Journal of Lie Theory
Volume **14** (2004) 523–535

On Dimension Formulas for $\mathfrak{gl}(m|n)$ Representations

E. M. Moens and J. Van der Jeugt

Communicated by H. Schlosser

Abstract. We investigate new formulas for the dimension and superdimension of covariant representations V_λ of the Lie superalgebra $\mathfrak{gl}(m|n)$. The notion of t-dimension is introduced, where the parameter t keeps track of the $\mathbb{Z}$-grading of V_λ. Thus when $t=1$, the t-dimension reduces to the ordinary dimension, and when $t=-1$ it reduces to the superdimension. An interesting formula for the t-dimension is derived from a recently obtained new formula for the supersymmetric Schur polynomial $s_\lambda(x/y)$, which yields the character of V_λ. It expresses the t-dimension as a simple determinant. For a special choice of λ, the new t-dimension formula gives rise to a Hankel determinant identity.

1. Introduction

Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{gl}(m|n)$. The general linear Lie superalgebra is one of the standard families of classical Lie superalgebras. Lie superalgebras are characterized by a $\mathbb{Z}_2$-grading $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus g_{\bar 1}$. For the general theory on classical Lie superalgebras and their representations, we refer to [5, 6, 16].

Let $\mathfrak{h}\subset\mathfrak{g}$ be the Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_{+1}$ be the $\mathbb{Z}$-grading that is consistent with the $\mathbb{Z}_2$-grading of $\mathfrak{g}$. Note that $\mathfrak{g}_0=\mathfrak{g}_{\bar 0}=\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)$. The dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ has a natural basis $\{\epsilon_1,\ldots,\epsilon_m,\delta_1,\ldots,\delta_n\}$, and the roots of $\mathfrak{g}$ can be expressed in terms of this basis. We shall work here with the so-called *distinguished choice* [5] for a triangular decomposition of $\mathfrak{g}$. In that case, the positive even roots are given by $\{\epsilon_i-\epsilon_j|1\le i<j\le m\}\cup\{\delta_i-\delta_j|1\le i<j\le n\}$, and the positive odd roots by $\{\epsilon_i-\delta_j|1\le i\le m,\ 1\le j\le n\}$.

Representation theory of Lie superalgebras, and in particular of $\mathfrak{gl}(m|n)$ or its simple counterpart $\mathfrak{sl}(m|n)$, is not a straightforward copy of the corresponding theory for simple Lie algebras. It is mainly due to the existence of *atypical* representations [6] that problems occur [22, 23, 24], in particular to compute the character. Only recently a solution has been proposed to some of these problems [17, 3] for $\mathfrak{gl}(m|n)$. In this paper, however, we shall be dealing only with the so-called covariant representations of $\mathfrak{gl}(m|n)$, for which an explicit character formula is known.

Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$. Such modules are $\mathfrak{h}$-diagonalizable with weight decomposition $V=\oplus_\mu V(\mu)$, and the character is defined to be $\operatorname{ch} V = \sum_\mu \dim V(\mu)\, e^\mu$, where e^μ ($\mu \in \mathfrak{h}^*$) is the formal exponential. Let Λ be the highest weight of V. We shall consider the specialization of $\operatorname{ch} V$ determined by

$$\begin{aligned} F(e^{\epsilon_i}) &= 1 \quad (i = 1, \ldots, m) \\ F(e^{\delta_j}) &= t \quad (j = 1, \ldots, n). \end{aligned} \tag{1}$$

This specialization is consistent with the $\mathbb{Z}$-grading of $\mathfrak{g}$, and the corresponding $\mathbb{Z}$-grading of V. The specialization of the character of V under F is referred to as the t-dimension of V and denoted by $\dim_t(V)$:

$$\dim_t(V) = F(\operatorname{ch} V) = \sum_\mu \dim V(\mu) F(e^\mu). \tag{2}$$

Often, the t-dimension would be defined [7, §10] as $F(e^{-\Lambda} \operatorname{ch} V)$, with Λ the highest weight of V; but here (2) is more convenient. The t-dimension of V stands for the polynomial

$$F(e^\Lambda) \sum_{j \in \mathbb{Z}_+} \dim V_{-j}\, t^j, \tag{3}$$

where $V = V_0 \oplus V_{-1} \oplus V_{-2} \oplus \cdots$ is the $\mathbb{Z}$-grading of V. Note that for the $\mathbb{Z}_2$-grading $V = V_{\bar 0} \oplus V_{\bar 1}$ we have $V_{\bar 0} = V_0 \oplus V_{-2} \oplus \cdots$ and $V_{\bar 1} = V_{-1} \oplus V_{-3} \oplus \cdots$. Therefore, the dimension of V is found by putting $t = 1$ in the expression for the t-dimension, whereas the superdimension of V is found by putting $t = -1$. So the t-dimension can also be seen as an extension of the notion of dimension and superdimension.

