Journal of Lie Theory
Volume **15** (2005) 135–143

The only global contact transformations of order two or more are point transformations

Ricardo J. Alonso-Blanco* **and David Blázquez-Sanz**

Communicated by P. Olver

Abstract. Let us consider J^k_mM as the space of k-jets of m-dimensional submanifolds of a smooth manifold M. Our purpose is to show that every contact transformation of J^k_mM, $k \geq 2$, is induced by a diffeomorphism of M (point transformations). It is also derived that a first order contact transformation can not be globally prolonged to higher order jets except when it is a point transformation. This holds true as well for jets of sections of a regular projection. The Legendre transformation gives us an example of this property.

Subject Matter Classification 2000: 58A20 ; 58A3
Keywords: Jet, contact system, contact transformation

Let M be a smooth manifold. The spaces of k-jets of m-dimensional submanifolds of M, J^k_mM, are provided with a canonical structure: the *contact system*. As a consequence, the natural automorphisms of J^k_mM will be the diffeomorphisms preserving such structure. These are the so called *contact* (or *Lie*) *transformations*. It is usually common as well as useful to consider the local versions of the mentioned transformations. For instance, the transformations of the open subset $J^k\pi \subset J^k_mM$ determined by the jets of sections of a regular projection $\pi\colon M \to B$, $\dim B = m$.

Bäcklund [2] had earlier proved that every contact transformation is the prolongation of a first order transformation ($k = 1$). Furthermore, if m is different than $\dim M - 1$, then, every contact transformation is the prolongation of a diffeomorphism of M (*point transformations*). Such a result will hold also in the case of local transformations.

In this paper we are going to prove that every contact transformation of J^k_mM (resp. $J^k\pi$), $k \geq 2$, is a point transformation:

$$\mathrm{Aut}_{\mathrm{Lie}} J^k_mM = \mathrm{Aut}\, M, \quad \mathrm{Aut}_{\mathrm{Lie}} J^k\pi = \mathrm{Aut}\, \pi, \quad k \geq 2, \forall\, m.$$

This way, we will eliminate the exceptional case $m = \dim M - 1$ for global higher order transformations. In order to get our result we will prove first that each global projectable contact transformation is a point transformation. This property combined with the theorem of Bäcklund will lead us to the desired result. This will be the content of Section 3.

* The first author was partially founded by Junta de Castilla y León under contract SA077/03

ISSN 0949–5932 / \$2.50

Previously, Section 1. covers the basic properties of jet spaces which will be needed later on. In Section 2., we will recall some facts on the Legendre transformation. That particular example led us to develop this paper.

Notation and conventions. For the rest of this paper, M will be a smooth manifold of dimension n and 'submanifold' will mean 'locally closed submanifold'.

The characters α, β, ..., will be reserved to denote multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$, $\beta = (\beta_1, \dots, \beta_m)$, ..., $\in \mathbb{N}^m$. Besides, we will denote by 1_i the multi-index $(1_i)_j = \delta_{ij}$. As usual, $|\alpha| = \alpha_1 + \cdots + \alpha_m$, $\alpha! = \alpha_1! \cdots \alpha_m!$ and, for each collection of numbers (or operators, partial derivatives, etc.) $u_1, \dots, u_m$, the expression u^α will denote the product $u_1^{\alpha_1} \cdots u^{\alpha_m}$.

1. Preliminaries on Jet spaces

In this section we will recall the basic notions and properties of jets of submanifolds. For the proofs we refer to [7, 8, 4].

Let us consider an m-dimensional submanifold $X \subset M$, and a point $p \in X$. The class of the submanifolds having at p a contact of order k with X is called the k-th order jet of X at p and it will be denoted by $j_p^k X$. The set $J_m^k M = \{j_p^k X \mid X \subseteq M\}$ is called the *space of* (m,k)*-jets of* M, *space of* k*-jets of* m*-dimensional submanifolds of* M or still, (m,k)*-Grassmann manifold associated to* M.

