Journal of Lie Theory
Volume **15** (2005) 197–218

Correspondences between Jet Spaces and PDE Systems

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Communicated by P. Olver

Abstract. Following some of Lie's ideas, we define between jet spaces canonical correspondences which allow us to associate with each first order PDE system another one with a single unknown function which contains as solutions that of the original system as well as its intermediate integrals. We also show for some systems of PDE that their integration is equivalent to that of their associated ones.
Subject Classification: 58A20, 35A30.
Keywords: jet, Lie correspondence, system of partial differential equations, involutive system, intermediate integral, Lie system.

Introduction

In a long paper published in 1895 [11], S. Lie attempted to *reduce, as far as possible, the general theory of partial differential equations of arbitrary order to that of first order ones, thereby making its treatment amenable from the theory of groups* (page 327). He devotes the second chapter to such a reduction, making a detailed study of the systems of two second order equations with two independent variables and only one unknown function. In [5, page 109] Goursat admits the method proposed by Lie to be ingenious and deep. However, as far as we know, these ideas by Lie have not been continued.

The most important achievement in [11] is the idea for the reduction, which one can guess from the statements and proofs, wrapped into the unavoidable imprecision caused by the state of the art at that time. This idea consists in using some natural correspondences between jet spaces that apply submanifolds of a space to submanifolds of another one, and therefore systems of partial differential equations of one kind to systems of another kind. The aim of this paper is to develop a theory of correspondences between certain jet spaces and apply it to systems of partial differential equations, thus clarifying and completing some of

*The research of the author was partially funded by Junta de Castilla y León under Project SA077/03
†The research of the author was partially funded by Junta de Castilla y León under Project SA077/03

ISSN 0949–5932 / $2.50

the partial results announced by Lie in [11].

For a better understanding of Lie's ideas it is convenient to think of jets of a manifold M as ideals of its ring of smooth functions. This point of view, which was introduced by Muñoz, Muriel and Rodríguez in [13], is a natural continuation of Weil's theory of near points [18] and it allows describing the process of prolongation, the affine structures, the contact system, etc., in terms of the ring of smooth functions of the original manifold, making the fibration of M unnecessary and simplifying essentially the calculus in local coordinates. Some applications of this theory to different topics can be found in [2, 14, 15, 16].

In the two first sections of this paper we explain the basic topics about jets and partial differential equations which we use later on; since we need essentially first order jets only, we focus our attention in them, though most of the results are generalizable to higher order jets.

In Section 3 we define canonical correspondences between some spaces of jets of a smooth manifold by means of the relation of inclusion of ideals. Ehresmann's point of view is not appropriate for this theory, because our correspondences involve jets of different dimensions (different numbers of "independent variables"); when the manifold is fibred over a fixed base manifold all the jets have the same dimension, and hence the correspondences cannot be established. As shown in [3], a jet is essentially the same object than the value of the contact system at it; using this fact we characterize our correspondences in terms of inclusions between contact systems.

The use of these correspondences, which we call Lie correspondences, allows us to associate with each first order system $\mathcal{R}$ of partial differential equations another one, $\mathcal{R}^*$, with only one unknown function; the properties of this kind of systems are well known. We establish a relationship between the solutions of both of them: each solution of $\mathcal{R}$ is also a solution (in the generalized Lie sense) of $\mathcal{R}^*$. We also clarify the meaning of the intermediate integrals of $\mathcal{R}$, which are obtained when they exist as solutions of $\mathcal{R}^*$.

Finally, we apply the theory to involutive PDE systems whose symbol equals zero and to systems of two second–order PDE's in two independent variables and one unknown function, obtaining that their integration is equivalent to that of their associated first order systems.

Some of the results of this paper were announced, without proofs, in [6].

1. Jets of submanifolds

This section contains some basic ideas and results about jet spaces from a point of view related to Weil bundles. We restrict ourselves to jets of submanifolds; a more general theory of Weil jets and the proofs of the results can be found in [1, 13].

Let M be an n-dimensional smooth manifold; if X is an m-dimensional submanifold of M, for each point $p \in X$ we define the (m,ℓ)-jet of X at p as the class of all m-dimensional submanifolds of M which have a contact of order ℓ with X at p. If X is a closed submanifold defined by an ideal I_X of $C^\infty(M)$, then we can associate with its (m,ℓ)-jet at p the ideal $\mathfrak{p}^\ell_m = I_X + \mathfrak{m}_p^{\ell+1}$, where $\mathfrak{m}_p$ is the ideal of the smooth functions on M vanishing at p. This gives a bijection between the set of (m,ℓ)-jets of M and the set of ideals $\mathfrak{p}^\ell_m$ of $C^\infty(M)$ such that the factor ring $C^\infty(M)\,/\mathfrak{p}^\ell_m$ is isomorphic to the Weil algebra $\mathbb{R}^\ell_m$ of polynomials of degree $\leq \ell$ in m variables. When $m = \dim M$, each (m, ℓ)-jet of M has the form $\mathfrak{m}_p^{\ell+1}$, where $p \in M$.

We will denote by $J_m^\ell M$ the set of all (m, ℓ)-jets of M. There is a canonical projection $\pi^\ell \colon J_m^\ell M \longrightarrow M$ which assigns to each jet $\mathfrak{p}_m^\ell$ the unique maximal ideal $\mathfrak{p}_m^0 = \mathfrak{m}_p$ of $C^\infty(M)$ containing $\mathfrak{p}_m^\ell$; the point $p \in M$ corresponding to this ideal is called the source of $\mathfrak{p}_m^\ell$.

Remark 1.1. When $\ell = 1$ we can give a geometric description of jets. Each first order jet $\mathfrak{p}_m^1 \in J_m^1 M$ is the ideal of the functions vanishing at a point $p \in M$ that are annihilated by m linearly independent tangent vectors $D_{1p}, \dots, D_{mp} \in T_pM$. Thus, $\mathfrak{p}_m^1$ can be thought of as an m–dimensional linear subspace $L_{\mathfrak{p}_m^1}$ of T_pM. Hence, $J_m^1 M$ is the Grassmann manifold of m-planes of M.

