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Extensions of Super Lie Algebras

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Abstract. We study (non-abelian) extensions of a super Lie algebra and identify a cohomological obstruction to the existence, parallel to the known one for Lie algebras. An analogy to the setting of covariant exterior derivatives, curvature, and the Bianchi identity in differential geometry is shown.

Keywords: Super Lie algebras, extensions of super Lie algebras, cohomology of super Lie algebras

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1. Introduction. The theory of group extensions and their interpretation in terms of cohomology is well known, see, e.g., [3], [6], [4], [2]. Analogous results for Lie algebras are dispersed in the literature, see [5], [15], [19]. The case of Lie algebroids is treated in [14], we owe this information to Kirill Mackenzie.

The present paper gives a unified and coherent account of this subject for super Lie algebras stressing a certain analogy with concepts from differential geometry: covariant exterior derivatives, curvature and the Bianchi identity.

In an unpublished preliminary version of this paper [1], the analogous results for Lie algebras were developed.

2. Super Lie algebras. (See [8], or [16] for an introduction) A *super Lie algebra* is a 2-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, together with a graded Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of degree 0. That is, $[\ ,\ ]$ is a bilinear map with $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j(\mathrm{mod} 2)}$, and such that for homogeneous elements $X \in \mathfrak{g}_x$, $Y \in \mathfrak{g}_y$, and $Z \in \mathfrak{g}_z$ the identities

$$[X,Y] = -(-1)^{xy}[Y,X] \qquad \text{(graded antisymmetry)}$$
$$[X,[Y,Z]] = [[X,Y],Z] + (-1)^{xy}[Y,[X,Z]] \qquad \text{(graded Jacobi identity)}$$

hold. The graded Jacobi identity, shorter $\sum_{\mathrm{cyclic}}(-1)^{xz}[X,[Y,Z]] = 0$, says that $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, $Y \mapsto [X,Y]$ is a graded derivation of degree x, so that $\mathrm{ad}_X[Y,Z] = [\mathrm{ad}_X Y, Z] + (-1)^{xy}[X, \mathrm{ad}_X Z]$. We denote by $\mathrm{der}(\mathfrak{g})$ the super Lie algebra of graded derivations of $\mathfrak{g}$. The notion of homomorphism is as usual, homomorphisms are always of degree 0.

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3. Describing extensions, first part. Consider any exact sequence of homomorphisms of super Lie algebras:

$$0 \to \mathfrak{h} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$$

Consider a graded linear mapping $s : \mathfrak{g} \to \mathfrak{e}$ of degree 0 with $p \circ s = \mathrm{Id}_{\mathfrak{g}}$. Then s induces mappings

$$\alpha : \mathfrak{g} \to \mathrm{der}(\mathfrak{h}) \quad (\textit{super connection}) \text{ by } \quad \alpha_X(H) = [s(X), H], \tag{3.1}$$

$$\rho : \bigwedge_{\text{graded}}^{2} \mathfrak{g} \to \mathfrak{h} \quad (\textit{curvature}) \text{ by } \quad \rho(X,Y) = [s(X), s(Y)] - s([X,Y]) \tag{3.2}$$

which are easily seen to be of degree 0 and to satisfy:

$$[\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \mathrm{ad}_{\rho(X,Y)} \tag{3.3}$$

$$\sum_{\text{cyclic}} (-1)^{xz} \Big(\alpha_X \rho(Y,Z) - \rho([X,Y],Z) \Big) = 0 \tag{3.4}$$

Property (3.4) is equivalent to the graded Jacobi identity in $\mathfrak{e}$.