The space $J_m^k M$ is naturally endowed with a differentiable structure. Let $\mathfrak{p}^k = j_p^k X$ and choose a local chart $\{x_1, \dots, x_m, y_1, \dots, y_{n-m}\}$ on a neighborhood of p such that X has the local equations $y_j = f_j(x_1, \dots, x_m)$, $1 \le j \le n-m$. For each multi-index $\alpha = (\alpha_1, \dots, \alpha_m)$, $|\alpha| \le k$, let us define

$$y_{j\alpha}(\mathfrak{p}^k) := \left(\frac{\partial^{|\alpha|} f_j}{\partial x^\alpha} \right)_p, \quad 1 \le j \le n-m,$$

(if $\alpha = 0$, then $y_{j0}(\mathfrak{p}^k) = y_j(p)$). The functions $\{x_i, y_{j\alpha}\}$ give local charts on $J_m^k M$.

Remark 1.1. When M is provided with a regular projection $\pi \colon M \to B$, $\dim B = m$, the sections of π are m-dimensional submanifolds of M. This way, it is possible to take the k-jets of sections of π so that we have a subset $J^k\pi \subset J_m^k M$. The space $J^k\pi$ is a dense open subset of $J_m^k M$.

For arbitrary integers $k \ge r \ge 0$, there are natural projections

$$\pi_{k,r} \colon J_m^k M \to J_m^r M, \qquad \mathfrak{p}^k = j_p^k X \mapsto \pi_{k,r}(\mathfrak{p}^k) := j_p^r X,$$

which in the above local coordinates are expressed by

$$\pi_{k,r}(x_i, y_{j\alpha})_{|\alpha| \le k} = (x_i, y_{j\alpha})_{|\alpha| \le r}.$$

For each given m-dimensional submanifold $X \subseteq M$, the submanifold

$$J_m^k X = \{j_p^k X \mid p \in X\} \subseteq J_m^k M, \tag{1}$$

is called the *k-jet prolongation of X*. Let $y_j = f_j(x_1, \ldots, x_m)$ be the local equations of X. Then, $J^k_m X \subseteq J^k_m M$ is given by

$$y_{j\alpha} = \frac{\partial^{|\alpha|} f_j(x)}{\partial x^\alpha}, \quad 1 \leq j \leq n-m, \ |\alpha| \leq k. \tag{2}$$

and the tangent space $T_{\mathfrak{p}^k} J^k_m X$ at the point $\mathfrak{p}^k = j^k_p X$ is spanned by the m vectors:

$$D_i = \left(\frac{\partial}{\partial x_i}\right)_{\mathfrak{p}^k} + \sum_{\substack{1\leq j\leq n-m \\ |\alpha|\leq k}} \left(\frac{\partial^{|\alpha+1_i|} f_j(x)}{\partial x^{\alpha+1_i}}\right)_p \left(\frac{\partial}{\partial y_{j\alpha}}\right)_{\mathfrak{p}^k}, \tag{3}$$

(observe that if $|\alpha| \leq k-1$, the coefficient of $\frac{\partial}{\partial y_{j\alpha}}$ equals $y_{j\alpha+1_i}(\mathfrak{p}^k)$).

Definition 1.2. For a given jet $\mathfrak{p}^k \in J^k_m M$, let us consider all the submanifolds X such that $\mathfrak{p}^k = j^k_p X$. Let also $\mathcal{C}_{\mathfrak{p}^k}$ be the subspace of $T_{\mathfrak{p}^k} J^k_m M$ spanned by the tangent spaces $T_{\mathfrak{p}^k} J^k_m X$. The distribution $\mathcal{C} = \mathcal{C}(J^k_m M)$ defined by $\mathfrak{p}^k \mapsto \mathcal{C}_{\mathfrak{p}^k}$ is the so-called *contact* or *Cartan distribution* on $J^k_m M$. The Pfaffian system associated with $\mathcal{C}$ will be called the *contact system* on $J^k_m M$ and denoted by $\Omega = \Omega(J^k_m M)$. Alternatively, Ω can be defined as the set of 1-forms which vanish on every prolonged submanifold $J^k_m X$.