In [13] $J_m^\ell M$ is endowed with a smooth structure as a quotient of the space of regular (m, ℓ)-velocities of M. Local coordinates may be described as follows:

Let $\mathfrak{p}_m^\ell \in J_m^\ell M$ be the (m, ℓ)-jet at $p \in M$ of a closed submanifold X of M. We can find a neighbourhood of p coordinated by functions $x_1, \dots, x_m, y_1, \dots, y_{n-m}$ such that the local equations of X can be written in the form

$$y_j = f_j(x_1, \dots, x_m); \qquad (1 \leq j \leq n - m)$$

then $\mathfrak{p}_m^\ell$ is the sum of $\mathfrak{m}_p^{\ell+1}$ and the ideal spanned by the $n - m$ functions

$$y_j - \sum_{|\alpha| \leq \ell} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f_j}{\partial x^\alpha}(p)\ (x - x(p))^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multi-index and

$$(x - x(p))^\alpha = (x_1 - x_1(p))^{\alpha_1} \cdot \ldots \cdot (x_m - x_m(p))^{\alpha_m}.$$

The functions $x_i, y_{j,\alpha}$ $(1 \leq i \leq m,\ 1 \leq j \leq n - m,\ |\alpha| \leq \ell)$ defined by

$$x_i(\mathfrak{p}_m^\ell) = x_i(p), \qquad y_{j,\alpha}(\mathfrak{p}_m^\ell) = \frac{\partial^{|\alpha|} f_j}{\partial x^\alpha}(p)$$

are local coordinates in an open subset of $J_m^\ell M$ (note that $y_{j,0} = y_j$).

Remark 1.2. In the notations above, $J_m^\ell M$ is locally the space of ℓ–jets of sections of the projection $(x_i, y_j) \longmapsto x_i$. This is the reason why we use the usual notations x_i, y_j, thus establishing a distinction between the "base coordinates" and the "fibre coordinates". Nevertheless, such a distinction is only formal. If we have local coordinates $x_1, \dots, x_n$ in an open subset of M, we can think of m of them as base coordinates and the remainder ones as fibre coordinates, but in a dynamical way, without fixing them. This idea is due to Lie (see [10]).

Let X be an m-dimensional closed submanifold of M; the prolongation of X to $J_m^\ell M$, $J_m^\ell X$, is the submanifold of $J_m^\ell M$ whose points are the jets of the form $I_X + \mathfrak{m}_p^{\ell+1}$, where p runs through X. It is easy to see that if, in the above local coordinates, the local equations of X are $y_j - f_j(x_1, \dots, x_m) = 0$ $(1 \leq j \leq n - m)$, then the local equations of its prolongation are

$$\begin{aligned} y_j &= f_j(x_1, \dots, x_m), && (1 \leq j \leq n - m) \\ y_{j,\alpha} &= \frac{\partial^{|\alpha|} f_j}{\partial x^\alpha}, && (1 \leq j \leq n - m; 1 \leq |\alpha| \leq \ell) \end{aligned}$$

There is a canonical Pffaf system in $J^\ell_m M$, the contact system (also called Cartan system), $\Omega(J^\ell_m M)$, which measures in some sense when a submanifold of $J^\ell_m M$ is the prolongation of an m–dimensional submanifold of M. In the above local coordinates $\Omega(J^\ell_m M)$ is locally spanned by the 1–forms

$$\omega_{j,\alpha} = dy_{j,\alpha} - \sum_{i=1}^{m} y_{j,\alpha+1_i}\, dx_i, \qquad (1 \leq i \leq m, |\alpha| \leq \ell - 1),$$

where 1_i is the m–index with 1 in the ith component and 0 in the remaining ones. Its associated distribution of vector fields is spanned by the total derivatives $\partial_i^{(\ell)}$ $(1 \leq i \leq m)$ and $\frac{\partial}{\partial y_{j,\alpha}}$ $(1 \leq j \leq n - m,\ |\alpha| = \ell)$.

Remark 1.3. Different coordinate free definitions of the contact system in jet spaces can be found in the literature. For first order it may be described easily: Given a jet $\mathfrak{p}^1_m \in J^1_m M$ with source $p \in M$, from the above expressions in local coordinates we get that $\Omega(J^1_m M)_{\mathfrak{p}^1_m}$ is spanned by the 1–forms $(\pi^1)^*\, d_p\varphi_j$ $(1 \leq j \leq n - m)$, where $\varphi_j = y_j - \sum\limits_{i=1}^{m} y_{j,i}(\mathfrak{p}^1_m)\, (x_i - x_i(p))$ is a function of the jet $\mathfrak{p}^1_m$ itself, and $(\pi^1)^*$ is the pull back by the projection $\pi^1 : J^1_m M \longrightarrow M$. Since $\mathfrak{p}^1_m = (\varphi_1, \dots, \varphi_{n-m}) + \mathfrak{m}_p^2$, we obtain

$$\Omega(J^1_m M)_{\mathfrak{p}^1_m} = (\pi^1)^*\, d_p \mathfrak{p}^1_m.$$

In other words, for first order the value of the contact system at a jet and the jet itself are essentially the same object.

This property remains being valid in a suitable sense for higher order and for more general jet spaces (see [3]), but along the paper we will use it only for first order jets.

2. First–order PDE systems with one unknown function

In the following section we shall deal with correspondences between jet spaces that allow us to associate with a given PDE system a first order one with only one unknown function. We shall now recall briefly some basic facts about this kind of systems and the spaces where they live, namely, $J^1_{n-1}M$ and T^*M, where M is an n–dimensional smooth manifold. A detailed treatment can be found in [12].

As we have seen, each jet $\mathfrak{p}^1_{n-1} \in J^1_{n-1}M$ is the ideal of functions of $C^\infty(M)$ vanishing at $p(= \mathfrak{p}^0_{n-1}) \in M$ and annihilated by $n-1$ linearly independent tangent vectors $D_1, \dots, D_{n-1} \in T_pM$. That is to say, the set of the functions $f \in \mathfrak{m}_p$ such that d_pf annihilates $D_1, \dots, D_{n-1}$. Thus, $\mathfrak{p}^1_{n-1} / \mathfrak{m}_p^2$ is a line in T^*_pM and $J^1_{n-1}M = \mathbb{P}(T^*M)$.

In T^*M there is a well–known canonical 1–form θ (see [12], for instance): for each $\alpha_p \in T^*_pM$, the value of θ at α_p is the lift to T^*M of the α_p itself via the projection $T^*M \longrightarrow M$. The 2–form $d\theta$ endows T^*M with a symplectic structure; the Lagrangian submanifolds of T^*M are the 1–forms (sections of $T^*M \longrightarrow M$) that are locally exact and also those deduced from them by means of canonical transformations (transformations which preserve the symplectic structure).