4. Motivation: Lie algebra extensions associated with a principal bundle. In the case of Lie algebras, the extension

$$0 \to \mathfrak{h} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$$

appears in the following geometric situation. Let $\pi : P \to M = P/K$ be a principal bundle with structure group K. Then the Lie algebra of infinitesimal automorphisms $\mathfrak{e} = \mathfrak{X}(P)^K$, i.e. the Lie algebra of K-invariant vector fields on P, is an extension of the Lie algebra $\mathfrak{g} = \mathfrak{X}(M)$ of all vector fields on M by the Lie algebra $\mathfrak{h} = \mathfrak{X}_{\mathrm{vert}}(P)^K$ of all vertical K-invariant vector fields, i.e., infinitesimal gauge transformations. In this case we have simultaneously an extension of $C^\infty(M)$-modules. A section $s : \mathfrak{g} \to \mathfrak{e}$ which is simultaneously a homomorphism of $C^\infty(M)$-modules can be considered as a connection, and ρ, defined as in 3.2, is the curvature of this connection. This geometric example is a guideline for our approach. It works also for super Lie algebras. See [9], section 11 for more background information. This analogy with differential geometry has also been noticed in [10] and [11] and has been used used extensively in the theory of Lie algebroids, see [14].

5. Algebraic theory of connections, curvature, and cohomology. We want to interpret 3.4 as $\delta_\alpha \rho = 0$ where δ_α is an analogon of the graded version of the Chevalley coboundary operator, but with values in the non-representation $\mathfrak{h}$; we shall see that this is exactly the notion of a *super exterior covariant derivative*. Namely, let $L^{p,y}_{\mathrm{gskew}}(\mathfrak{g};\mathfrak{h})$ be the space of all graded antisymmetric p-linear mappings $\Phi : \mathfrak{g}^p \to \mathfrak{h}$ of degree y, i.e.

$$\Phi(X_1, \dots, X_p) \in \mathfrak{h}_{y+x_1+\dots+x_p},$$
$$\Phi(X_1, \dots, X_p) = -(-1)^{x_i x_{i+1}} \Phi(X_1, \dots, X_{i+1}, X_i, \dots, X_p).$$

In order to treat the graded Chevalley coboundary operator we need the following notation, which is similar to the one used in [12], 3.1: Let $\mathbf{x} = (x_1, \dots, x_k) \in (\mathbb{Z}_2)^k$ be a multi index of binary degrees $x_i \in \mathbb{Z}_2$ and let $\sigma \in \mathcal{S}_k$ be a permutation of k symbols. Then we define the *multigraded sign* $\operatorname{sign}(\sigma, \mathbf{x})$ as follows: For a transposition $\sigma = (i, i+1)$ we put $\operatorname{sign}(\sigma, \mathbf{x}) = -(-1)^{x_i\, x_{i+1}}$; it can be checked by combinatorics that this gives a well defined mapping $\operatorname{sign}(\quad, \mathbf{x}) : \mathcal{S}_k \to \{-1, +1\}$. In fact one may define directly $\operatorname{sign}(\sigma, \mathbf{x}) = \operatorname{sign}(\sigma) \operatorname{sign}(\sigma_{|x_1|,\dots,|x_k|})$, where $|\quad| : \mathbb{Z}_2 \to \mathbb{Z}$ is the embedding and where $\sigma_{|x_1|,\dots,|x_k|}$ is that permutation of $|x_1| + \dots + |x_k|$ symbols which moves the i-th block of length $|x_i|$ to the position σi, and where $\operatorname{sign}(\sigma)$ denotes the ordinary sign of a permutation in $\mathcal{S}_k$. Let us write $\sigma x = (x_{\sigma 1}, \dots, x_{\sigma k})$, then we have

$$\operatorname{sign}(\sigma \circ \tau, \mathbf{x}) = \operatorname{sign}(\sigma, \mathbf{x}). \operatorname{sign}(\tau, \sigma \mathbf{x}),$$

and $\Phi \in L^{p,y}_{\mathrm{gskew}}(\mathfrak{g}; \mathfrak{h})$ satisfies

$$\Phi(X_{\sigma 1}, \dots, X_{\sigma p}) = \operatorname{sign}(\sigma, \mathbf{x}) \Phi(X_1, \dots, X_p)$$