It can be derived from (3) that $\mathcal{C}$ is generated by the tangent fields

$$\frac{d}{dx_i} := \frac{\partial}{\partial x_i} + \sum_{\substack{1\leq j\leq n-m \\ |\alpha|\leq k-1}} y_{j\alpha+1_i} \frac{\partial}{\partial y_{j\alpha}} \quad \text{and} \quad \frac{\partial}{\partial y_{j\sigma}}, \tag{4}$$

where $1 \leq i \leq n, 1 \leq j \leq n-m$, and $|\sigma| = k$.

In the same way, the contact system Ω is generated by the 1-forms

$$\omega_{j\alpha} := dy_{j\alpha} - \sum_i y_{j\alpha+1_i} dx_i. \tag{5}$$

The local expressions (3) and (4) show the following lemma.

Lemma 1.3. *If $\mathfrak{p}^k = j^k_p X$ then $(\pi_{k,k-1})_* \mathcal{C}_{\mathfrak{p}^k} = T_{\mathfrak{p}^{k-1}} J^{k-1}_m X$.*

Definition 1.4. A tangent vector $D_{\mathfrak{p}^k} \in T_{\mathfrak{p}^k} J^k_m M$ is *$\pi_{k,k-1}$-vertical* if we have $(\pi_{k,k-1})_* D_{\mathfrak{p}^k} = 0$. The $\pi_{k,k-1}$-vertical tangent vectors define a distribution $Q = Q(J^k_m M)$.

Q is a sub-distribution of $\mathcal{C}$ whose maximal solutions are the fibers of $\pi_{k,k-1}$. In local coordinates,

$$Q = \left\langle \frac{\partial}{\partial y_{j\sigma}} \right\rangle_{\substack{1\leq j\leq n-m \\ |\sigma|=k}} \subset \ \mathcal{C} = \left\langle \frac{d}{dx_i}, \frac{\partial}{\partial y_{j\sigma}} \right\rangle_{\substack{1\leq i\leq m \\ 1\leq j\leq n-m \\ |\sigma|=k}}$$

Definition 1.5. A diffeomorphism $\phi^k \colon J^k_m M \to J^k_m M$ such that $(\phi^k)^*\Omega = \Omega$ is said to be a *contact transformation*.

For integers $h > k \geq 0$, let us consider the diagram

$$\begin{tikzcd} J^h_m M \arrow[r, "\phi^h"] \arrow[d, "\pi_{h,k}"'] & J^h_m M \arrow[d, "\pi_{h,k}"] \\ J^k_m M \arrow[r, "\phi^k"'] & J^k_m M. \end{tikzcd} \tag{6}$$

Definition 1.6. (1) A contact transformation ϕ^k on $J^k_m M$ is *prolongable* to $J^h_m M$ if there is a contact transformation ϕ^h such that the diagram (6) is commutative.
(2) A diffeomorphism ϕ^h of $J^h_m M$ is *projectable* to $J^k_m M$ if there is a diffeomorphism ϕ^k such that the diagram (6) is commutative.

Lemma 1.7. *The following properties hold*
(1) *When it exists, the prolongation of a contact transformation is unique.*
(2) *If a contact transformation ϕ^h is projectable to ϕ^k then ϕ^k is also a contact transformation. Moreover, ϕ^h is the prolongation of ϕ^k.*
(3) *The prolongation ϕ^h of a contact transformation ϕ^k is projectable to ϕ^k.*

Lemma 1.8. *Each diffeomorphism $\phi\colon M \to M$ can be prolonged to a contact transformation $\phi^{(k)}\colon J^k_m M \to J^k_m M$ in the following way:*

$$\phi^{(k)}(j^k_p X) := j^k_{\phi(p)}(\phi(X)), \quad \forall\, j^k_p X \in J^k_m M.$$

The map $\phi^{(k)}$ is called the k-jet prolongation *of ϕ and, both of them, are said to be* point transformations*.*

The following theorem is due to Bäcklund [2] (see also [1, Ch. I]). A modern proof is given in [8] (or [4, Ch. 3]) by using geometrical arguments combined with the computation of the solutions of the contact system having maximal dimension. Another proof can be seen in [6, Ch. 11], which is based on the Cartan equivalence method.