Let us take local coordinates $x_1, \dots, x_{n-1}, y_1$ in M; these and the conjugated ones $p_1, \dots, p_{n-1}, q$ in T^*M, and $x_1, \dots, x_{n-1}, y_1, y_{1,1}, \dots, y_{1,n-1}$ in $J^1_{n-1}M$. The local equations for the projection $\pi\colon T^*M \longrightarrow J^1_{n-1}M$ are

$$(x_i, y_1, p_i, q) \longrightarrow (x_i, y_1, y_{1,i} = -\frac{p_i}{q}),$$

the canonical 1–form is $\theta = p_1 dx_1 + \cdots + p_{n-1}dx_{n-1} + q dy_1$, and the contact system $\Omega(J^1_{n-1}M)$ is spanned by a unique 1–form $\omega = dy_1 - y_{1,1}dx_1 - \cdots - y_{1,n-1}dx_{n-1}$. Thus,

$$\pi^*(\omega) = dy_1 + \sum_{k=1}^{n-1} \frac{p_k}{q} dx_k = \frac{1}{q}\theta$$

Proposition 2.1. *The Pfaff system spanned in T^*M by the canonical 1–form θ is projectable, and its projection onto JnM is $\Omega(J^1_{n-1}M)$.*

A *system of partial differential equations of order ℓ, in m independent variables, over M*, is a locally closed submanifold $\mathcal{R}$ of $JMLV$. A *classical solution of* $\mathcal{R}$ is an m-dimensional submanifold $X \subseteq M$ such that $J^\ell_m X \subseteq \mathcal{R}$. A *generalized solution* is an m–dimensional submanifold $\overline{X} \subseteq J^\ell_m M$ solution of the contact system such that $\overline{X} \subseteq \mathcal{R}$.

A *first order system with only one unknown function* is either a locally closed submanifold $\mathcal{R} \subseteq J^1_{n-1}M$ or, if 'the unknown function does not appear explicitly', a locally closed submanifold $\mathcal{F} \subseteq T^*M$. Let $\mathcal{R} \subseteq J^1_{n-1}M$; the solutions of $\mathcal{R}$ in the generalized Lie sense are the Legendre submanifolds of $J^1_{n-1}M$ contained in $\mathcal{R}$. Among them, there are the classical solutions: hypersurfaces $X_{n-1} \subseteq M$ such that $J^1_{n-1}(X) \subseteq \mathcal{R}$. Passing from $J^1_{n-1}M$ to T^*M, the submanifolds $\mathcal{F} \subseteq T^*M$ are the first order systems with only one unknown function which does not appear explicitly. A classical solution of $\mathcal{F}$ is an exact 1–form dV which, as a section of $T^*M \longrightarrow M$, values in $\mathcal{F}$, and a generalized solution is a Lagrangian submanifold $X_n \subseteq \mathcal{F}$.

3. Correspondences between jet spaces

In this section, M will be an n-dimensional fixed manifold and all the jet spaces are referred to it. Hence, we shall simplify the notation by omitting M when no confusion can arise. Thus, Ω^ℓ_m will denote $\Omega(J^\ell_m M)$, for example.

Since each jet in M is an ideal of $C^\infty(M)$, the relation of inclusion between ideals gives canonical correspondences between jet spaces. Focusing on the case in which we are interested, we give the:

Definition 3.1. Given the integers $0 \le m \le r \le n$, the *Lie correspondence* $\bigwedge_{m,r} = \bigwedge_{m,r}(M)$ is the subset of the fibred product $J^1_m M \times_M J^1_r M$ consisting of the pairs of jets $(\mathfrak{p}^1_m, \mathfrak{p}^1_r)$ (with the same source $p = \mathfrak{p}^0_m = \mathfrak{p}^0_r$) such that $\mathfrak{p}^1_m \supseteq \mathfrak{p}^1_r$ (inclusion as ideals of $C^\infty(M)$).

A geometric interpretation of these correspondences results from thinking of each first order jet as a linear subspace of T_pM: the inclusion between ideals becomes an (reversed) inclusion between linear subspaces ($L_{\mathfrak{p}^1_m} \subseteq L_{\mathfrak{p}^1_r}$).

Remark 3.2. At this point it is essential to stop thinking of jets as 'jets of cross-sections of a fibred manifold', because when M is fibred over a manifold X, all the jets have the same dimension ($= \dim X$) and the above correspondences cannot be established.

Since the value at $\mathfrak{p}^1_m$ of the contact system Ω^1_m is (see §1) the set $\{d_p f,\ f \in \mathfrak{p}^1_m\}$ $(p = \mathfrak{p}^0_m)$, we can understand the Lie correspondences in terms of inclusions between contact systems:

Proposition 3.3. **(Basic Lemma).** *The neccessary and sufficient condition for a couple $(\mathfrak{p}^1_m, \mathfrak{p}^1_r) \in J^1_m M \times_M J^1_r M$ to be in $\bigwedge_{m,r}$ is that the following inclusion*

$$(\Omega^1_m)_{\mathfrak{p}^1_m} \supseteq (\Omega^1_r)_{\mathfrak{p}^1_r} \qquad (\textit{lifted to } J^1_m M \times_M J^1_r M)$$

holds.

For each first order PDE system $\mathcal{R} \subseteq J^1_m M$, the restriction of the correspondence $\bigwedge_{m,r}$ to $\mathcal{R}$ will be denoted by $\mathcal{R}_{m,1}$, that is:

$$\mathcal{R}_{m,1} = \{(\mathfrak{p}^1_m, \mathfrak{p}^1_r) \in \mathcal{R} \times_M J^1_r M : \mathfrak{p}^1_m \supseteq \mathfrak{p}^1_r\}$$

Projecting the correspondence $\mathcal{R}_{m,1}$ to $J^1_r M$ we can associate with $\mathcal{R}$ a first order PDE system in r independent variables. From now on we will restrict ourselves to the case $r = n-1$; the submanifolds $\mathcal{F} \subseteq J^1_{n-1} M$ are the first order systems with only one unknown function and the properties of this kind of systems are well known (see [12], for instance).

In [11], S. Lie associates with some systems of partial differential equations a first order system with an unique unknown function which does not appear explicitly, that is to say, a submanifold of T^*M; since $J^1_{n-1}M$ is the projectivized manifold of T^*M, for the correspondences $\bigwedge_{m,n-1}$ we can replace the second factor in $J^1_m M \times_M J^1_{n-1} M$ by T^*M, which we will do in the sequel. Thus, we shall denote by $\bigwedge_{m,*}(M)$ the subset of $J^1_m M \times_M T^*M$ defined by

$$\bigwedge\nolimits_{m,*}(M) = \left\{(\mathfrak{p}^1_m, \alpha_p) \in J^1_m M \times_M T^*M : \alpha_p \in \mathfrak{p}^1_m/\mathfrak{m}^2_p\right\}.$$

From the Basic Lemma and Proposition 2.1 it follows:

Proposition 3.4. *The intrinsic equation of $\bigwedge_{m,*}$ as a submanifold of $J^1_m M \times_M T^*M$ is $\theta \in \Omega^1_m$ (lifted to $J^1_m M \times_M T^*M$), where θ is the canonical 1-form in T^*M.*

The above proposition provides an effective method for calculating the local equations of the above correspondence $\bigwedge_{m,*}$ as submanifold of $J^1_m M \times_M T^*M$.