Given a super connection $\alpha : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})$ as in 3.1, we define the graded version of the covariant exterior derivative by

$$\begin{aligned}
&\delta_\alpha : L^{p,y}_{\mathrm{gskew}}(\mathfrak{g}; \mathfrak{h}) \to L^{p+1,y}_{\mathrm{gskew}}(\mathfrak{g}; \mathfrak{h}) \\
(\delta_\alpha \Phi)(X_0, \dots, X_p) &= \sum_{i=0}^{p} (-1)^{x_i y + a_i(\mathbf{x})} \alpha_{X_i}(\Phi(X_0, \dots, \widehat{X_i}, \dots, X_p)) \\
&\quad + \sum_{i<j} (-1)^{a_{ij}(\mathbf{x})} \Phi([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p) \\
a_i(\mathbf{x}) &= x_i(x_1 + \dots + x_{i-1}) + i \\
a_{ij}(\mathbf{x}) &= a_i(\mathbf{x}) + a_j(\mathbf{x}) + x_i x_j
\end{aligned}$$

for cochains Φ with coefficients in the non-representation $\mathfrak{h}$ of $\mathfrak{g}$. In fact, δ_α has the formal property of a *super covariant exterior derivative*, namely:

$$\delta_\alpha(\psi \wedge \Phi) = \delta\psi \wedge \Phi + (-1)^q \psi \wedge \delta_\alpha \Phi$$

for $\Phi \in L^{p,y}_{\mathrm{gskew}}(\mathfrak{g}; \mathfrak{h})$ and $\psi \in L^{q,z}_{\mathrm{gskew}}(\mathfrak{g}; \mathbb{R})$ a form of degree q and weight z (we put $\mathbb{R}$ of degree 0), where

$$(\delta\psi)(X_0, \dots, X_q) = \sum_{i<j} (-1)^{a_{ij}(\mathbf{x})} \Phi([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_q)$$

is the super analogon of the Chevalley coboundary operator for cochains with values in the trivial $\mathfrak{g}$-representation $\mathbb{R}$, and where the module structure is given by

$$\begin{aligned}
&(\psi \wedge \Phi)(X_1 \dots, X_{q+p}) = \\
&= \frac{1}{q!p!} \sum_{\sigma \in \mathcal{S}_{q+p}} \operatorname{sign}(\sigma, \mathbf{x}) (-1)^{y b_q(\sigma, \mathbf{x})} \psi(X_{\sigma 1}, \dots, X_{\sigma q}) \Phi(X_{\sigma(q+1)}, \dots, X_{\sigma(q+p)}),
\end{aligned}$$

where $b_i(\sigma, \mathbf{x}) = |x_{\sigma 1}| + \dots + |x_{\sigma i}|$.

Moreover for $\Phi \in L^{p,y}_{\mathrm{gskew}}(\mathfrak{g};\mathfrak{h})$ and $\Psi \in L^{q,z}_{\mathrm{gskew}}(\mathfrak{g};\mathfrak{h})$ we put

$$[\Phi,\Psi]_\wedge(X_1,\dots,X_{p+q}) =$$
$$= \frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}(\sigma,\mathbf{x})(-1)^{zb_p(\sigma,\mathbf{x})}[\Phi(X_{\sigma 1},\dots,X_{\sigma p}),\Psi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})]_{\mathfrak{h}}.$$

(5.1) The bracket $[\quad,\quad]_\wedge$ is a $\mathbb{Z}\times\mathbb{Z}_2$-graded Lie algebra structure on

$$L^*_{\mathrm{skew}}(V,\mathfrak{h}) = \bigoplus_{p\in\mathbb{Z}_{\ge 0},y\in\mathbb{Z}_2} L^{p,y}_{\mathrm{gskew}}(\mathfrak{g};\mathfrak{h})$$

which means that the analoga of the properties of section 2 hold for the signs $(-1)^{p_1p_2+y_1y_2}$. See [12] for more details.

A straightforward computation shows that for $\Phi \in L^{p,y}_{\mathrm{gskew}}(\mathfrak{g};\mathfrak{h})$ we have

$$\delta_\alpha\delta_\alpha(\Phi) = [\rho,\Phi]_\wedge. \tag{5.2}$$

Note that 5.2 justifies the use of the super analogon of the Chevalley cohomology if $\alpha : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})$ is a homomorphism of super Lie algebras or $\alpha : \mathfrak{g} \to \operatorname{End}(V)$ is a representation in a graded vector space. See [12] for more details.