Theorem 1.9. *Every contact transformation $\phi^k\colon J^k_m M \to J^k_m M$ is the prolongation of a first order contact transformation $\phi^1\colon J^1_m M \to J^1_m M$. Moreover, when $m \neq n-1$, $n = \dim M$, ϕ^k is the k-jet prolongation of a point transformation $\phi\colon M \to M$.*

Remark 1.10. The above theorem is local. That is, under the appropriate substitutions, it holds when ϕ^k is only defined for an open set of $J^k_m M$. For instance, if ϕ^k is just defined on jets of sections of a regular projection $M \to B$.

2. A comment on the prolongation of the Legendre transformation

In this section we will consider the prolongation of the Legendre transformation to the space of second order jets J^2. We will see that this prolongation is not possible on the whole of J^2 but just on a dense open subset of it.

Let $\mathbb{R}^{m+1}$ be coordinated by $\{x_1, \dots, x_m, u\}$ and consider the projection on the first m coordinates $\pi \colon \mathbb{R}^{m+1} \to \mathbb{R}^m$, $\pi(x_1, \dots, x_m, u) = (x_1, \dots, x_m)$. The induced system of global coordinates for $J^1\pi$ is given by $\{x_i, u, p_i\}$, $1 \le i \le m$, where, if $y_1 := u$ then we put $p_i := y_{11_i}$ (see the beginning of Section 1.). This way, the contact system on $J^1\pi$ is generated by the 1-form $\omega = dz - \sum_i p_i dx_i$. Similarly, on $J^2\pi$ we have the coordinates $\{x_i, u, p_i, p_{ij}\}$, $1 \le i, j \le m$, where $p_{ij} := y_{11_i+1_j}$ and the contact system is spanned by ω and $\omega_j = dp_j - \sum_i p_{ij} dx_i$.

The *Legendre transformation* is the diffeomorphism

$$J^1\pi \xrightarrow{L} J^1\pi$$

given in coordinates by

$$\begin{cases} L^* x_i = p_i \\ L^* u = u - \sum_{i=1}^m p_i x_i \\ L^* p_i = -x_i \end{cases} \tag{7}$$

By direct computation it is not difficult to prove that L is a (first order) contact transformation. That is to say, $L^*\omega = \lambda\omega$ for a suitable function λ. In addition, L is not a point transformation because both L^*x_i and L^*u involve the "first order" coordinates p_i.

Now we will try to prolong L to second order jets by obtaining a contact transformation

$$J^2\pi \xrightarrow{L^1} J^2\pi$$

The conditions that L^1 should satisfy are the following two:

1. L^1 projects to L (in particular, $(L^1)^*\omega = \lambda\omega$)
2. $(L^1)^*\omega_i = \mu_i\omega + \sum_j \mu_{ij}\,\omega_j$ for suitable functions μ_i, μ_{ij}.

Hence, $\overline{x}_i := (L^1)^*x_i$, $\overline{u} := (L^1)^*u$ and $\overline{p}_i := (L^1)^*p_i$ are completely determined by the first condition above. The remainder coordinates, $\overline{p}_{ij} := (L^1)^*p_{ij}$, must be obtained from the second condition. By introducing the local expressions for ω_i we get

$$(L^1)^*\omega_i = -dx_i - \sum_j \overline{p}_{ij} dp_j$$

from which it is easy to see that $\mu_i = 0$, $\mu_{ij} = -\overline{p}_{ij}$ and

$$-dx_i - \sum_j \overline{p}_{ij} dp_j = -\sum_j \overline{p}_{ij} dp_j + \sum_{j,k} \overline{p}_{ij} p_{jk} dx_k.$$

By equating coefficients we finally arrives to $-\sum_j \overline{p}_{ij} p_{jk} = -\delta_{ik}$. If we define the matrices $P := \begin{pmatrix} p_{ij} \end{pmatrix}$ and $\overline{P} := \begin{pmatrix} \overline{p}_{ij} \end{pmatrix}$, then the above equations can be shortened to

$$\overline{P}P = -I,$$

where I denotes the identity matrix. As a consequence, the $\overline{p}_{ij}$'s are obtained from

$$\overline{P} = -P^{-1}$$

which is possible if and only if $\det P \neq 0$.

