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The Height Function on the 2-Dimensional Cohomology of a Flag Manifold

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Abstract. Let G/T be the flag manifold of a compact semisimple Lie group G modulo a maximal torus $T \subset G$. We express the height function on the 2-dimensional integral cohomology $H^2(G/T)$ of G/T in terms of the geometry of the root systems of the Lie groups.
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1. Introduction

Let G be a compact connected semi-simple Lie group of rank n with a fixed maximal torus $T \subset G$. The homogeneous space G/T is a smooth manifold, known as the *complete flag manifold* of G. In general, if H is the centralizer of a one-parameter subgroup of G, the homogeneous space G/H is called a *generalized flag manifold* of G.

Let $H^*(G/T)$ be the integral cohomology of G/T . The *height function* $h_G : H^2(G/T) \to \mathbb{Z}$ on the 2-dimensional cohomology is defined by

$$h_G(x) = \max\{m \mid x^m \neq 0\}.$$

We study the following problem: Evaluate the function $h_G : H^2(G/T) \to \mathbb{Z}$ for a given G.

For the case $G = SU(n)$, *the special unitary group* of order n, the problem has been studied by several authors. Monk [13], Ewing and Liulevicius [9] independently described the set $h^{-1}_{SU(n)}(n-1)$. This partial result was used by Hoffman [10] in classifying endomorphisms of the ring $H^*(SU(n)/T)$. Broughton, Hoffman and Homer computed the function $h_{SU(n)}$ in [1], and the result was applied in [11] to characterize the action of cohomology automorphisms of $SU(n)/H$ on $H^2(SU(n)/H)$.

A thorough understanding of the function h_G may lead to a general way to study the cohomology endomorphisms of flag manifolds G/H. In this paper, we

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solve the problem for all compact semi-simple G in terms of the geometry of root system of G.

It is worth to mention that, by describing the cohomology in terms of root system invariants, S. Papadima classified the automorphisms of the algebra $H^*(G/T)$ in [14].

Write $L(G)$ for the Lie algebra of G and let $\exp : L(G) \to G$ be the exponential map. The Cartan subalgebra of G relative to T is denoted by $L(T) \subset L(G)$. Fix a set $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset L(T)$ of simple roots of G (cf.[12], p.47).

Consider the set $\Phi_n = \{K | K \subset \{1, 2, \cdots, n\}\}$. For a $K \in \Phi_n$ let $H_K \subset G$ be the centralizer of $\exp(\bigcap_{i \notin K} L_{\alpha_i}) \subset G$, where $L_{\alpha_i} \subset L(T)$ is the hyperplane perpendicular to $\alpha_i \in \Delta$. If $K = \{1, \cdots, n\}$, we regard $H_K = T$ (the maximal torus). Define the function $f_G : \Phi_n \to \mathbb{Z}$ by setting

$$f_G(K) = \frac{1}{2}(\dim_{\mathbb{R}} G - \dim_{\mathbb{R}} H_K) \ , \ K \in \Phi_n .$$

Lemma 4 in Section 3 shows that the f_G can be effectively calculated in concrete situations.

Let $\omega_1, \omega_2, \cdots, \omega_n \in H^2(G/T)$ be the *fundamental dominant weights* associated to the set Δ of simple roots (cf. Lemma 1 in Section 2 or [5], [8]). The ω_i form an additive basis for $H^2(G/T)$ (which also generate multiplicatively the rational cohomology algebra $H^*(G/T; \mathbb{Q})$).

Let W_G be the Weyl group of G. The canonical action of W_G on G/T induces an W_G-action on $H^2(G/T)$ ([5],[4]). Let $\mathbb{Z}^+$ be the set of non-negative integers. Our main result is

Theorem . *For each $x \in H^2(G/T)$, there is $w \in W_G$ such that*

$$w(x) = \sum_{1 \le i \le n} \lambda_i \omega_i , \quad \lambda_i \in \mathbb{Z}^+ .$$

Further if we set $K = \{i | \lambda_i \neq 0, 1 \le i \le n\} (\in \Phi_n)$, then $h_G(x) = f_G(K)$.