This paper is dealing with the computation of the t-dimension of a particular class of finite-dimensional irreducible representations of $\mathfrak{gl}(m|n)$, namely the covariant representations. These were introduced by Berele and Regev [2], and Sergeev [18]. They showed that the tensor product of N copies of the natural $(m+n)$-dimensional representation of $\mathfrak{gl}(m/n)$ is completely reducible, and that the irreducible components V_λ can be labeled by a partition λ of N such that λ is inside the (m,n)-hook, i.e. such that $\lambda_{m+1} \leq n$. Berele and Regev not only introduced these representations, they also gave a character formula for them. The character of V_λ is known as a supersymmetric Schur function [2, 9, 20]. It is a polynomial in variables x_i ($i = 1, \ldots, m$) and y_j ($j = 1, \ldots, n$) with $x_i = e^{\epsilon_i}$ and $y_j = e^{\delta_j}$, and denoted by

$$\operatorname{ch} V_\lambda = s_\lambda(x/y). \tag{4}$$

There exist a number of expressions for $s_\lambda(x/y)$. One is a combinatorial expression, by means of supertableaux [2, 9]. Another expression is a formula due to Sergeev and Pragacz [14, 21, 15]. These two formulas, however, are less convenient to determine the t-dimension. In order to compute the t-dimension, there are two useful formulas. The first is the classical formula relating the supersymmetric Schur function $s_\lambda(x/y)$ to the determinant of elementary or complete supersymmetric polynomials. These formulas go back to [4, 1], see also [9]. The second is a new determinantal formula for supersymmetric Schur polynomials [12].

For the first formula, consider the complete supersymmetric functions defined by

$$h_r(x/y) = \sum_{k=0}^{r} h_{r-k}(x) e_k(y), \tag{5}$$

where h_{r-k} and e_k are the complete and elementary symmetric functions [11] respectively. Then the supersymmetric Schur polynomial is given by

$$s_\lambda(x/y) = \det_{1\le i,j\le \ell(\lambda)} \Big(h_{\lambda_i - i + j}(x/y) \Big), \tag{6}$$

where $\ell \equiv \ell(\lambda)$ is the length of the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. The polynomials $s_\lambda(x/y)$ are identically zero when $\lambda_{m+1} > n$.

Since $x_i = e^{\epsilon_i}$ and $y_j = e^{\delta_j}$, the specialization (1) corresponds to putting each $x_i = 1$ and $y_j = t$ in $s_\lambda(x/y)$. For the elementary and complete symmetric functions, such specializations are well-known:

$$h_r(x_1, \ldots, x_m)\Big|_{x_i=1} = \binom{m+r-1}{r} = \binom{m+r-1}{m-1}, \tag{7}$$

$$e_r(x_1, \ldots, x_m)\Big|_{x_i=1} = \binom{m}{r}. \tag{8}$$

Thus it follows from (5) and (6) that

Proposition 1.1. *The t-dimension of V_λ is given by the determinant*

$$\dim_t V_\lambda = \det_{1\le i,j\le \ell(\lambda)} \left(\sum_{k=0}^{\lambda_i - i + j} \binom{m + \lambda_i - i + j - k - 1}{\lambda_i - i + j - k} \binom{n}{k} t^k \right). \tag{9}$$

Although this formula is simple to derive, it should be observed that in general the matrix elements in the right hand side of (9) do not have a “closed form” expression [13]: they remain polynomials in t. Even for $t = 1$, the expression $\sum_{k=1}^{r} \binom{m+r-k-1}{r-k}\binom{n}{k}$ cannot be simplified in general. Only for $t = -1$ we have

$$\sum_{k=0}^{r} \binom{m+r-k-1}{r-k}\binom{n}{k}(-1)^k = \binom{m-n-1+r}{r}.$$

This is related to the fact that

$$\sum_{k=0}^{r} \binom{m+r-k-1}{r-k}\binom{n}{k} t^k = \binom{m+r-1}{r}\, {}_2F_1\left({-r, -n \atop -m-r+1} ; -t \right), \tag{10}$$

in terms of the ${}_2F_1$ hypergeometric function [13, 19], and the terminating ${}_2F_1$ series – with general parameters – is summable only with argument 1.

This implies that for $t = -1$, i.e. the superdimension formula $\operatorname{sdim} V_\lambda$, the expression (9) can be simplified:

$$\operatorname{sdim} V_\lambda = \det_{1\le i,j\le \ell(\lambda)} \left(\binom{m-n-1+\lambda_i - i + j}{\lambda_i - i + j} \right) \tag{11}$$

$$= \frac{\prod_{i<j}(\lambda_i - i - \lambda_j + j)}{\prod_i (\lambda_i - i + l(\lambda))} \prod_i (m-n+1-i)_{\lambda_i}. \tag{12}$$

Herein, $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol [19], and the determinant in (11) can be written in closed form using [10, (3.11)]. So in general the superdimension has a closed form expression (12), whereas the dimension has not.

Observe that (12) yields: if $m \leq n$ then $\operatorname{sdim} V_\lambda = 0$ when $\lambda_1 + m > n$ and $\operatorname{sdim} V_\lambda \neq 0$ when $\lambda_1 + m \leq n$; if $m > n$ then $\operatorname{sdim} V_\lambda = 0$ when $\lambda'_1 + n > m$ and $\operatorname{sdim} V_\lambda \neq 0$ when $\lambda'_1 + n \leq m$ (where λ' is the conjugate of λ).

In the following section we shall consider the new determinantal formula for supersymmetric Schur polynomials [12], and use it to compute the t-dimension. This time, the expression for $\dim_t(V_\lambda)$ is quite different from (9): it reduces again to a determinant, but now the matrix elements are closed forms in t instead of hypergeometric series in t. We shall then simplify this expression, and discuss some applications.