6. Describing extensions, continued. Continuing the discussion of section 3, we now can describe completely the super Lie algebra structure on $\mathfrak{e} = \mathfrak{h}\oplus s(\mathfrak{g})$ in terms of α and ρ:

$$\begin{aligned}&[H_1 + s(X_1), H_2 + s(X_2)] = \\ &\qquad = ([H_1,H_2] + \alpha_{X_1}H_2 - (-1)^{h_1x_2}\alpha_{X_2}H_1 + \rho(X_1,X_2)) + s[X_1,X_2].\end{aligned} \tag{6.1}$$

If $\alpha : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})$ and $\rho : \bigwedge^2_{\mathrm{graded}}\mathfrak{g} \to \mathfrak{h}$ satisfy (3.3) and (3.4) then one checks easily that formula (6.1) gives a super Lie algebra structure on $\mathfrak{h}\oplus s(\mathfrak{g})$.

If we change the linear section s to $s' = s + b$ for linear $b : \mathfrak{g} \to \mathfrak{h}$ of degree zero, then we get

$$\alpha'_X = \alpha_X + \operatorname{ad}^{\mathfrak{h}}_{b(X)} \tag{6.2}$$

(6.3)
$$\begin{aligned}\rho'(X,Y) &= \rho(X,Y) + \alpha_X b(Y) - (-1)^{xy}\alpha_Y b(X) - b([X,Y]) + [bX,bY] \\ &= \rho(X,Y) + (\delta_\alpha b)(X,Y) + [bX,bY].\\ \rho' &= \rho + \delta_\alpha b + \tfrac12[b,b]_\wedge.\end{aligned}$$

7. Proposition. *Let $\mathfrak{h}$ and $\mathfrak{g}$ be super Lie algebras.*

Then isomorphism classes of extensions of $\mathfrak{g}$ over $\mathfrak{h}$, i.e. short exact sequences of Lie algebras $0\to\mathfrak{h}\to\mathfrak{e}\to\mathfrak{g}\to 0$ modulo the equivalence described by commutative diagrams of super Lie algebra homomorphisms *correspond bijectively to equivalence classes of data of the following form:*

(7.1) *a linear mapping* $\alpha : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})$ *of degree 0,*

(7.2) *a graded skew-symmetric bilinear mapping* $\rho : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ *of degree 0,*

such that

$$[\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \operatorname{ad}_{\rho(X,Y)} \tag{7.3}$$

$$\sum_{\text{cyclic}} (-1)^{xz} \Big(\alpha_X \rho(Y, Z) - \rho([X, Y], Z) \Big) = 0. \tag{7.4}$$

On the vector space $\mathfrak{e} := \mathfrak{h} \oplus \mathfrak{g}$ *a Lie algebra structure is given by*

$$\begin{aligned}(7.5)\quad & [H_1 + X_1, H_2 + X_2]_{\mathfrak{e}} = \\ & = ([H_1, H_2]_{\mathfrak{h}} + \alpha_{X_1} H_2 - (-1)^{x_2 h_1} \alpha_{X_2} H_1 + \rho(X_1, X_2)) + [X_1, X_2]_{\mathfrak{g}},\end{aligned}$$

and the associated exact sequence is

Two data (α, ρ) *and* (α', ρ') *are equivalent if there exists a linear mapping* $b : \mathfrak{g} \to \mathfrak{h}$ *of degree 0 such that*

(7.6)
$$\begin{gathered}\alpha'_X = \alpha_X + \operatorname{ad}^{\mathfrak{h}}_{b(X)} \quad \text{and} \\ \rho'(X, Y) = \rho(X, Y) + \alpha_X b(Y) - (-1)^{xy} \alpha_Y b(X) - b([X, Y]) + [b(X), b(Y)],\end{gathered}$$

$$\rho' = \rho + \delta_\alpha b + \tfrac{1}{2}[b, b]_\wedge, \tag{7.7}$$

the corresponding isomorphism being

$$\mathfrak{e} = \mathfrak{h} \oplus \mathfrak{g} \to \mathfrak{h} \oplus \mathfrak{g} = \mathfrak{e}', \qquad H + X \mapsto H - b(X) + X.$$