Let us take local coordinates $x_1, \ldots, x_m, y_1, \ldots, y_{n-m}$ in M, these and $y_{1,1}, \ldots, y_{1,m}, \ldots, y_{n-m,1}, \ldots, y_{n-m,m}$ in $J^1_m M$, and $x_1, \ldots, x_m, y_1, \ldots, y_{n-m}$ and the 'conjugated' ones $p_1, \ldots, p_m, q_1, \ldots, q_{n-m}$ in T^*M. The contact system Ω^1_m is spanned by the 1-forms

$$\omega_j = dy_j - \sum_{i=1}^m y_{j,i}\, dx_i \qquad (1 \le j \le n-m)$$

and the canonical 1-form θ of T^*M lifted to $J^1_m M \times_M T^*M$ is

$$\theta = \sum_{i=1}^{m} p_i \ dx_i + \sum_{j=1}^{n-m} q_j \ dy_j = \sum_{i=1}^{m} \left(p_i + \sum_{j=1}^{n-m} q_j y_{j,i} \right) \ dx_i + \sum_{j=1}^{n-m} q_j \ \omega_j$$

Hence, the condition for θ to be in the contact system is that the following equations hold:

$$p_i + \sum_{j=1}^{n-m} q_j y_{j,i} = 0 \qquad (1 \leq i \leq m) \tag{1}$$

These are the local equations of $\bigwedge_{m,*}$ as a submanifold of $J^1_m M \times_M T^*M$.

For each system $\mathcal{R} \subseteq J^1_m M$, $\mathcal{R}_{m,*}$ will denote the restriction of $\bigwedge_{m,*}$ to $\mathcal{R}$. The fibre of $\mathcal{R}_{m,*}$ over $\mathfrak{p}^1_m$ is the set of the differentials at p of all the functions of $\mathfrak{p}^1_m$, or, what it is the same, $(\Omega^1_m)_{\mathfrak{p}^1_m}$. The projectivized space of this fibre is the fibre of $\mathcal{R}_{m,n-1}$, which is the collection of hyperplanes of T_pM passing through $L_{\mathfrak{p}^1_m}$.

Definition 3.5. The projection of $\mathcal{R}_{m,*}$ in T^*M will be called the *first order system of partial differential equations associated with* $\mathcal{R}$ and it will be denoted by $\mathcal{R}^*$.

The forms ω_j, dx_i $(1 \leq i \leq m, 1 \leq j \leq n-m)$ are linearly independent when specialized to any submanifold fibred over M. Hence, equations (1) give also the condition for the specialization of θ to such a submanifold of $J^1_m M \times_M T^*M$ to be in the specialization of the contact system Ω^1_m; therefore, if $\mathcal{R} \subseteq J^1_m M$ is a first order PDE system given by

$$F_k(x_i, y_j, y_{j,i}) = 0, \qquad (1 \leq k \leq s) \tag{2}$$

the local equations of $\mathcal{R}_{m,*}$ as a submanifold of $J^1_m M \times_M T^*M$ are

$$\left\{ \begin{array}{ll} p_i + \sum\limits_{j=1}^{n-m} q_j \ y_{j,i} = 0 & \qquad (1 \leq i \leq m) \\ F_k(x_i, y_j, y_{j,i}) = 0 & \qquad (1 \leq k \leq s) \end{array} \right. \tag{3}$$

and that of its first order associated system $\mathcal{R}^* \subseteq T^*M$ are obtained by eliminating the derivatives $y_{j,i}$ from the equations of $\mathcal{R}_{m,*}$.

Remark 3.6. (1) A priori, $\mathcal{R}^*$ might not be a submanifold of T^*M. The above definitions and results can be extended to the complex framework; when one is working with complex algebraic manifolds and $\mathcal{R}$ is an algebraic submanifold of $J^1_m M$, then $\mathcal{R}^*$ contains a dense open subset that is a manifold.

(2) Observe that if the number of equations of $\mathcal{R}$ is not enough to eliminate the $y_{j,i}$, $\mathcal{R}^*$ may be the whole T^*M .

Example 3.7. Let $X \subseteq M$ be an m-dimensional submanifold; $\mathcal{R} = J^1_m X \subseteq J^1_m M$ is a system of partial differential equations whose unique solution is X (and its open subsets). Each jet $\mathfrak{p}^1_m \in J^1_m X$ can be identified with T_pX (where $p = \mathfrak{p}^0_m$

is the source of $\mathfrak{p}_m^1$). We can think of each $\mathfrak{p}_{n-1}^1 \subseteq \mathfrak{p}_m^1$ as a hyperplane $H_p \subseteq T_pM$ containing $L_{\mathfrak{p}_m^1} = T_pX$. Thus, $\mathcal{R}_{m,n-1}$ is the collection of all the hyperplanes H_p tangent to X: it is *the manifold of contact elements of* X in the Lie terminology. $(JmX)^*$ is a Lagrangian submanifold of T^*M; in fact, it is homogeneous, that is, a solution of $\theta = 0$.

In the above notation assume that X is locally given by

$$y_j - f_j(x_1, \dots, x_m) = 0, \qquad (1 \le j \le n-m). \tag{4}$$

From (3) it follows that the local equations of $(J_m^1 X)^*$ as a submanifold of T^*M are

$$\begin{cases} p_i + \displaystyle\sum_{j=1}^{n-m} q_j \frac{\partial f_j}{\partial x_i} = 0, & (1 \le i \le m) \\ y_j - f_j(x_1, \dots, x_m) = 0, & (1 \le j \le n-m) \end{cases} \tag{5}$$

Set $V = -\sum\limits_{j=1}^{n-m} f_j\, q_j$; the equations (5) are written as

$$\begin{cases} p_i = \dfrac{\partial V}{\partial x_i}, & (1 \le i \le m) \\ y_j = -\dfrac{\partial V}{\partial q_j}, & (1 \le j \le n-m) \end{cases} \tag{6}$$

A trivial verification proves that the functions $p_i - \frac{\partial V}{\partial x_i}$, $y_j + \frac{\partial V}{\partial q_j}$ $(1 \le i \le m,\ 1 \le j \le n-m)$ are in involution with respect to the usual Poisson structure in T^*M.