By summarizing,

The prolongation of the Legendre transformation L to $J^2\pi$ is only possible for the dense open subset $\{\det P \neq 0\} \subset J^2\pi$.

Remark 2.1. As it is well known, the Legendre transformation is a very useful tool to study partial differential equations (PDE). When we consider a second order PDE and try to use the Legendre transformation we automatically remove the possible solutions on which $\det P = 0$. In the case of $m = 2$, that is the equation of the developable surfaces. See [3, Vol II, pp. 32] for details.

So, it is natural to ask if this fact is due to the particular structure of the Legendre transformation or a more general property. The answer confirms the second possibility. Indeed, we will prove even more: there are no global higher order contact transformations except the point transformations.

3. The main result

In this section we will prove the statement enunciated by the title of this paper. The sequence of arguments is the following. As we will see below, from Lemmas 3.1, 3.2, it is derived that the projection ϕ^{k-1} of a contact transformation ϕ^k, if it exists, preserves the distribution Q on $J^{k-1}_m M$. This way, ϕ^{k-1} is also projectable and the induction proves that ϕ^k is a point transformation (Proposition 3.3). Taking into account Theorem 1.9 we will get the aforementioned result.

Lemma 3.1. *Let $\phi^k\colon J^k_m M \to J^k_m M$ be a contact transformation which is projectable to $\phi^{k-1}\colon J^{k-1}_m M \to J^{k-1}_m M$ and let X be a submanifold of M such that $\mathfrak{p}^{k-1} = j^{k-1}_p X$, $p \in X$, and $\bar{\mathfrak{p}}^{k-1} := \phi^{k-1}(\mathfrak{p}^{k-1})$. Then there exists a submanifold Y, $\bar{\mathfrak{p}}^{k-1} = j^{k-1}_{\bar{p}} Y$, such that*

$$(\phi^{k-1})_*(T_{\mathfrak{p}^{k-1}} J^{k-1}_m X) = T_{\bar{\mathfrak{p}}^{k-1}}(J^{k-1}_m Y).$$

Proof. Let $\mathfrak{p}^k$ be the k-jet of X at the point p and let us consider the following commutative diagram of tangent spaces and maps

$$\begin{array}{ccc}
T_{\mathfrak{p}^k} J^k_m X & \xrightarrow{(\phi^k)_*} & (\phi^k)_*(T_{\mathfrak{p}^k} J^k_m X) \\
{\scriptstyle (\pi_{k,k-1})_*}\downarrow & & \downarrow{\scriptstyle (\pi_{k,k-1})_*} \\
T_{\mathfrak{p}^{k-1}} J^{k-1}_m X & \xrightarrow[(\phi^{k-1})_*]{} & (\phi^{k-1})_*(T_{\mathfrak{p}^{k-1}} J^{k-1}_m X)
\end{array}$$

Then, $(\phi^{k-1})_*(T_{\mathfrak{p}^{k-1}} J^{k-1}_m X)$ has dimension m because $(\phi^{k-1})_*$ is an isomorphism. On the other hand, if $\phi^k(\mathfrak{p}^k)$ equals $j^k_{\bar{p}} Y$ then $(\pi_{k,k-1})_* \mathcal{C}_{\bar{\mathfrak{p}}^k} = T_{\bar{\mathfrak{p}}^{k-1}} J^{k-1}_m Y$ (Lemma 1.3). In addition, $(\phi^k)_*(T_{\mathfrak{p}^k} J^k_m X) \subset \mathcal{C}_{\bar{\mathfrak{p}}^k}$. It follows that

$$(\phi^{k-1})_*(T_{\mathfrak{p}^{k-1}} J^{k-1}_m X) \subseteq T_{\bar{\mathfrak{p}}^{k-1}} J^{k-1}_m Y.$$

The spaces involved in the above inclusion have the same dimension and so the inclusion is an equality. ∎

The following lemma characterizes the vectors in Q amongst those of the contact distribution $\mathcal{C}$.