Moreover, a datum (α, ρ) *corresponds to a split extension (a semidirect product) if and only if* (α, ρ) *is equivalent to a datum of the form* $(\alpha', 0)$ *(then* α' *is a homomorphism). This is the case if and only if there exists a mapping* $b : \mathfrak{g} \to \mathfrak{h}$ *such that*

$$\rho = \delta_\alpha b - \tfrac{1}{2}[b, b]_\wedge. \tag{7.8}$$

Proof. Direct computations. ■

8. Corollary. *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be super Lie algebras such that* $\mathfrak{h}$ *has no (graded) center. Then isomorphism classes of extensions of* $\mathfrak{g}$ *over* $\mathfrak{h}$ *correspond bijectively to homomorphisms of super Lie algebras*

$$\bar{\alpha} : \mathfrak{g} \to \operatorname{out}(\mathfrak{h}) = \operatorname{der}(\mathfrak{h}) / \operatorname{ad}(\mathfrak{h}).$$

Proof. Choose a linear lift $\alpha : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})$ of $\bar\alpha$. Since $\bar\alpha : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h})$ is a homomorphism, there is a uniquely defined skew symmetric linear mapping $\rho : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ such that $[\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \operatorname{ad}_{\rho(X,Y)}$. Condition (7.4) is then automatically satified. For later use we record the simple proof:

$$\begin{aligned}
&\sum_{\text{cyclic}X,Y,Z} (-1)^{xz}\Big[\alpha_X\rho(Y,Z) - \rho([X,Y],Z), H\Big] \\
&= \sum_{\text{cyclic}X,Y,Z} (-1)^{xz}\Big(\alpha_X[\rho(Y,Z),H] - (-1)^{(x(y+z))}[\rho(Y,Z),\alpha_X H] - \\
&\qquad\qquad\qquad\qquad\qquad\qquad - [\rho([X,Y],Z),H]\Big) \\
&= \sum_{\text{cyclic}X,Y,Z} (-1)^{xz}\Big(\alpha_X[\alpha_Y,\alpha_Z] - \alpha_X\alpha_{[Y,Z]} - (-1)^{(x(y+z))}[\alpha_Y,\alpha_Z]\alpha_X + \\
&\qquad\qquad + (-1)^{(x(y+z))}\alpha_{[Y,Z]}\alpha_X - [\alpha_{[X,Y]},\alpha_Z] + \alpha_{[[X,Y]Z]}\Big)H \\
&= \sum_{\text{cyclic}X,Y,Z} (-1)^{xz}\Big([\alpha_X,[\alpha_Y,\alpha_Z]] - [\alpha_X,\alpha_{[Y,Z]}] - [\alpha_{[X,Y]},\alpha_Z] + \alpha_{[[X,Y]Z]}\Big)H \\
&= 0.
\end{aligned}$$

Thus (α,ρ) describes an extension, by Proposition 7. The rest is clear. ■

9. Remark. If the super Lie algebra $\mathfrak{h}$ has no center and a homomorphism $\bar\alpha : \mathfrak{g} \to \operatorname{out}(\mathfrak{h}) = \operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h})$ is given, the extension corresponding to $\bar\alpha$ is given by the pullback diagram

where $\operatorname{der}(\mathfrak{h}) \times_{\operatorname{out}(\mathfrak{h})} \mathfrak{g}$ is the Lie subalgebra

$$\operatorname{der}(\mathfrak{h}) \times_{\operatorname{out}(\mathfrak{h})} \mathfrak{g} := \{(D,X) \in \operatorname{der}(\mathfrak{h}) \times \mathfrak{g} : \pi(D) = \bar\alpha(X)\} \subset \operatorname{der}(\mathfrak{h}) \times \mathfrak{g}.$$

We owe this remark to E. Vinberg.

If the super Lie algebra $\mathfrak{h}$ has no center and satisfies $\operatorname{der}(\mathfrak{h}) = \mathfrak{h}$, and if $\mathfrak{h}$ is an ideal in a super Lie algebra $\mathfrak{e}$, then $\mathfrak{e} \cong \mathfrak{h} \oplus \mathfrak{e}/\mathfrak{h}$, since $\operatorname{Out}(\mathfrak{h}) = 0$.