2. Proof of the Theorem

Equip $L(G)$ with an inner product $(,)$ so that the adjoint representation of G acts as isometries of $L(G)$.

Let $\pi : \widetilde{G} \to G$ be the universal cover of G and $\widetilde{T} \subset \widetilde{G}$, the maximal torus of $\widetilde{G}$ corresponding to T (i.e. $\pi(\widetilde{T}) = T$). The tangent map of π at the unit $e \in \widetilde{G}$ yields isomorphisms of algebras

$$L(T) \cong L(\widetilde{T}), \ L(G) \cong L(\widetilde{G}).$$

Equip $L(\widetilde{G})$ with the metric $(,)$ so that the identifications are also isometries of Euclidean spaces.

The *fundamental dominant weights* $\Omega_i \in L(T) = L(\widetilde{T})$, $1 \le i \le n$, of G relative to Δ (cf. [12], p.67) generate (over the integers) the *weight lattice* $\Gamma \subset L(T) = L(\widetilde{T})$ of G and $\widetilde{G}$. Since $\widetilde{G}$ is simply connected, any $z \in \Gamma$ gives rise to a homomorphism $\widetilde{z} : \widetilde{T} \to S^1$ onto the cycle group, characterized uniquely by the property that its derivative $d\widetilde{z} : L(\widetilde{T}) = L(T) \to \mathbb{R}$ at the group unit $e \in \widetilde{T}$ satisfies

$d\widetilde{z}(h) = \frac{2(z,h)}{(h,h)}$, $h \in L(\widetilde{T})$.

The cohomology class in $H^1(\widetilde{T})$ determined by the $\widetilde{z}$ is denoted by $[z]$ (we have made use of the standard fact from homotopy theory: for any manifold (or complex) X the set of homotopy classes of maps $X \to S^1$ are in one-to-one correspondence with $H^1(X)$, the first integral cohomology of X).

Let $\beta : H^1(\widetilde{T}) \to H^2(G/T)$ be the transgression for the fibration $\widetilde{T} \subset \widetilde{G} \to \widetilde{G}/\widetilde{T} = G/T$ (cf. [4], [5]). The geometric origin of the classes ω_i employed by the Theorem (where they were also called *fundamental dominant weights*) can be seen from the next result (cf. [5], p.489).

Lemma 1. *With respect to the standard $W_G = W_{\widetilde{G}}$ action on Γ and G/T, the correspondence $\Gamma \to H^2(G/T)$, $z \to \beta[z]$, is a W_G-isomorphism.*

In particular, if we put $\beta[\Omega_i] = \omega_i$, then the ω_i constitute an additive basis for $H^2(G/T)$.

Remark 1. The classes $\omega_i \in H^2(G/T)$, $1 \leq i \leq n$, are of particular interests in the algebraic intersection theory on G/T. With respect to the Schubert cell-decomposition of the space G/T [2], they are precisely *the special Schubert classes* on G/T [8].

For a $K \in \Phi_n$ consider the standard fibration $p_K : G/T \to G/H_K$. It was shown in [5], p.507 that

Lemma 2. *The induced map $p_K^* : H^*(G/H_K) \to H^*(G/T)$ is injective and satisfies*

$$p_K^*(H^2(G/H_K)) = span_{\mathbb{Z}}\{\omega_i \mid i \in K\}.$$

In particular, if we let $\tau_i \in H^2(G/H_K)$, $i \in K$, be the classes so that

$$p_K^*(\tau_i) = \omega_i \in H^2(G/T),\ i \in K,$$

then the τ_i, $i \in K$, form an additive basis for $H^2(G/H_K)$.

Canonically, the flag manifolds G/H_K admit complex structures. It is natural to ask which $\kappa \in H^2(G/H_K)$ can appear as Kaehler classes on G/H_K. A partial answer to this question is known. With the notation developed above, Corollary 14.7 in [5] may be rephrased as

Lemma 3. *If $\kappa = \sum_{i \in K} \lambda_i \tau_i$ with $\lambda_i > 0$ for all $i \in K$, then κ is a Kaehler class on G/H_K. In particular, $\kappa^{f_G(K)} \neq 0$.*

The last clause in Lemma 3 is verified by $f_G(K) = \dim_{\mathbb{C}} G/H_K$.

