to our context in a future work.

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