2. A formula for the t-dimension

The starting point of our new t-dimension formula is the recently introduced determinantal formula for the supersymmetric Schur function $s_\lambda(x/y)$ [12], deduced using a character formula of Kac and Wakimoto [8]. Let $x = x^{(m)} = (x_1, \ldots, x_m)$ and $y = y^{(n)} = (y_1, \ldots, y_n)$; let λ be a partition with $\lambda_{m+1} \leq n$ (i.e. λ is inside the (m,n)-hook), and let k be the (m,n)-index of λ:

$$k = \min\{j | \lambda_j + m + 1 - j \leq n\}; \tag{13}$$

see [12] for its meaning: in particular, $m - k + 1$ is the *atypicality* of the representation V_λ. As usual, λ' denotes the conjugate of λ. The new formula reads:

$$s_\lambda(x/y) = \pm D^{-1} \det \begin{pmatrix} \left(\frac{1}{x_i + y_j}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} & \left(x_i^{\lambda_j + m - n - j}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq k-1}} \\ \left(y_j^{\lambda'_i + n - m - i}\right)_{\substack{1\leq i\leq n-m+k-1 \\ 1\leq j\leq n}} & 0 \end{pmatrix} \tag{14}$$

with

$$D = \frac{\prod_{i<j}(x_i - x_j) \prod_{i<j}(y_i - y_j)}{\prod_{i,j}(x_i + y_j)}.$$

Observe that the sign in (14) is $(-1)^{mn-m+k-1}$; since its role is not essential here, we shall usually just write $\pm$.

In order to deduce a t-dimension formula from (14) we will need some simple properties of symmetric polynomials and a careful analysis of the determinant in (14) using row and column operations.

We have already mentioned the complete and elementary symmetric functions. Another class that we need here are the monomial symmetric functions $m_\lambda(x)$ [11]. The number of terms in $m_\lambda(x)$ is easy to count, so that we have the following counterpart of (8):

$$m_{(0^{r_0}1^{r_1}\ldots k^{r_k})}(x_1, \ldots, x_m)\Big|_{x_i=1} = \frac{m!}{r_0! r_1! \ldots r_k!} \text{ where } \sum_{i=0}^{k} r_i = m. \tag{15}$$

The following lemma gives some simple decomposition properties of symmetric functions:

Lemma 2.1. *Let $x = x' + x''$ be a decomposition of $x = (x_1, \ldots, x_m)$ in two disjoint subsets. Then*

$$h_r(x) = \sum_{k=0}^{r} h_k(x')h_{r-k}(x'') \quad \text{and} \quad m_\lambda(x) = \sum_{\mu\cup\nu=\lambda} m_\mu(x')m_\nu(x'').$$

Proof. The proof for the $h_r(x)$ polynomials follows immediately from the generating function for these polynomials [11, (I.2.5)]. For the m_λ, we use induction on $|x''|$. First, let $x'' = (x_m)$. By the definition of $m_\lambda(x)$, with $\lambda = (\lambda_1, \lambda_2, \ldots)$, if follows that

$$m_\lambda(x) = m_\lambda(x') + \sum_{\lambda_i\cup\mu=\lambda} m_\mu(x')m_{\lambda_i}(x_m) = \sum_{\mu\cup\nu=\lambda} m_\mu(x')m_\nu(x_m).$$

Now assume that the property holds for $|x''| \leq q$. Let $\bar{x}' = x' \setminus \{x_i\}$ and $\bar{x}'' = x'' \cup \{x_i\}$ for a certain $x_i \in x'$. Then, using the induction hypothesis:

$$\begin{aligned} m_\lambda(x) &= \sum_{\tau\cup\kappa=\lambda} m_\tau(x')m_\kappa(x'') = \sum_{\tau\cup\kappa=\lambda} \Bigg(\sum_{\mu\cup\eta=\tau} m_\mu(\bar{x}')m_\eta(x_i) \Bigg) m_\kappa(x'') \\ &= \sum_{\mu\cup\nu=\lambda} m_\mu(\bar{x}') \Bigg(\sum_{\eta\cup\kappa=\nu} m_\eta(x_i)m_\kappa(x'') \Bigg) = \sum_{\mu\cup\nu=\lambda} m_\mu(\bar{x}')m_\nu(\bar{x}''). \qquad \blacksquare \end{aligned}$$

Next, we shall use a number of times the same sequence of elementary row or column operations in matrices. So it is convenient to fix these in an algorithm:
Algorithm 1 Given a matrix with at least m rows, with R_i denoting row i. The algorithm consists of the following row operations:

$$\begin{array}{lll} \text{Step 1:} & R_i \longrightarrow \dfrac{R_i - R_1}{x_i - x_1}, & \text{for } 1 < i \leq m; \\ \text{Step 2:} & R_i \longrightarrow \dfrac{R_i - R_2}{x_i - x_2}, & \text{for } 2 < i \leq m; \\ \quad\vdots & & \\ \text{Step } m-1: & R_m \longrightarrow \dfrac{R_m - R_{m-1}}{x_m - x_{m-1}}. & \end{array}$$

So the total number of row operations is $m(m-1)/2$.
Algorithm 2 Given a matrix with at least n columns, with C_j denoting column j. This algorithm consists of the following $n(n-1)/2$ column operations:

$$\begin{array}{lll} \text{Step 1:} & C_j \longrightarrow \dfrac{C_j - C_1}{y_j - y_1}, & \text{for } 1 < j \leq n; \\ \text{Step 2:} & C_j \longrightarrow \dfrac{C_j - C_2}{y_j - y_2}, & \text{for } 2 < j \leq n; \\ \quad\vdots & & \\ \text{Step } n-1: & C_n \longrightarrow \dfrac{C_n - C_{n-1}}{y_n - y_{n-1}}. & \end{array}$$