Lemma 3.2. *Let $\mathfrak{p}^k \in J^k_m M$ and denote by $S_{\mathfrak{p}^k}$ the set $\bigcup_{j^k_p X=\mathfrak{p}^k} T_{\mathfrak{p}^k} J^k_m X$. Then,*

$$(1)\quad \{0\} = Q_{\mathfrak{p}^k} \cap S_{\mathfrak{p}^k} \qquad \text{and} \qquad (2)\quad \mathcal{C}_{\mathfrak{p}^k} = Q_{\mathfrak{p}^k} \cup S_{\mathfrak{p}^k}.$$

Proof. (1) If $D_{\mathfrak{p}^k} \in Q_{\mathfrak{p}^k}$ then $(\pi_{k,k-1})_* D_{\mathfrak{p}^k} = 0$. On the other hand, given a submanifold X with $\mathfrak{p}^k = j^k_p X$, the projection $(\pi_{k,k-1})_*$ defines an isomorfism when restricted to $T_{\mathfrak{p}^k} J^k_m X$. As a consequence, $D_{\mathfrak{p}^k} \neq 0$ cannot be in $T_{\mathfrak{p}^k} J^k_m X$.

(2) Let $D_{\mathfrak{p}^k} \in \mathcal{C}_{\mathfrak{p}^k}$, $D_{\mathfrak{p}^k} \neq 0$, $D_{\mathfrak{p}^k} \notin Q_{\mathfrak{p}^k}$ (so that $(\pi_{k,k-1})_* D_{\mathfrak{p}^k} \neq 0$).

We choose coordinates around p such that $x_i(\mathfrak{p}^k) = y_{j\alpha}(\mathfrak{p}^k) = 0$. In such a local chart we have

$$D_{\mathfrak{p}^k} = \sum_{i=1}^{m} a_i \left(\frac{\partial}{\partial x_i}\right)_{\mathfrak{p}^k} + \sum_{j=1}^{n-m} \sum_{|\sigma|=k} b_{j\sigma} \left(\frac{\partial}{\partial y_{j\sigma}}\right)_{\mathfrak{p}^k},$$

for appropriate $a_i, b^i_{j\sigma} \in \mathbb{R}$ (see (4)). In addition, at least one a_i is non zero because $(\pi_{k,k-1})_* D_{\mathfrak{p}^k} = \sum_i a_i \left(\frac{\partial}{\partial x_i}\right)_{\mathfrak{p}^{k-1}} \neq 0$. A linear change (of the type $x'_i = \sum_r \lambda_{ir} x_r$) allows us to assume

$$D_{\mathfrak{p}^k} = \left(\frac{\partial}{\partial x_1}\right)_{\mathfrak{p}^k} + \sum_{j=1}^{n-m} \sum_{|\sigma|=k} b_{j\sigma} \left(\frac{\partial}{\partial y_{j\sigma}}\right)_{\mathfrak{p}^k}.$$

The submanifold X defined by

$$y_j = \sum_{|\sigma|=k} b_{j\sigma} \frac{x^{\sigma+1_1}}{(\sigma+1_1)!}, \qquad 1 \leq j \leq n-m,$$

satisfies $\mathfrak{p}^k = j^k_p X$ and $D_{\mathfrak{p}^k} \in T_{\mathfrak{p}^k} J^k_m X$ (see the expressions (3)). ■

Proposition 3.3. *Let $\phi^k\colon J^k_m M \to J^k_m M$ be a contact transformation which is projectable to $\phi^{k-1}\colon J^{k-1}_m M \to J^{k-1}_m M$. Then, ϕ^k is the k-jet prolongation of a diffeomorphism $\phi\colon M \to M$.*