Each inclusion $\mathcal{S} \subseteq \mathcal{R}$ between submanifolds of $J_m^1 M$ gives rise to another one $\mathcal{S}^* \subseteq \mathcal{R}^*$. In particular, when $X \subseteq M$ is a solution of $\mathcal{S}$, from $J_m^1 X \subseteq \mathcal{R}$ it follows that $(J_m^1 X)^* \subseteq \mathcal{R}^*$; we can state the result of [11, page 351] as follows:

Theorem 3.8. **Lie** *If X is a solution of the PDE system $\mathcal{R} \subseteq J_m^1 M$, X is also a solution, in the generalized Lie sense, of the first order system $\mathcal{R}^*$; that is, $(J_m^1 X)^*$ is a Lagrangian submanifold of $\mathcal{R}^*$.*

Examples 3.9. (1) Let us consider $\mathbb{R}^4$ with coordinates x_1, x_2, y_1, y_2 and $J_2^1\mathbb{R}^4$ with coordinates x_1, x_2, y_1, y_2 and the derivatives $y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}$; $T^*\mathbb{R}^4$ is coordinated by x_1, x_2, y_1, y_2 and their conjugated ones p_1, p_2, q_1, q_2.

The equations of the correspondence $\bigwedge_{2,*}(\mathbb{R}^4) \subseteq J_2^1\mathbb{R}^4 \times_{\mathbb{R}^4} T^*\mathbb{R}^4$ are (see above)

$$\begin{aligned} p_1 + q_1\, y_{1,1} + q_2\, y_{2,1} = 0 \\ p_2 + q_1\, y_{1,2} + q_2\, y_{2,2} = 0 \end{aligned} \tag{7}$$

Let $\mathcal{R} \subseteq J_2^1\mathbb{R}^4$ be the PDE system given by

$$y_{j,i} = f_{j,i}(x_1, x_2, y_1, y_2) \qquad (1 \le i, j \le 2). \tag{8}$$

The equations of its associated first order system $\mathcal{R}^* \subseteq T^*\mathbb{R}^4$ are obtained by eliminating the derivatives $y_{j,i}$ $(1 \le i, j \le 2)$ from (7) and (8). That is, the equations of $\mathcal{R}^*$ are

$$\begin{aligned} p_1 + q_1\, f_{1,1} + q_2\, f_{2,1} = 0 \\ p_2 + q_1\, f_{1,2} + q_2\, f_{2,2} = 0 \end{aligned} \tag{9}$$

Now, we look for the solutions of $\mathcal{R}^*$ of the form $(J^1_2 X)^*$, where $X \subseteq \mathbb{R}^4$ is given by $y_1 = \varphi_1(x_1, x_2)$, $y_2 = \varphi_2(x_1, x_2)$. Hence Equations (9) must be satisfied by $(J^1_2 X)^*$, whose equations are

$$\begin{aligned}
&y_1 = \varphi_1(x_1, x_2)\\
&y_2 = \varphi_2(x_1, x_2)\\
&p_1 + q_1 \frac{\partial \varphi_1}{\partial x_1} + q_2 \frac{\partial \varphi_2}{\partial x_1} = 0\\
&p_2 + q_1 \frac{\partial \varphi_1}{\partial x_2} + q_2 \frac{\partial \varphi_2}{\partial x_2} = 0
\end{aligned}$$

So we have $\frac{\partial \varphi_j}{\partial x_i} = f_{j,i}$, what proves that X is also a solution of $\mathcal{R}$. Here the converse of the above theorem also occurs, but this is not always true.

(2) In the same notation of the previous example let us consider the system $\mathcal{R} \subseteq J^1_2 \mathbb{R}^4$ given by

$$y_{1,1} = y_{2,2},\ y_{1,2} = y_{2,2},\ y_{2,1} = y_{2,2} \tag{10}$$

The equations of its associated first order system $\mathcal{R}^* \subseteq T^*\mathbb{R}^4$ are

$$p_1 - p_2 = 0 \tag{11}$$

As before, we look for solutions of $\mathcal{R}^*$ of the form $(J^1_2 X)^*$; among them there will be the solutions of $\mathcal{R}$. We find easily that X is given by $y_1 = \varphi_1(x_1 + x_2)$, $y_2 = \varphi_2(x_1 + x_2)$, where φ_1, φ_2 are arbitrary functions of one variable. In order to obtain a solution of $\mathcal{R}$ we must impose the additional condition $\varphi_1' = \varphi_2'$; so, the solutions of $\mathcal{R}$ are

$$\begin{aligned}
&y_1 = \varphi(x_1 + x_2)\\
&y_2 = \varphi(x_1 + x_2) + c
\end{aligned}$$

where φ is an arbitrary smooth function of one variable and c a constant.

The latter example proves that the converse of the above theorem is not true in general: there may exist solutions of $\mathcal{R}^*$ which project to M with dimension equal to m and they are not solutions of $\mathcal{R}$.

The PDE systems considered by Lie in [11] are those for which it is possible to 'solve the parametric derivatives' in $\mathcal{R}$ from the equations of the correspondence. This condition becomes that its associated first order system $\mathcal{R}^*$ parametrizes the correspondence $\mathcal{R}_{m,*}$. So, we give the following

Definition 3.10. A system $\mathcal{R}$ is a *Lie system* when the projection of $\mathcal{R}_{m,*}$ over $\mathcal{R}^*$ is an isomorphism.

Examples 3.11. (1) For each m–dimensional submanifold $X \subseteq M$, $J^1_m X$ is a Lie system because $J^1_m X \simeq X$.

(2) The system in the example (1) above is a Lie system. The same remains being valid for every system $\mathcal{R} \subseteq J^1_m M$ whose symbol is equal to zero, since locally $\mathcal{R} \simeq M$.

(3) The system in the example (2) above is a Lie system wherever $q_1 + q_2 \neq 0$.

Remark 3.12. For a Lie system, each $\mathfrak{p}^1_{n-1}$ contained (as an ideal) in a $\mathfrak{p}^1_m \in \mathcal{R}$, is in only one of them. In other words: each contact element $H_p \in J^1_{n-1}M$ that contains an m-dimensional $P_p \subseteq T_pM$ determining a jet in $\mathcal{R}$, contains only one of them.

When $\mathcal{R}$ is a Lie system, the composition of $\mathcal{R}^* \approx \mathcal{R}_{m,*}$ with the projection $\mathcal{R}_{m,*} \longrightarrow \mathcal{R}$ gives a parametrization, λ, of $\mathcal{R}$ by its first order associated system $\mathcal{R}^*$:

$$\begin{tikzcd} & \mathcal{R}_{m,*} \arrow[dl, "\approx"'] \arrow[dr] & \\ \mathcal{R}^* \arrow[rr, "\lambda"] & & \mathcal{R} \end{tikzcd}$$

The situation is as follows: we have a first order system $\mathcal{R}^*$ and a smooth map $\lambda : \mathcal{R}^* \longrightarrow \mathcal{R}$ such that for each $\alpha_p \in \mathcal{R}^*$, $\alpha_p \in \lambda(\alpha_p)/\mathfrak{m}^2_p$; by Proposition 3.4 this is equivalent to the condition $\theta \in \lambda^*(\Omega^1_m)$, θ being the specialization to $\mathcal{R}^*$ of the canonical 1-form in T^*M.