10. Theorem. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be super Lie algebras and let*

$$\bar\alpha : \mathfrak{g} \to \operatorname{out}(\mathfrak{h}) = \operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h})$$

be a homomorphism of super Lie algebras. Then the following are equivalent:

(10.1) *For one (equivalently: any) linear lift $\alpha : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})$ of degree 0 of $\bar\alpha$ choose $\rho : \bigwedge^2_{\text{graded}} \mathfrak{g} \to \mathfrak{h}$ of degree 0 satisfying $([\alpha_X, \alpha_Y] - \alpha_{[X,Y]}) = \operatorname{ad}_{\rho(X,Y)}$. Then the $\delta_{\bar\alpha}$-cohomology class of $\lambda = \lambda(\alpha,\rho) := \delta_\alpha\rho : \bigwedge^3 \mathfrak{g} \to Z(\mathfrak{h})$ in $H^3(\mathfrak{g}; Z(\mathfrak{h}))$ vanishes.*

(10.2) *There exists an extension $0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$ inducing the homomorphism $\bar\alpha$.*

If this is the case then all extensions $0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$ *inducing the homomorphism* $\bar\alpha$ *are parameterized by* $H^2(\mathfrak{g}, (Z(\mathfrak{h}), \bar\alpha))$, *the second graded Chevalley cohomology space of the super Lie algebra* $\mathfrak{g}$ *with values in the graded* $\mathfrak{g}$*-module* $(Z(\mathfrak{h}), \bar\alpha)$.

Proof. It follows from the computation in the proof of corollary 8 that

$$\operatorname{ad}(\lambda(X,Y,Z)) = \operatorname{ad}(\delta_\alpha \rho(X,Y,Z)) = 0$$

so that $\lambda(X,Y,Z) \in Z(\mathfrak{h})$. The super Lie algebra $\operatorname{out}(\mathfrak{h}) = \operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h})$ acts on the center $Z(\mathfrak{h})$, thus $Z(\mathfrak{h})$ is a graded $\mathfrak{g}$-module via $\bar\alpha$, and $\delta_{\bar\alpha}$ is the differential of the Chevalley cohomology. Using 5.2, then 5.1 we see

$$\delta_{\bar\alpha}\lambda = \delta_\alpha\delta_\alpha\rho = [\rho,\rho]_\wedge = -(-1)^{2\cdot 2+0\cdot 0}[\rho,\rho]_\wedge = 0,$$

so that $[\lambda] \in H^3(\mathfrak{g}; Z(\mathfrak{h}))$.

Let us check next that the cohomology class $[\lambda]$ does not depend on the choices we made. If we are given a pair (α, ρ) as above and we take another linear lift $\alpha' : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})$ then $\alpha'_X = \alpha_X + \operatorname{ad}_{b(X)}$ for some linear $b : \mathfrak{g} \to \mathfrak{h}$. We consider

$$\rho' : \bigwedge^2_{\text{graded}} \mathfrak{g} \to \mathfrak{h}, \quad \rho'(X,Y) = \rho(X,Y) + (\delta_\alpha b)(X,Y) + [b(X), b(Y)].$$

Easy computations show that

$$\begin{gathered}
[\alpha'_X, \alpha'_Y] - \alpha'_{[X,Y]} = \operatorname{ad}_{\rho'(X,Y)} \\
\lambda(\alpha,\rho) = \delta_\alpha\rho = \delta_{\alpha'}\rho' = \lambda(\alpha',\rho')
\end{gathered}$$

so that even the cochain did not change. So let us consider for fixed α two linear mappings

$$\rho, \rho' : \bigwedge^2_{\text{graded}} \mathfrak{g} \to \mathfrak{h}, \quad [\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \operatorname{ad}_{\rho(X,Y)} = \operatorname{ad}_{\rho'(X,Y)}.$$

Then $\rho - \rho' =: \mu : \bigwedge^2_{\text{graded}} \mathfrak{g} \to Z(\mathfrak{h})$ and clearly $\lambda(\alpha,\rho) - \lambda(\alpha,\rho') = \delta_\alpha\rho - \delta_\alpha\rho' = \delta_{\bar\alpha}\mu$.