Lemma 2.2. *Let $(r_1, r_2, \ldots)$ be a sequence of (non-negative) integers, and consider matrices*

$$A = \Big(h_{r_j}(x_i) \Big)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}, \qquad B = \Big(h_{r_i}(y_j) \Big)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}.$$

Then Algorithm 1 transforms A into $A^\star$, and Algorithm 2 transforms B into $B^\star$, with

$$A^\star = \Big(h_{r_j-i+1}(x_1, \ldots, x_i) \Big)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}, \qquad B^\star = \Big(h_{r_i-j+1}(y_1, \ldots, y_j) \Big)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}.$$

Proof. It is sufficient to give the proof for A only (so we assume $p \geq m$). Denote by $A^{(s)}$ the matrix obtained after step s of the algorithm. We shall prove that the (i,j)-element of $A^{(s)}$ is given by $A^{(s)}_{i,j} = h_{r_j-s}(x_1, \ldots, x_s, x_i)$, by induction on s. Clearly, in the first step the elements $h_{r_j}(x_i)$ are replaced by

$$\frac{x_i^{r_j} - x_1^{r_j}}{x_i - x_1} = x_i^{r_j-1} + x_i^{r_j-2}x_1 + \ldots + x_i x_1^{r_j-2} + x_1^{r_j-1} = h_{r_j-1}(x_1, x_i).$$

Now we can assume that after step s we have $A^{(s)}_{i,j} = h_{r_j-s}(x_1, \ldots, x_s, x_i)$ for all $i > s$. Step $s+1$ consist of the operations $R_i \longrightarrow (R_i - R_{s+1})/(x_i - x_{s+1})$ for all $i > s+1$. Thus the element $A^{(s+1)}_{i,j}$ becomes, using Lemma 2.1 a number of times:

$$\begin{aligned}
&\frac{h_{r_j-s}(x_1, \ldots, x_s, x_i) - h_{r_j-s}(x_1, \ldots, x_s, x_{s+1})}{x_i - x_{s+1}} = \\
&= \sum_{l=0}^{r_j-s-1} h_l(x_1, \ldots, x_s) \frac{x_i^{r_j-s-l} - x_{s+1}^{r_j-s-l}}{x_i - x_{s+1}} \\
&= \sum_{l=0}^{r_j-s-1} h_l(x_1, \ldots, x_s)(x_i^{r_j-s-l-1} + x_i^{r_j-s-l-2}x_{s+1} + \ldots \\
&\qquad\qquad + x_i x_{s+1}^{r_j-s-l-2} + x_{s+1}^{r_j-s-l-1}) \\
&= \sum_{l=0}^{r_j-s-1} h_l(x_1, \ldots, x_s) h_{r_j-s-l-1}(x_{s+1}, x_i) = h_{r_j-s-1}(x_1, \ldots, x_{s+1}, x_i).
\end{aligned}$$

Since the algorithm applies in total $i-1$ row transformations on row i, it follows that $A^\star_{i,j} = h_{r_j-i+1}(x_1, \ldots, x_i)$. ∎

Lemma 2.3. *Algorithm 1 transforms $R = \left(\dfrac{1}{x_i + y_j} \right)_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ into*

$$R^\star = \left(\frac{(-1)^{i-1}}{\prod_{l=1}^{i}(x_l + y_j)} \right)_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}.$$

Proof. Denote by $R^{(s)}$ the matrix obtained after step s of the algorithm. We shall prove that the (i,j)-element of $R^{(s)}$ is given by

$$R^{(s)}_{i,j} = \frac{(-1)^s}{\prod_{l=1}^s (x_l+y_j)(x_i+y_j)}.$$

In the first step, the operations are $R_i \longrightarrow \frac{R_i - R_1}{x_i - x_1}$ for $i>1$, so

$$R^{(1)}_{i,j} = \left(\frac{1}{x_i+y_j} - \frac{1}{x_1+y_j}\right)\frac{1}{x_i-x_1} = \frac{-1}{(x_1+y_j)(x_i+y_j)}.$$

Next we use induction on s. One finds:

$$\begin{aligned} R^{(s+1)}_{i,j} &= \left(\frac{(-1)^s}{\prod_{l=1}^s (x_l+y_j)(x_i+y_j)} - \frac{(-1)^s}{\prod_{l=1}^s (x_l+y_j)(x_{s+1}+y_j)}\right)\frac{1}{x_i-x_{s+1}} \\ &= \frac{(-1)^s}{\prod_{l=1}^s (x_l+y_j)} \cdot \frac{(-1)}{(x_{s+1}+y_j)(x_i+y_j)}. \end{aligned}$$

Since the algorithm applies in total $i-1$ row transformations on row i, the result follows. ∎

The following is a technical lemma on partitions, using the reverse lexicographic ordering [11, §I.1] for partitions of the same integer. So when we write $\lambda \le \mu$, this means that λ and μ are partitions of the same integer (i.e. $|\lambda| = |\mu|$) with either $\lambda = \mu$ or else the first non-vanishing difference $\lambda_i - \mu_i$ negative.