When $\mathcal{R}$ is a Lie system, the isomorphism $\mathcal{R}^* \simeq \mathcal{R}_{m,*}$ yields a relationship between the dimension g of the symbol of $\mathcal{R}$(=tangent space to the fibers of the projection $\mathcal{R} \longrightarrow M$), its number m of independent variables and the number of independent equations of $\mathcal{R}^*$.

In the above notation, we assume that the rank of the projection $\mathcal{R} \longrightarrow M$ is the highest one possible ($= n$) at all the points in $\mathcal{R}$. Hence, we can solve (locally) from the equations of the system s of the $(n-m)m$ coordinates $y_{1,1}, \dots, y_{n-m,m}$ as functions of the remaining ones and $x_1, \dots, x_m, y_1, \dots, y_{n-m}$. If g denotes the dimension of the symbol of $\mathcal{R}$, then $s = (n-m)m - g$ and hence, since $\dim(J^1_mM \times_M T^*) = 2n + (n-m)m$, we have

$$\dim \mathcal{R}_{m,*} = 2n + (n-m)m - m - (n-m)m + g = 2n - m + g.$$

Let r be the number of independent equations of $\mathcal{R}^*$ in T^*M. Since $\dim \mathcal{R}^* = \dim \mathcal{R}_{m,*}$, we obtain $m - g = r$, and since $r \geq 0$, then $m \geq g$. Note that when $m = g+1$, the first order associated system $\mathcal{R}^*$ is a single partial differential equation.

Proposition 3.13. *Let $\mathcal{R} \subseteq J^1_mM$ be a first order PDE system with m independent variables and $n-m$ unknown functions and let g be the dimension of its symbol. If $\mathcal{R}$ is a Lie system, its associated system $\mathcal{R}^*$ has $m-g$ equations. Hence, $m \geq g$.*

On the other hand, the characteristic vector fields are known to play an important role in the integration of first order systems with only one unknown function. In the remainder of this section we relate the characteristic systems of $\mathcal{R}$ and $\mathcal{R}^*$.

In the above notation, let $\mathcal{R} \subseteq J^1_mM$ be a Lie system, which we may assume written locally in the form:

$$y_{j,k} - F_{j,k}(x_i, y_j, y_{h,\ell}) = 0;$$

the couples (j,k) corresponding to the derivatives which we can solve run through a set I of indexes of lenght s. Let us denote by J the set of pairs (h,ℓ) corresponding to the remaining (parametric) derivatives. The functions $x_1, \ldots, x_m, y_1, \ldots, y_{n-m}$ together with the g parametric derivatives are local coordinates in $\mathcal{R}$.

The local equations of $\mathcal{R}_{m,*}$ are (see above):

$$\begin{aligned} p_i + \sum_{j=1}^{n-m} q_j y_{j,i} &= 0 \qquad (1 \le i \le m) \\ y_{j,k} - F_{j,k} &= 0 \qquad ((j,k) \in I) \end{aligned} \tag{12}$$

To simplify the calculations below it will be convenient to take as local coordinates in $\mathcal{R}^*(\simeq \mathcal{R}_{m,*})$ the functions $x_i, y_j, y_{h,\ell}, q_j$ $(1 \le i \le m, 1 \le j \le n-m, (h,\ell) \in J)$, solving p_i $(1 \le i \le m)$ from (12). In these local coordinates the equations of λ are

$$\begin{aligned} \lambda : \mathcal{R}^* &\longrightarrow \mathcal{R} \\ (x_i, y_j, y_{h,\ell}, q_j) &\longrightarrow (x_i, y_j, y_{h,\ell}) \end{aligned} \tag{13}$$

Let θ be the canonical 1-form in T^*M specialized to $\mathcal{R}^*$ and let Ω be the contact system in $\mathcal{R}$. Ω is spanned by the 1-forms

$$\omega_j = dy_j - \sum_{i=1}^{m} y_{j,i}\, dx_i \qquad (1 \le j \le n-m),$$

where $y_{j,i}$ is replaced by $F_{j,i}$ for $(j,i) \in I$.

We have:

$$\begin{aligned} \theta &= \sum_{i=1}^{m} p_i\ dx_i + \sum_{j=1}^{n-m} q_j\ dy_j \\ &= \sum_{\substack{1\le j\le n-m \\ 1\le i\le m \\ (j,i)\in J}} (-q_j\ y_{j,i})\ dx_i + \sum_{\substack{1\le j\le n-m \\ 1\le i\le m \\ (j,i)\in I}} (-q_j\ F_{j,i})\ dx_i + \sum_{j=1}^{n-m} q_j\ dy_j = \sum_{j=1}^{n-m} q_j\ \omega_j, \end{aligned}$$

where the right hand side in the first line is understood restricted to $\mathcal{R}^*$.
And its differential:

$$d\theta = \sum_{j=1}^{n-m} dq_j \wedge \omega_j + \sum_{j=1}^{n-m} q_j\ d\omega_j$$

Now, let us take a vector field D in the characteristic system of Ω and let $\overline{D}$ be a vector field in the characteristic system of $\lambda^*\Omega$ which projects onto it. Then,

$$i_{\overline{D}} d\theta = \sum_{j=1}^{n-m} \overline{D}(q_j)\ \omega_j - \sum_{j=1}^{n-m} \omega_j(\overline{D})\ dq_j + \sum_{j=1}^{n-m} q_j\ i_{\overline{D}} d\omega_j$$

Since $\omega_j(\overline{D}) = 0$ and $i_{\overline{D}} d\omega_j \in \lambda^*\Omega$ $(1 \le j \le n-m)$, $i_{\overline{D}} d\theta \in \lambda^*\Omega$. On the other hand, for each vector field D^v tangent to $\mathcal{R}^*$ and vertical for the projection λ we have

$$i_{D^v} d\theta = \sum_{j=1}^{n-m} D^v(q_j)\ \omega_j$$

Thus, by adding up to $\overline{D}$ a suitable vertical vector field we obtain a vector field $D^* \in \operatorname{rad} d\theta$ which projects by λ_* onto D.