Proof of the Theorem: We start by verifying the first assertion in the Theorem. In view of the W_G-isomorphism in Lemma 1, it suffices to show that

(2.1) *for any $z \in \Gamma \subset L(T)$ there is a $w \in W_G$ such that $w(z) = \sum_{1 \leq i \leq n} \lambda_i \Omega_i$ with $\lambda_i \in \mathbb{Z}^+$.*

The closure of the fundamental Weyl chamber Λ relative to the set Δ of simple roots [12], p.49 can be described in terms of the weights Ω_i as

$$\overline{\Lambda} = \{\sum_{1 \leq i \leq n} \lambda_i \Omega_i \mid \lambda_i \in \mathbb{R}^+\}.$$

Since W_G acts transitively on Weyl chambers, for any $z \in L(T)$ there is a $w \in W_G$ such that $w(z) \in \overline{\Lambda}$. Further, if $z \in \Gamma$ we must have, in the expression $w(z) = \sum_{1\leq i\leq n} \lambda_i \Omega_i$, that $\lambda_i \in \mathbb{Z}^+$ because of the standard fact $w(\Gamma) = \Gamma$. This verifies (2.1), hence the first part of the Theorem.

For a $x \in H^2(G/T)$ we can now assume that $w(x) = \sum_{1\leq i\leq n} \lambda_i \omega_i$ with $\lambda_i \in \mathbb{Z}^+$ for some $w \in W_G$. Since the w acts as automorphism of the ring $H^*(G/T)$ we have

(2.2) $\quad h_G(x) = h_G(w(x))$.

Let $K = \{i | \lambda_i \neq 0, 1 \leq i \leq n\}$ and consider in $H^2(G/H_K)$ the class $\kappa = \sum_{i\in K} \lambda_i \tau_i$. Since the ring map p_K^* is injective and satisfies $p_K^*(\kappa) = w(x)$ by Lemma 2, we have

(2.3) $\quad h_G(w(x)) = \max\{m \mid \kappa^m \neq 0\} = f_G(K)$,

where the last equality follows from Lemma 3. Combining (2.2) with (2.3) completes the proof.

3. Computations

The Theorem reduces the evaluation of h_G to that of f_G. The latter can be effectively calculated, as the following recipe shows. Denote by D_G *the Dynkin diagram* of G (whose vertices consist of a set of simple roots $\alpha_1, \alpha_2, \cdots, \alpha_n$ of G [12], p 57). For a $K \in \Phi_n$ we have (cf. [5], 13.5-13.6)

(3.1) putting $T_K = \exp(\bigcap_{i\notin K} L_{\alpha_i}) \subset G$, then T_K is a torus group of dimension $|K|$ (the cardinality of K);

(3.2) letting $\overline{H}_K$ be the semi-simple part of H_K, then H_K admits a factorization into *semi-product* of subgroups as $H_K = T_K \cdot \overline{H}_K$ with $T_K \cap \overline{H}_K$ finite;

(3.3) the Dynkin diagram of $\overline{H}_K$ can be obtained from D_G by deleting all the vertices α_i with $i \in K$ as well as the edges incident to them.

Summarizing the function f_G can be computed as follows. For any connected Dynkin diagram D, define

$$p(D) = \frac{1}{2}(\dim_{\mathbb{R}} G - rankG),$$

where G is a compact semisimple Lie group with $D_G = D$. It is then easy to compute $p(D)$ for the connected Dynkin diagrams of the usual classification:

D	$p(D)$
A_n	$\frac{1}{2}n(n+1)$
B_n, C_n	n^2
D_n	$n(n-1)$
E_6	36
E_7	63
E_8	120
F_4	24
G_2	6

Now for an arbitrary Dynkin diagram, let $p(D)$ be the sum of p applied to each connected component.

Lemma 4. $f_G(K) = p(D_G) - p(D_{\bar{H}_K})$, *where* $\bar{H}_K$ *is defined in (3.2).*

In short, the function f_G can be read directly from the Dynkin diagram of G. In what follows we present some computational examples based on Lemma 4. Let $SU(n)$ be the special unitary group of order n, $SO(n)$ the special orthogonal group of order n, and $Sp(n)$ the symplectic group of order n. The five exceptional Lie groups are denoted as usual by G_2, F_4, E_6, E_7, E_8.