Lemma 2.4. *Assume that α, β, ν, μ are partitions with $\ell(\alpha) = s+1$, $\ell(\beta) = s+2$ and $\ell(\nu) = 2$. Then, for $i, s, t \in \mathbb{N}$:*

$$\alpha \le (i, 1^s), \quad \mu \cup (t) = \alpha, \quad \nu \le (t,1) \qquad \Leftrightarrow \qquad \beta = \mu \cup \nu \le (i, 1^{s+1}).$$

Proof. Assume that $\alpha \le (i,1^s)$, $\mu \cup (t) = \alpha$ and $\nu \le (t,1)$, then $|\beta| = |\mu| + |\nu| = (i+s-t) + (t+1) = |(i, 1^{s+1})|$. Furthermore $\beta_1 = \max(\mu_1, \nu_1) \le \max(\mu_1, t) = \alpha_1 \le i$, so $\beta \le (i, 1^{s+1})$.
Conversely, assume that $\beta = \mu \cup \nu \le (i, 1^{s+1})$, then ν is of the form $\nu = (\beta_k, \beta_l)$ ($\beta_l > 0$), so $\nu \le (\beta_k + \beta_l - 1, 1)$. Put $t = \beta_k + \beta_l - 1$ and $\alpha = \mu \cup (t)$. Then $|\alpha| = |\mu| + |(t)| = (i+s+1-\beta_k-\beta_l) + (\beta_k+\beta_l-1) = i+s$. Since $\ell(\mu) = s$ we have that $|\mu| \ge s$, and $|(t)| \le i$. So $\alpha_1 = \max(\mu_1, t) \le \max(\mu_1, i) \le i$, thus $\alpha \le (i, 1^s)$. ∎

This technical lemma is needed in the following:

Lemma 2.5. *Let $Y_j = \frac{1}{1+y_j}$ and consider the matrix*

$$R = \left(\frac{(-1)^{i+1}}{(1+y_j)^i}\right)_{\substack{1\le i\le m\\1\le j\le n}} = \left((-1)^{i+1} Y_j^i\right)_{\substack{1\le i\le m\\1\le j\le n}}.$$

Algorithm 2 transforms R into

$$R^\star = \left((-1)^{i+j} \sum_{\alpha \le (i, 1^{j-1})} m_\alpha(Y_1, \dots, Y_j)\right)_{\substack{1\le i\le p\\1\le j\le n}}.$$

Proof. Observe that $1/(y_i - y_j) = Y_iY_j/(Y_j - Y_i)$. Denote, as usual, by $R^{(s)}$ the matrix obtained after step s of the algorithm. We shall prove that

$$R_{i,j}^{(s)} = (-1)^{i+s+1} \sum_{\alpha \leq (i,1^s)} m_\alpha(Y_1, \ldots, Y_s, Y_j), \qquad \text{for all } j > s.$$

Step 1 consist of the column operations $C_j \longrightarrow \dfrac{C_j - C_1}{Y_1 - Y_j} Y_1 Y_j$, so $R_{i,j}^{(1)}$ is given by

$$\begin{aligned}
&\Big((-1)^{i+1}Y_j^i - (-1)^{i+1}Y_1^i\Big)\frac{Y_1Y_j}{Y_1 - Y_j}\\
&= (-1)^{i+2}(Y_j^iY_1 + Y_j^{i-1}Y_1^2 + \ldots + Y_j^2Y_1^{i-1} + Y_jY_1^i)\\
&= (-1)^{i+2} \sum_{\alpha \leq (i,1)} m_\alpha(Y_1, Y_j).
\end{aligned}$$

Next we use induction on s. This yields, using Lemma 2.1:

$$\begin{aligned}
&R_{i,j}^{(s+1)}\\
&= (-1)^{i+s+1} \sum_{\alpha \leq (i,1^s)} \Big(m_\alpha(Y_1, \ldots, Y_s, Y_j) - m_\alpha(Y_1, \ldots, Y_s, Y_{s+1})\Big)\frac{Y_jY_{s+1}}{Y_{s+1} - Y_j}\\
&= (-1)^{i+s+1} \sum_{\alpha \leq (i,1^s)} \sum_{\mu \cup (t) = \alpha} m_\mu(Y_1, \ldots, Y_s)(Y_j^t - Y_{s+1}^t)\frac{Y_jY_{s+1}}{Y_{s+1} - Y_j}\\
&= (-1)^{i+s+2} \sum_{\alpha \leq (i,1^s)} \sum_{\mu \cup (t) = \alpha} m_\mu(Y_1, \ldots, Y_s) \sum_{\nu \leq (t,1)} m_\nu(Y_{s+1}, Y_j).
\end{aligned}$$

Next, we use Lemma 2.4 and finally Lemma 2.1 again:

$$\begin{aligned}
R_{i,j}^{(s+1)} &= (-1)^{i+s+2} \sum_{\beta \leq (i,1^{s+1})} \Big(\sum_{\mu \cup \nu = \beta} m_\mu(Y_1, \ldots, Y_s) m_\nu(Y_{s+1}, Y_j)\Big)\\
&= (-1)^{i+s+2} \sum_{\beta \leq (i,1^{s+1})} m_\beta(Y_1, \ldots, Y_{s+1}, Y_j).
\end{aligned}$$

Since the algorithm applies in total $j-1$ column transformations on column j, the result follows. ■

The next lemma is about the specialization of such matrix elements. By $y = 1$ we mean the substitution $(y_1 = 1, \ldots, y_j = 1)$.