We summarize the discussion above in

Proposition 3.14. *Let $\mathcal{R} \subseteq J^1_m M$ be a Lie system. Let Ω be the contact system in $J^1_m M$ specialized to $\mathcal{R}$ and let θ be the canonical 1–form in T^*M specialized to $\mathcal{R}^*$. Then, for each vector field D in the characteristic system of Ω there exists another one, D^*, in rad $d\theta$ that projects onto it.*

If $\mathcal{R}$ is a Lie system, the number r of independent equations for $\mathcal{R}^*$ is $m-g$ (see Proposition 3.13) and consequently the dimension of rad $d\theta$ is at most $m-g$. It follows immediately

Corollary 3.15. *Let $\mathcal{R} \subseteq J^1_m M$ be a Lie system and let g be the dimension of its symbol. The dimension of the characteristic system of the contact one in $\mathcal{R}$ is at most $m-g$; if the equality holds, $\mathcal{R}^*$ is an involutive system and its characteristic system projects onto that of $\mathcal{R}$.*

Remark 3.16. The above corollary gives a condition for the dimension g of the symbol of a Lie system $\mathcal{R}$ in order to expect the existence of Cauchy characteristic vector fields: $g < m$(=number of independent variables). As far as we know this result does not seem to be in the literature. Note that the condition for a system to be a Lie one occurs often in the practice which makes this result valid for a wide class of PDE systems.

Example 3.17. Let us consider again the example (2) in 3.9. Recall that the associated first order system $\mathcal{R}^*$ is in this case the single linear equation $p_1 - p_2 = 0$. Its characteristic system is spanned by the hamiltonian vector field

$$D^* = D_{p_1-p_2} = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$$

As a consequence of the linearity of $p_1 - p_2 = 0$, D^* is the lift to $T^*\mathbb{R}^4$ of the vector field $\overline{D}$ in $\mathbb{R}^4$ whose expression in local coordinates is $\overline{D} = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$. The classical solutions of $\mathcal{R}^*$ are computed easily: they are first integrals of $\overline{D}$, i.e., they are $V(x_1 + x_2, y_1, y_2)$, where V is an arbitrary smooth function.

From Proposition 3.14 we have that the dimension of the characteristic system of $\mathcal{R}$ is at the most 1. The candidate to span it is the vector field $D = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ tangent to $\mathcal{R}$, projection of D^* to $\mathcal{R}$. It is easily checked that D is in fact a characteristic vector field.

In this particular case $\mathcal{R}^*$ agrees with the system whose solutions are the first integrals of the projection to $\mathbb{R}^4$ of the characteristic system of $\mathcal{R}$. Observe that the solutions $V \in C^\infty(\mathbb{R}^4)$ of $\mathcal{R}^*$ satisfy the following property: they are functions such that for every constant c, $V = c$ can be foliated by solutions of $\mathcal{R}$. In the terminology of the next section V is an *intermediate integral* of order 0 of $\mathcal{R}$.

4. Higher order PDE systems. Intermediate integrals.

The theory of the Lie correspondences in the way we have dealt with it is applied to PDE systems with any number of independent variables and unknown functions, but only to first order systems. However, each PDE system can be written in this form due to the natural inclusions $J^\ell_m M \subseteq J^1_m(J^{\ell-1}_m M)$; each system $\mathcal{R} \subseteq J^\ell_m M$

must be considered as a submanifold of $J^1_m(J^{\ell-1}_m M)$, and the base–manifold for the Lie correspondence is $J^{\ell-1}_m M$. Accordingly, $\mathcal{R}^*$ is a subset of $T^*J^{\ell-1}_m M$.

Let $\mathcal{R} \subseteq J^\ell_m M$ be a Lie system such that $\mathcal{R} \longrightarrow J^{\ell-1}_m M$ is onto. We have the following commutative diagram:

$$\begin{tikzcd}
& \mathcal{R}_{m,*} \arrow[dl, "\approx"'] \arrow[dr] & \\
T^*J^{\ell-1}_m M \supseteq \mathcal{R}^* \arrow[rr, "\lambda"] \arrow[dr] & & \mathcal{R} \arrow[dl] \\
& J^{\ell-1}_m M &
\end{tikzcd}$$

We shall denote by θ the canonical 1–form in $T^*J^{\ell-1}_m M$ specialized to $\mathcal{R}^*$ and by Ω the contact system in $J^1_m(J^\ell_m M)$ specialized to $\mathcal{R}$. From Proposition 3.4 it follows that

$$\theta \in \lambda^*(\Omega). \tag{14}$$

A classical solution of $\mathcal{R}^*$ is an exact 1–form in $J^{\ell-1}_m M$ which values in $\mathcal{R}^*$ as a section of $T^*J^{\ell-1}_m M \longrightarrow J^{\ell-1}_m M$. Given a solution dV of $\mathcal{R}^*$, λ transports the section dV to a section $\sigma = \lambda \circ dV : J^{\ell-1}_m M \longrightarrow \mathcal{R}$:

$$\begin{tikzcd}
\mathcal{R}^* \arrow[rr, "\lambda"] \arrow[dr] & & \mathcal{R} \arrow[dl] \\
& J^{\ell-1}_m M \arrow[ul, bend left, "dV"] \arrow[ur, bend right, "\sigma"'] &
\end{tikzcd}$$

Since $(dV)^*\theta = dV$, (14) implies that

$$dV = (dV)^*\theta \in (dV)^*\lambda^*\Omega = \sigma^*(\Omega),$$

which gives that V is a first integral of the distribution of tangent vector fields associated with $\sigma^*(\Omega)$. We have thus proved:

Proposition 4.1. *Let $\mathcal{R} \subseteq J^\ell_m M$ be a Lie system such that $\mathcal{R} \longrightarrow J^{\ell-1}_m M$ is onto. Then, for each solution dV of $\mathcal{R}^*$, V is a first integral of $(\sigma^*\Omega)^\perp$, where $\sigma = \lambda \circ dV$ and λ is the projection $\mathcal{R}^* \longrightarrow \mathcal{R}$.*

Next we study the relationship between the intermediate integrals of a given system and the solutions of its associated first order one.

Definition 4.2. Let $\mathcal{R} \subseteq J^\ell_m M$ be a system of partial differential equations of order ℓ. An *intermediate integral of order* $\ell - 1$ of $\mathcal{R}$ is a hypersurface $\mathcal{F} \subseteq J^{\ell-1}_m M$ (a single PDE of order $\ell - 1$) that admits a complete integral formed by common solutions with $\mathcal{R}$.

Lemma 4.3. *Let $\mathcal{F} \subseteq J^{\ell-1}_m M$ be an intermediate integral of order $\ell - 1$ of $\mathcal{R} \subseteq J^\ell_m M$ whose local equation is $F = 0$. Then, for each $\mathfrak{p}^{\ell-1}_m \in \mathcal{F}$, $d_{\mathfrak{p}^{\ell-1}_m} F \in \mathcal{R}^*$.*

Proof. The necessary and sufficient condition for $d_{\mathfrak{p}_m^{\ell-1}}F \in \mathcal{R}^*$ is that there exists $\mathfrak{p}_m^\ell \in \mathcal{R}$ in the fibre of $\mathfrak{p}_m^{\ell-1}$ such that $(\mathfrak{p}_m^\ell, d_{\mathfrak{p}_m^{\ell-1}}F) \in \bigwedge_{m,*}(J_m^{\ell-1}M)$; that is, $d_{\mathfrak{p}_m^{\ell-1}}F \in \mathfrak{p}_m^\ell \Big/ \mathfrak{m}^2_{\mathfrak{p}_m^{\ell-1}}$ $(\mathfrak{p}_m^\ell \in J_m^1(J_m^{\ell-1}M))$.