If there exists an extension inducing $\bar\alpha$ then for any lift α we may find ρ as in proposition 7 such that $\lambda(\alpha,\rho) = 0$. On the other hand, given a pair (α,ρ) as in (1) such that $[\lambda(\alpha,\rho)] = 0 \in H^3(\mathfrak{g}, (Z(\mathfrak{h}), \bar\alpha))$, there exists $\mu : \bigwedge^2 \mathfrak{g} \to Z(\mathfrak{h})$ such that $\delta_{\bar\alpha}\mu = \lambda$. But then

$$\operatorname{ad}_{(\rho-\mu)(X,Y)} = \operatorname{ad}_{\rho(X,Y)}, \quad \delta_\alpha(\rho - \mu) = 0,$$

so that $(\alpha, \rho - \mu)$ satisfy the conditions of 7 and thus define an extension which induces $\bar\alpha$.

Finally, suppose that (10.1) is satisfied, and let us determine how many extensions there exist which induce $\bar{\alpha}$. By proposition 7 we have to determine all equivalence classes of data (α, ρ) as described there. We may fix the linear lift α and one mapping $\rho : \bigwedge^2_{\text{graded}} \mathfrak{g} \to \mathfrak{h}$ which satisfies (7.3) and (7.4), and we have to find all ρ' with this property. But then $\rho - \rho' = \mu : \bigwedge^2_{\text{graded}} \mathfrak{g} \to Z(\mathfrak{h})$ and

$$\delta_{\bar{\alpha}}\mu = \delta_\alpha \rho - \delta_\alpha \rho' = 0 - 0 = 0$$

so that μ is a 2-cocycle. Moreover we may still pass to equivalent data in the sense of proposition 7 using some $b : \mathfrak{g} \to \mathfrak{h}$ which does not change α, i.e. $b : \mathfrak{g} \to Z(\mathfrak{h})$. The corresponding ρ' is, by (7.7), $\rho' = \rho + \delta_\alpha b + \frac{1}{2}[b, b]_\wedge = \rho + \delta_{\bar{\alpha}} b$. Thus only the cohomology class of μ matters. ∎

11. Corollary. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be super Lie algebras such that $\mathfrak{h}$ is abelian. Then isomorphism classes of extensions of $\mathfrak{g}$ over $\mathfrak{h}$ correspond bijectively to the set of all pairs $(\alpha, [\rho])$, where $\alpha : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h}) = \operatorname{der}(\mathfrak{h})$ is a homomorphism of super Lie algebras and $[\rho] \in H^2(\mathfrak{g}, \mathfrak{h})$ is a graded Chevalley cohomology class with coefficients in the $\mathfrak{g}$-module $\mathfrak{h}$.*

Proof. This is obvious from theorem 10. ∎

12. An interpretation of the class λ. Let $\mathfrak{h}$ and $\mathfrak{g}$ be super Lie algebras and let a homomorphism of super Lie algebras $\bar{\alpha} : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h})$ be given. We consider the extension

$$0 \to \operatorname{ad}(\mathfrak{h}) \to \operatorname{der}(\mathfrak{h}) \to \operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h}) \to 0$$

and the following diagram, where the bottom right hand square is a pullback (compare with remark 9):

The left hand vertical column describes $\mathfrak{h}$ as a central extension of $\operatorname{ad}(\mathfrak{h})$ with abelian kernel $Z(\mathfrak{h})$ which is moreover killed under the action of $\mathfrak{g}$ via $\bar{\alpha}$; it is given by a cohomology class $[\nu] \in H^2(\mathrm{ad}(\mathfrak{h}); Z(\mathfrak{h}))^{\mathfrak{g}}$. In order to get an extension $\mathfrak{e}$ of $\mathfrak{g}$ with kernel $\mathfrak{h}$ as in the third row we have to check that the cohomology class $[\nu]$ is in the image of $i^* : H^2(\tilde{\mathfrak{e}}; Z(\mathfrak{h})) \to H^2(\mathrm{ad}(\mathfrak{h}); Z(\mathfrak{h}))^{\mathfrak{g}}$. It would be interesting to interpret this in terms of the super analogon of the Hochschild-Serre spectral sequence from [7].

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