If G is one of the above groups, we assume that a set of simple roots of G is given and ordered as the vertices of Dynkin diagram of G pictured in [12], p.58.

For a $K \in \Phi_n$ write $\overline{K} \in \Phi_n$ for the complement of K in $\{1, \cdots, n\}$. Note that any $K \in \Phi_n$ splits into disjoint union of some consecutive segments $K = K_1 \sqcup \cdots \sqcup K_m$. For example if $K = \{2,3,5\} \in \Phi_6$, then

(1) $\overline{K} = \{1,4,6\}$;

(2) $K = \{2,3\} \sqcup \{5\}$, $m = 2$.

Example 1. If $G = SU(n)$, the function $f_G : \Phi_{n-1} \to \mathbb{Z}$ is given by

$$f_G(K) = \tfrac{n(n-1)}{2} - \sum_{i=1}^{m} \tfrac{(|\overline{K}_i|+1)|\overline{K}_i|}{2},$$

where $\overline{K} = \overline{K}_1 \sqcup \cdots \sqcup \overline{K}_m$.

Example 2. If $G = SO(2n)$, the function $f_G : \Phi_n \to \mathbb{Z}$ is given by three cases.

i) If $n \in K$, then $f_G(K) = n(n-1) - \sum\limits_{i=1}^{m} \frac{(|\overline{K}_i|+1)|\overline{K}_i|}{2}$.

ii) If $n-1 \in K$, let $\sigma : \{1,2,\cdots,n\} \to \{1,2,\cdots,n\}$ be the transposition of $n-1$ and n. Then $f_G(K) = f_G(\sigma(K))$.

iii) If $n, n-1 \notin K$, then $f_G(K) = n(n-1) - \sum\limits_{i=1}^{m-1} \frac{(|\overline{K}_i|+1)|\overline{K}_i|}{2} - |\overline{K}_m|(|\overline{K}_m| - 1)$.

Example 3. If $G = SO(2n+1)$ or $Sp(n)$, the function $f_G : \Phi_n \to \mathbb{Z}$ is given by two cases.

i) If $n \in K$, then $f_G(K) = n^2 - \sum\limits_{i=1}^{m} \frac{(|\overline{K}_i|+1)|\overline{K}_i|}{2}$.

ii) If $n \notin K$, then $f_G(K) = n^2 - \sum\limits_{i=1}^{m-1} \frac{(|\overline{K}_i|+1)|\overline{K}_i|}{2} - |\overline{K}_m|^2$.

Example 4. If $G = E_6$ the function $f_G : \Phi_6 \to \mathbb{Z}$ is given by the table below.

K	$f_G(K)$	K	$f_G(K)$	K	$f_G(K)$	K	$f_G(K)$	K	$f_G(K)$
$\emptyset$	0	{24}	30	{134}	32	{346}	33	{2345}	34
{1}	16	{25}	29	{135}	32	{356}	32	{2346}	34
{2}	21	{26}	26	{136}	30	{456}	32	{2356}	34
{3}	25	{34}	31	{145}	33	{1234}	33	{2456}	33
{4}	29	{35}	31	{146}	33	{1235}	34	{3456}	34
{5}	25	{36}	29	{156}	30	{1236}	33	{12345}	35
{6}	16	{45}	31	{234}	32	{1245}	34	{12346}	35
{12}	26	{46}	31	{235}	32	{1246}	34	{12356}	35
{13}	26	{56}	26	{236}	32	{1256}	33	{12456}	35
{14}	31	{123}	30	{245}	32	{1345}	34	{13456}	35
{15}	29	{124}	32	{246}	32	{1346}	34	{23456}	35
{16}	24	{125}	32	{256}	30	{1356}	33	{123456}	36
{23}	29	{126}	30	{345}	33	{1456}	34		

Example 5. For the five exceptional Lie groups, the values of h_G on the fundamental dominant weights $\omega_1, \cdots, \omega_n$ (with $n = 2, 4, 6, 7$ and 8 respectively) are given in the table below.