Lemma 2.6.

$$R_{i,j} = \sum_{\alpha \leq (i,1^{j-1})} m_\alpha(y_1, \ldots, y_j)\Big|_{y=1} = \binom{i+j-2}{j-1}.$$

Proof. It is easy to verify (e.g. using (15)) that $R_{1,j} = 1$ and $R_{i,1} = 1$. Now,

$$\begin{aligned}
R_{i,j} &= \sum_{\alpha \leq (i,1^{s+1})} m_\alpha(y_1, \ldots, y_j)\Big|_{y=1}\\
&= \Big(\Big(\sum_{\mu \leq (i,1^{j-2})} m_\mu(y_1, \ldots, y_{j-1})\Big)y_j + \Big(\sum_{\nu \leq (i-1,1^{j-1})} m_\nu(y_1, \ldots, y_j)\Big)y_j\Big)\Big|_{y=1}\\
&= R_{i,j-1} + R_{i-1,j}.
\end{aligned}$$

Hence the result follows. ■

Now we have all ingredients to determine the specialization of (14).

Theorem 2.7. *The t-dimension of V_λ is given by* $\dim_t(V_\lambda) = \pm(1+t)^{mn} R(\lambda)$ *with*

$$R(\lambda) = \det \begin{pmatrix} \left(\frac{(-1)^{i+j}}{(1+t)^{i+j-1}} \binom{i+j-2}{j-1} \right)_{\substack{1\le i\le m\\ 1\le j\le n}} & \left(\binom{\lambda_j+m-n-j}{i-1} \right)_{\substack{1\le i\le m\\ 1\le j\le k-1}} \\ \left(t^{\lambda'_i+n-m-i-j+1} \binom{\lambda'_i+n-m-i}{j-1} \right)_{\substack{1\le i\le n-m+k-1\\ 1\le j\le n}} & 0 \end{pmatrix}. \tag{16}$$

Proof. Consider the determinant in (14) and apply Algorithm 1 on the corresponding matrix. From Lemmas 2.2 and 2.3 it follows that the first m rows of this matrix become

$$\begin{pmatrix} \left(\frac{(-1)^{i-1}}{\prod_{l=1}^{i}(x_l+y_j)} \right)_{\substack{1\le i\le m\\ 1\le j\le n}} & \left(h_{\lambda_j+m-n-i-j+1}(x_1,\dots,x_i) \right)_{\substack{1\le i\le m\\ 1\le j\le k-1}} \end{pmatrix}$$

while the determinant has been multiplied by a factor $\prod_{i>j}(x_i-x_j)$. Now we can make the substitution $x_i = 1$; then (14) becomes

$$\pm \frac{\prod_j (1+y_j)^m}{\prod_{i<j}(y_i-y_j)} \det \begin{pmatrix} \left(\frac{(-1)^{i-1}}{(1+y_j)^i} \right)_{\substack{1\le i\le m\\ 1\le j\le n}} & \left(\binom{\lambda_j+m-n-j}{i-1} \right)_{\substack{1\le i\le m\\ 1\le j\le k-1}} \\ \left(y_j^{\lambda'_i+n-m-i} \right)_{\substack{1\le i\le n-m+k-1\\ 1\le j\le n}} & 0 \end{pmatrix}.$$

Next apply Algorithm 2 on the first n columns of this matrix. Using Lemmas 2.2 and 2.5, this becomes

$$\prod_j (1+y_j)^m \det \begin{pmatrix} \left((-1)^{i+j} \sum_{\alpha\le(i,1^{j-1})} m_\alpha(Y_1,\dots,Y_j) \right)_{\substack{1\le i\le m\\ 1\le j\le n}} & \left(\binom{\lambda_j+m-n-j}{i-1} \right)_{\substack{1\le i\le m\\ 1\le j\le k-1}} \\ \left(h_{\lambda'_i+n-m-i-j+1}(y_1,\dots,y_j) \right)_{\substack{1\le i\le n-m+k-1\\ 1\le j\le n}} & 0 \end{pmatrix}.$$

Finally, substituting $y_j = t$, using Lemma 2.6, and the fact that we are dealing with homogeneous symmetric polynomials, leads to the result. ∎

Compared to (9), (16) has the advantage that each matrix element is a simple binomial coefficient multiplied by a power of t or $(1+t)$, and no longer a finite series of type ${}_2F_1(-t)$. So in general (16) is easier to compute. Furthermore, its simple form is more appropriate to deduce certain properties of the t-dimension for particular V_λ, as we shall demonstrate in the following section.

3. Further simplifications, examples and applications

Let λ be a partition in the (m,n)-hook, and λ' its conjugate. Recall the definition of the (m,n)-index k of λ in (13), and let us also define the related integer r:

$$k = \min\{i | \lambda_i + m + 1 - i \le n\}, \qquad (1 \le k \le m+1);$$
$$r = n - m + k - \lambda_k - 1.$$

For the combinatorial meaning of r, see [12]. Since λ is in the (m,n)-hook, λ' is in the (n,m)-hook, and we can define its (n,m)-index k' and the corresponding number r':

$$k' = \min\{i|\lambda'_i + n + 1 - i \le m\}, \qquad (1 \le k' \le n+1);$$
$$r' = m - n + k' - \lambda'_{k'} - 1.$$

Applying the determinant formula (14) for $s_\lambda(x/y)$ and for $s_{\lambda'}(y/x)$ yields the same, with determinants of transposed matrices. Comparing the orders of the matrices implies that $n+k-1 = m+k'-1$, so we have

$$n + k = m + k', \qquad r = k' - \lambda_k - 1, \qquad r' = k - \lambda'_{k'} - 1. \tag{17}$$

Furthermore, from [12, Lemma 3.2] we know that $\lambda'_{\lambda_k+l} = k-1$ for all $1 \le l \le r$. So the binomials on the last r rows of the matrix in (16) take the values

$$\binom{\lambda'_i + n - m - i}{j-1} = \binom{r-l}{j-1} \qquad \text{for } 1 \le l \le r, \text{ and } i = \lambda_k + l.$$

By the triangularity of the matrix with such binomial coefficients as entries, the determinant in (16) can thus be reduced according to the last r rows.