Proof. Let us assume $k \geq 2$ (if $k = 1$ there is nothing to say). According to Lemma 3.1 we have $(\phi^{k-1})_* S_{\mathfrak{p}^{k-1}} = S_{\phi^{k-1}(\mathfrak{p}^{k-1})}$, for all $\mathfrak{p}^{k-1} \in J^{k-1}_m M$. As a consequence of Lemma 3.2 we derive

$$(\phi^{k-1})_* Q = Q.$$

Thus, ϕ^{k-1} sends each fiber of $J^{k-1}_m M \to J^{k-2}_m M$ to another one, because they are the maximal solutions of Q. This shows that ϕ^{k-1} is projectable to ϕ^{k-2} on $J^{k-2}_m M$. The induction proves that ϕ^k projects to a point transformation $\phi\colon M \to M$. By Lemma 1.7-(1), ϕ^k has to be $\phi^{(k)}$. ■

Theorem 3.4. *If $k \geq 2$, every contact transformation $\phi^k\colon J^k_m M \to J^k_m M$ is the k-jet prolongation of a point transformation $\phi\colon M \to M$.*

Proof. According to Theorem 1.9, ϕ^k is the prolongation of a first order contact transformation. In particular, ϕ^k is projectable to $J_m^{k-1}M$. Proposition 3.3 applied to ϕ^k finishes the proof. ∎

The following statement is a direct consequence.

Corollary 3.5. *If a first order contact transformation ϕ^1 is not a point transformation, then ϕ^1 can not be prolonged to second or higher order jets.*

Taking into account their proofs, Theorem 3.4 and Corollary 3.5 can be generalized to the case of contact transformations

$$\phi^k \colon \pi_{k,k-1}^{-1}(U) \to \pi_{k,k-1}^{-1}(U),$$

where U is an arbitrary open set $U \subset J_m^{k-1}M$. As a particular instance, we can consider $U = J^{k-1}\pi \subset J_m^{k-1}M$ so that $\pi_{k,k-1}^{-1}(U) = J^k\pi$ and, this way,

Theorem 3.6. *Let $\pi \colon M \to B$ be a regular projection. The only contact transformations*

$$\phi^k \colon J^k\pi \to J^k\pi, \quad k \geq 2$$

are the point transformations.

Moreover, a first order contact transformation $\phi^1 \colon J^1\pi \to J^1\pi$ can not be prolonged to second or higher order jets except when ϕ^1 is a point transformation.

Acknowledgements.

We would like to express our gratitude to Prof. J. Muñoz for his interest, advice and help in developing this paper. We also thank Profs. S. Jiménez and J. Rodríguez for many interesting discussions and suggestions.

References

[1] Anderson, R. L., N. H. Ibragimov, "Lie-Bäcklund transformations in applications," SIAM Studies in Applied Mathematics **1**, Philadelphia, 1979.

[2] Bäcklund, A. V., *Über Flächentransformationen*, Math. Ann. **9** (1876), 297–320.

[3] Courant, R., and D. Hilbert, "Methods of Mathematical Physics. Vol. II: Partial differential equations," Interscience Publishers, New York-London 1962.

[4] Krasil'shchik, J., and A. M. Vinogradov (Eds.), "Symmetries and Conservation Laws for Differential Equations of Mathematical Physics," Translations of Mathematical Monographs **182**, Amer. Math. Soc.

[5] Olver, P. J., "Applications of Lie groups to differential equations," Second Edition. Graduate Texts in Mathematics **107**, Springer-Verlag, New York, 1993.

[6] —, "Equivalence, invariants, and symmetry," Cambridge University Press, Cambridge, 1995.

[7] Vinogradov, A. M., *An informal introduction to the geometry of jet spaces*, Rend. Sem. Fac. Sci. Univ. Cagliari **58** (1988), 301–333.

[8] —, *Local symmetries and conservation laws*, Acta Appl. Math. **2** (1984), 21–78.

Ricardo J. Alonso-Blanco
Departamento de Matemáticas
Universidad de Salamanca
Plaza de la Merced 1-4
E-37008 Salamanca, Spain
ricardo@usal.es

David Blázquez-Sanz
Departamento de Matemáticas
Universidad de Salamanca
Plaza de la Merced 1-4
E-37008 Salamanca, Spain
telebrass@hotmail.com

Received February 24, 2004
and in final form May 11, 2004