G	$h_G(\omega_1)$	$h_G(\omega_2)$	$h_G(\omega_3)$	$h_G(\omega_4)$	$h_G(\omega_5)$	$h_G(\omega_6)$	$h_G(\omega_7)$	$h_G(\omega_8)$
G_2	5	5						
F_4	15	20	20	15				
E_6	16	21	25	29	25	16		
E_7	33	42	47	53	50	42	27	
E_8	78	92	98	106	104	97	83	57

4. Applications and extensions of the main result

Our method can be extended to compute the height function

$$h_{(G,H)} : H^2(G/H) \to \mathbb{Z},\ x \to \max\{m \mid x^m \neq 0\}.$$

for a generalized flag manifold G/H, where H is the centralizer of a one-parameter subgroup in G.

Since H is conjugate in G to one of the subgroups H_K, $K \in \Phi_n$ [2], and since the induced ring map $p_K^* : H^*(G/H_K) \to H^*(G/T)$ is injective and identifies $H^2(G/H_K)$ with the submodule $span_{\mathbb{Z}}\{\omega_i \mid i \in K\} \subset H^2(G/T)$ (by Lemma 2), we have

Proposition 1. *The function $h_{(G,H)} : H^2(G/H) \to \mathbb{Z}$ can be given by restricting h_G to the submodule $span_{\mathbb{Z}}\{\omega_i \mid i \in K\} \subset H^2(G/T)$.*

The Theorem enables one to recover and extend some relevant results previously known. For $G = SU(n)$, $SO(n)$ and $Sp(n)$ one has the following classical descriptions for the cohomology of G/T from Borel [3], 1953.

$H^*(U(n)/T;\mathbb{Z}) = \mathbb{Z}[t_1, \cdots, t_n]/\langle e_i(t_1, \cdots, t_n), 1 \leq i \leq n\rangle$;
$H^*(SO(2n+1)/T;\mathbb{R}) = \mathbb{R}[t_1, \cdots, t_n]/\langle e_i(t_1^2, \cdots, t_n^2), 1 \leq i \leq n\rangle$;
$H^*(Sp(n)/T;\mathbb{Z}) = \mathbb{Z}[t_1, \cdots, t_n]/\langle e_i(t_1^2, \cdots, t_n^2), i \leq n\rangle$;
$H^*(SO(2n)/T;\mathbb{R}) = \mathbb{R}[t_1, \cdots, t_n]/\langle t_1 \cdots t_n; e_i(t_1^2, \cdots, t_n^2), 1 \leq i \leq n-1\rangle$,

where $e_r(y_1, \cdots, y_n)$ is the r-th elementary symmetric function in $y_1, \cdots, y_n$, and where $t_i \in H^2(G/T)$. The transitions between the bases for $H^2(G/T)$ given by the t_i and by the ω_i in our theorem are seen as follows (cf. [8], Example 3)

(4.1) for $G = SU(n)$, $\omega_i = t_1 + \cdots + t_i$, $1 \leq i \leq n-1$;
(4.2) for $G = SO(2n+1)$, $\omega_i = t_1 + \cdots + t_i$, $1 \leq i \leq n-1$; and
$\omega_n = \frac{1}{2}(t_1 + \cdots + t_n)$;
(4.3) for $G = Sp(n)$, $\omega_i = t_1 + \cdots + t_i$, $1 \leq i \leq n$;
(4.4) for $G = SO(2n)$, $\omega_i = t_1 + \cdots + t_i$, $1 \leq i \leq n-2$;
$\omega_{n-1} = \frac{1}{2}(t_1 + \cdots + t_{n-1} - t_n)$; and
$\omega_n = \frac{1}{2}(t_1 + \cdots + t_{n-1} + t_n)$.