Completely analogous, the remaining determinant can be reduced according to the last r' columns. What remains is the determinant of a matrix of order $n+k-1-r-r'$, and we have

Corollary 3.1. *The t-dimension of V_λ is given by* $\dim_t(V_\lambda) = \pm(1+t)^{mn} R'(\lambda)$ *with*

$$R'(\lambda) = \det \begin{pmatrix} \left(\frac{(-1)^{i+j+r+r'}}{(1+t)^{i+j+r+r'-1}} \binom{i+j+r+r'-2}{j-1} \right)_{\substack{1\le i\le m-r' \\ 1 \le j \le n-r}} & \left(\binom{\lambda_j+m-n-j}{i+r'-1} \right)_{\substack{1\le i\le m-r' \\ 1\le j\le \lambda'_{k'}}} \\ \left(t^{\lambda'_i+n-m-i-j-r+1} \binom{\lambda'_i+n-m-i}{j+r-1} \right)_{\substack{1\le i\le \lambda_k \\ 1\le j\le n-r}} & 0 \end{pmatrix} \tag{18}$$

An interesting application follows from this formula for the special case of $\lambda = \left((n-a)^{(m-a)}\right)$, where $a = 0, 1, \ldots, \min(m,n)$. For such a rectangular λ, we have

$$k = m - a + 1, \ k' = n - a + 1, \ r = n - a, \ r' = m - a, \ \lambda_k = 0, \ \lambda'_{k'} = 0,$$

and so the determinant in (18) reduces:

$$\begin{aligned} &\dim_t(V_\lambda) \\ &= \pm(1+t)^{mn} \det_{1\le i,j\le a} \left(\frac{(-1)^{i+j+r+r'}}{(1+t)^{i+j+r+r'-1}} \binom{i+j+r+r'-2}{j-1} \right) \\ &= \pm(1+t)^{mn-a(r+r'-1)} \det_{1\le i,j\le a} \left(\frac{(-1)^{i+j}}{(1+t)^{i+j}} \binom{i+j+m+n-2a-2}{j-1} \right), \end{aligned}$$

The resulting determinant can be further simplified: in the corresponding matrix, multiply row i by $(-1)^{i+1}(1+t)^{i+1}$ for all $1 \le i \le a$, and then multiply column j by $(-1)^{1+j}(1+t)^{j-1}$ for all $1 \le j \le a$. This yields:

$$\dim_t(V_\lambda) = (1+t)^{(m-a)(n-a)} \det_{1\le i,j\le a} \left(\binom{i+j+m+n-2a-2}{j-1} \right).$$

Now the matrix elements have no longer a power of $(1+t)$, but only a binomial coefficient. The remaining determinant can easily be computed. Taking out common factors in rows and columns, it becomes

$$\prod_{i=1}^{a} \frac{(i+m+n-2a-1)!}{(i+n-a-1)!(i+m-a-1)!} \det_{1\le i,j\le a}\Big((m+n-2a+i)_{j-1}\Big).$$

The last determinant is of the form

$$\det_{1\le i,j\le a}\Big((x_i)_{j-1}\Big) = \det_{1\le i,j\le a}\left(x_i^{j-1}\right) = \prod_{1\le i<j\le a}(x_j-x_i),$$

see [10, (2.2)].

So we finally obtain, for $\lambda = \Big((n-a)^{(m-a)}\Big)$, that

$$\begin{aligned}\dim_t(V_\lambda) &= (1+t)^{(m-a)(n-a)} \prod_{i=0}^{a-1} \frac{(m+n-2a+i)!\, i!}{(n-a+i)!(m-a+i)!} \\ &= (1+t)^{(m-a)(n-a)} \prod_{i=0}^{a-1} \frac{\binom{m+n-2a+i}{n-a+i}}{\binom{m-a+i}{i}}\end{aligned}$$

Comparing this with (9), we obtain a closed form expression for determinants of the type (9) where $\lambda = \Big((n-a)^{(m-a)}\Big)$. Replacing m by $m+1$, $m-a$ by s, and reversing the order of the rows of the corresponding matrix, this yields, using the ${}_2F_1$ notation:

$$\begin{aligned}&\det_{0\le i,j\le s}\left(\binom{n+i+j}{m}\, {}_2F_1\left(\begin{matrix} m-n-i-j, -n \\ -n-i-j \end{matrix}; -t\right)\right) \\ &\quad = (-1)^{s(s+1)/2}(1+t)^{(s+1)(s+n-m)} \prod_{i=1}^{m-s} \frac{\binom{2s+n-m+i}{s+1}}{\binom{s+i}{s+1}} \qquad (s\le m).\end{aligned}$$