In each of the four cases we have

Lemma 5. *The set $\{\pm t_1, \cdots, \pm t_n\}$ agrees with the W_G-orbit through $\omega_1 = t_1$. Therefore, $h_G(t_i) = \dim_{\mathbb{C}} G/H_{\{1\}}$, $1 \leq i \leq n$ (by the Theorem).*

The manifolds $G/H_{\{1\}}$ can be identified with familiar spaces by the discussion at the beginning of Section 3. Let $\mathbb{C}P^{n-1}$ be the projective space of complex lines in $\mathbb{C}^n$; $H(n)$ the real Grassmannian of oriented 2-planes in $\mathbb{R}^n$; and let $\mathbb{H}P^{n-1}$ be the projective space of quaternionic lines in the n-quaternionic vector space $\mathbb{H}^n$. We have

$$G/H_{\{1\}} = \begin{cases} \mathbb{C}P^{n-1} & \text{if} \quad G = SU(n); \\ H(n) & \text{if} \quad G = SO(n); \\ E(n) & \text{if} \quad G = Sp(n), \end{cases}$$

where $E(n)$ is the total space of complex projective bundle associated to γ_n, the complex reduction of the canonical quaternionic line bundle over $\mathbb{H}P^{n-1}$ (cf. [6], Section 2.5). It can be verified directly from Lemma 4 that

Lemma 6. *Assume that G is one of the matrix groups $SU(n), SO(m)$ with $m \neq 4,5,6,8$, or $Sp(n)$ with $n > 2$. For all K we have $\dim_{\mathbb{C}} G/H_{\{1\}} \leq \dim_{\mathbb{C}} G/H_K$, where the equality holds if and only if*

1) $K = \{1\}$ if $G \neq SU(n)$;

2) $K = \{1\}, \{n-1\}$ if $G = SU(n)$.

Combining Lemma 5, 6 and the Theorem we show

Proposition 2. *Assume that G is one of the matrix groups $SU(n), SO(m)$ with $m \neq 4,5,8$, or $Sp(n)$ with $n \neq 2$. Then*

(i) $h^{-1}_{SU(n)}(n-1) = \{\lambda t_i \mid 1 \leq i \leq n,\ \lambda \in \mathbb{Z} \setminus \{0\}\}$;

(ii) $h^{-1}_{SO(2n)}(2n-2) = \{\lambda t_i \mid 1 \leq i \leq n,\ \lambda \in \mathbb{Z} \setminus \{0\}\}$;

(iii) $h^{-1}_{SO(2n+1)}(2n-1) = \{\lambda t_i \mid 1 \leq i \leq n,\ \lambda \in \mathbb{Z} \setminus \{0\}\}$;

(iv) $h^{-1}_{Sp(n)}(2n-1) = \{\lambda t_i \mid 1 \leq i \leq n,\ \lambda \in \mathbb{Z} \setminus \{0\}\}$.

Proof. In view of Lemma 5 and 6, it remains to show that, if $G = SU(n)$, $O(\omega_{n-1}) = O(\omega_1)$, where $O(x)$ is the W_G-orbit through $x \in H^2(G/T)$.

¿From the first relation $e_1(t_1, \cdots, t_n) = 0$ in $H^*(SU(n)/T)$ we get $\omega_{n-1} = -t_n$ from (4.1). It follows that $O(\omega_{n-1}) = O(\omega_1)$ by Lemma 5. ∎

Remark 2. Lemma 6 fails for $G = SO(4)$, $SO(5)$, $SO(6)$, $SO(8)$ or $Sp(2)$. For example if $G = SO(8)$, the solutions in $K \in \Phi_4$ to the equation $\dim_{\mathbb{C}} G/H_{\{1\}} = \dim_{\mathbb{C}} G/H_K$ are $K = \{1\}, \{3\}$ and $\{4\}$.

However, in these minor cases, one can work out the pre-image of h_G at its minimal non-zero values using the same method. For instance

$$\begin{aligned} h^{-1}_{SO(8)}(6) &= O(\lambda\omega_1) \cup O(\lambda\omega_3) \cup O(\lambda\omega_4) \\ &= \{\lambda t_i,\ \tfrac{\lambda}{2} \sum_{i=1}^{4} \pm t_i \mid 1 \leq i \leq 4,\ \lambda \in \mathbb{Z} \setminus \{0\}\}. \end{aligned}$$

Previously, item (i) was obtained independently by Monk [13], Ewing and Liulevicius [9]. (iii) and (iv) were shown by the authors in [7].

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