The change of order of the rows implies we are dealing with a Hankel determinant, and for such determinants the row and column indices are usually starting from 0. This determinant identity can be written in a number of alternative ways. E.g. applying a transformation on the ${}_2F_1$, and denoting $t/(t+1)$ by z, one can write

$$\det_{0\le i,j\le s}\left(A_{i+j}\right) = (-1)^{s(s+1)/2}(1-z)^{(s+1)(m-s)} \prod_{i=1}^{m-s} \frac{\binom{2s+n-m+i}{s+1}}{\binom{s+i}{s+1}}, \tag{19}$$

where

$$A_k = \binom{n+k}{m}\, {}_2F_1\left(\begin{matrix} -m, -n \\ -n-k \end{matrix}; z\right). \tag{20}$$

Since this is a polynomial identity in n, the condition that n must be an integer can be dropped. Replacing n by u and z by $-v$, one can write this in the following form:

Corollary 3.2. *Let m and s be positive integers with $s \le m$, u and v arbitrary variables, and*

$$A_k = \sum_{l=0}^{m} \binom{u+k-l}{m-l}\binom{u}{l} v^l.$$

Then the Hankel determinant is given by

$$\det_{0\le i,j\le s}(A_{i+j}) = (-1)^{s(s+1)/2}(1+v)^{(s+1)(m-s)} \prod_{i=1}^{m-s} \frac{\binom{2s+u-m+i}{s+1}}{\binom{s+i}{s+1}}.$$

It seems to be difficult to find an independent proof of this corollary, even with the methods of [10, §2.6]. Here, it is a simple consequence of the two different t-dimension formulas for a particular V_λ.

References

[1] Balantekin, A. B., and I. Bars, *Dimension and character formulas for Lie supergroups*, Journal of Mathematical Physics **22** (1981), 1149–1162.

[2] Berele, A., and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Advances in Mathematics **64** (1987), 118–175.

[3] Brundan, J., *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra* $\mathfrak{gl}(m|n)$, Journal of the American Mathematical Society **16** (2003), 185–231.

[4] Dondi, P. H., and P. D. Jarvis, *Diagram and superfield techniques in the classical superalgebras*, Journal of Physics A **14** (1981), 547–563.

[5] Kac, V. G., *Lie superalgebras*, Advances in Mathematics **26** (1977), 8–96.

[6] —, *Representations of classical Lie superalgebras*, Lecture Notes in Mathematics **676**, 597–626, Springer, Berlin, 1978.

[7] —, "Infinite dimensional Lie algebras," Cambridge University Press, Cambridge, 1990.

[8] Kac, V. G., and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*, Progress in Mathematics **123** (1994), 415–456.

[9] King, R. C., *Supersymmetric functions and the Lie supergroup* $U(m/n)$, Ars Combinatoria **16 A** (1983), 269–287.

[10] Krattenthaler, C., *Advanced determinant calculus*, Séminaire Lotharingien Combinatoire **42** (1999), Article B42q, 67 pp.

[11] Macdonald, I.G., "Symmetric functions and Hall polynomials," Oxford University Press, Oxford, 2nd edition, 1995.

[12] Moens, E. M., and J. Van der Jeugt, *A determinantal formula for supersymmetric Schur polynomials*, Journal of Algebraic Combinatorics **17** (2003), 283–307.

[13] Petkovšek, M., H. S. Wilf and D. Zeilberger, "A=B", A. K. Peters, Wellesley, Massachusetts, 1996.

[14] Pragacz, P., *Algebro–geometric applications of Schur S- and Q-polynomials*, Lecture Notes in Mathematics **1478**, pp. 130–191, Springer, Berlin, 1991.

[15] Pragacz, P., and A. Thorup, *On a Jacobi-Trudi identity for supersymmetric polynomials*, Advances in Mathematics **95** (1992), 8–17.

[16] Scheunert, M., "The theory of Lie superalgebras", Springer, Berlin, 1979.

[17] Serganova, V., *Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra* $\mathfrak{gl}(m|n)$, Selecta Mathematica **2** (1996), 607–651.

[18] Sergeev, A. N., *Tensor algebra of the identity representation as a module over the Lie superalgebras* $Gl(n,m)$ *and* $Q(n)$ *(Russian)*, Matematicheskiĭ Sbornik **123** (1984), 422–430.

[19] Slater, L. J., "Generalized Hypergeometric Functions," Cambridge University Press, Cambridge, 1966.

[20] Stembridge, J., *A characterization of supersymmetric polynomials*, Journal of Algebra **95** (1985), 439–444.

[21] Van der Jeugt, J., and V. Fack, *The Pragacz identity and a new algorithm for Littlewood-Richardson coefficients*, Computers and Mathematics with Applications **21** (1991), 39–47.

[22] Van der Jeugt, J., J. W. B. Hughes, R. C. King and J. Thierry-Mieg, *Character formulas for irreducible modules of the Lie superalgebra* $sl(m/n)$, Journal of Mathematical Physics **31** (1990), 2278–2304.

[23] —, *A character formula for singly atypical modules of the Lie superalgebra* $sl(m/n)$, Communications in Algebra **18** (1990), 3453–3480.

[24] Van der Jeugt, J., and R. B. Zhang, *Characters and composition factor multiplicities for the Lie superalgebra* $gl(m/n)$, Letters in Mathematical Physics **47** (1999), 49–61.

E. M. Moens and J. Van der Jeugt
Department of Applied Mathematics
and Computer Science
Ghent University
Krijgslaan 281-S9
B-9000 Gent
Belgium
ElsM.Moens@UGent.be
Joris.VanderJeugt@UGent.be

Received June 10, 2003
and in final form June 10, 2003