Since $\mathcal{F}$ is an intermediate integral of $\mathcal{R}$, it has a complete integral formed by common solutions with $\mathcal{R}$. Hence, for each $\mathfrak{p}_m^{\ell-1} \in \mathcal{F}$ there exists $X \subseteq M$, a solution of $\mathcal{R}$, such that $\mathfrak{p}_m^{\ell-1} \in J_m^{\ell-1}X \subseteq \mathcal{F}$. Therefore, $F \in I(J_m^{\ell-1}X)$, where $I(J_m^{\ell-1}X)$ is the ideal of $J_m^{\ell-1}X$ in $C^\infty(J_m^{\ell-1}M)$.

On the other hand, each $\mathfrak{p}_m^\ell \in J_m^{\ell-1}X \subseteq \mathcal{R}$ in the fibre of $\mathfrak{p}_m^{\ell-1}$ is of the form (as an ideal of $C^\infty(J_m^{\ell-1}M)$) $\mathfrak{p}_m^\ell = I(J_m^{\ell-1}X) + \mathfrak{m}^2_{\mathfrak{p}_m^{\ell-1}}$. Thus, $F \in \mathfrak{p}_m^\ell$, which is our assertion. ■

The above lemma gives us more. Namely, if $\mathcal{F}_c$ is a local fibration of $J_m^{\ell-1}M$ by intermediate integrals of $\mathcal{R}$, we obtain a local section of $T^*J_m^{\ell-1}M \longrightarrow J_m^{\ell-1}M$ valued in $\mathcal{R}^*$ in the following way: if $F = c$, c being a parameter and $F \in C^\infty(J_m^{\ell-1}M)$, is the local equation of $\mathcal{F}_c$, dF is such a section. Since these sections agree with the solutions of $\mathcal{R}^*$, we have thus proved:

Theorem 4.4. *Let $\mathcal{R} \subseteq J_m^{\ell-1}M$ be a PDE system. Each local fibration $\{\mathcal{F}_c\}$ of $J_m^{\ell-1}M$ by intermediate integrals of order $\ell-1$ of $\mathcal{R}$ gives a (local) solution of its associated first order system. Consequently, the (fibrations of $J_m^{\ell-1}M$ by) intermediate integrals of $\mathcal{R}$ of order $\ell-1$, when they exist, are among the solutions of $\mathcal{R}^*$.*

Remark 4.5. In the next section we shall prove in some particular cases that the (classical) solutions of $\mathcal{R}^*$ agree exactly with the (fibrations of $JMLLV$ by) intermediate integrals of $\mathcal{R}$ of order $\ell-1$.

5. Examples

A. PDE systems with symbol equal to zero.

PDE systems whose symbol equals zero were studied by Lie [10, pag. 171-183], who characterized those PDE systems whose solutions depend only on arbitrary constants. Is is known that the contact system specialized to such an involutive system is a completely integrable Pfaff system (see [10] for instance). Consequently, Frobenius's theorem allows us to integrate it by means of ordinary differential equations.

We shall now apply the theory of the Lie correspondences developed in §3 and §4. For each PDE system $\mathcal{R}$ as above we prove that the first integrals of the contact system specialized to $\mathcal{R}$ agree with the solutions of $\mathcal{R}^*$ and also with the intermediate integrals of $\mathcal{R}$.

Let $\mathcal{R} \subseteq J_m^\ell M$ be a PDE system whose symbol equals zero. As usual, the projection $\mathcal{R} \longrightarrow J_m^\ell M$ is assumed to be onto, and hence $\mathcal{R} \longrightarrow J_m^\ell M$ is a local isomorphism.

$\mathcal{R}$ can be considered as a first order system via the canonical immersion $J_m^\ell M \subseteq J_m^1(J_m^{\ell-1}M)$. Therefore, the base–manifold for the Lie correspondence is $J_m^{\ell-1}M$. As a consequence of the local isomorphism $\mathcal{R} \simeq J_m^{\ell-1}$, one has that $\mathcal{R}_{m,*} \subseteq \mathcal{R} \times_{J_m^{\ell-1}M} T^*J_m^{\ell-1}M$ is locally isomorphic to $\mathcal{R}^*$; that is, $\mathcal{R}$ is a Lie system.

Hence, we have the following commutative diagram:

$$\begin{tikzcd} & \mathcal{R}_{m,*} \arrow[dl, "\approx"'] \arrow[dr] & \\ T^*(J_m^{\ell-1}M) \supseteq \mathcal{R}^* \arrow[rr, "\lambda"] \arrow[dr] & & \mathcal{R} \arrow[dl, "\approx"] \\ & J_m^{\ell-1}M & \end{tikzcd}$$

We shall denote by θ the canonical 1-form in $T^*J_m^{\ell-1}M$ specialized to $\mathcal{R}^*$, and by Ω the contact system in $J_m^\ell M$ specialized to $\mathcal{R}$. Since Ω is a Pfaff system completely integrable, from Proposition 3.14 and its corollary it follows that $\mathcal{R}^*$ is an involutive system and that rad $d\theta$ projects onto $\Omega^\perp$ by λ_*.

On the other hand, Proposition 4.1 shows that for each solution dV of $\mathcal{R}^*$, V is a first integral of Ω (taken to $J_m^{\ell-1}M$ by the isomorphism $\mathcal{R} \longrightarrow J_m^{\ell-1}M$). Consequently, the family $\{V = c\}$, where c is a constant, is a (local) fibration of $J_m^{\ell-1}M$ by intermediate integrals of $\mathcal{R}$. Conversely, if $V = c$ is the local equation of such a fibration, from Theorem 4.4 it follows that dV is a solution of $\mathcal{R}^*$. We have thus proved:

Theorem 5.1. *Let $\mathcal{R} \subseteq J_m^\ell M$ be a formally compatible PDE system whose symbol equals zero at all the points in $\mathcal{R}$ and let Ω be the contact system in $J_m^\ell M$ specialized to $\mathcal{R}$. Then the following assertions hold:*

1. *$\mathcal{R}$ is a Lie system.*
2. *$\mathcal{R}^*$ is an involutive system.*
3. *The characteristic system of $\mathcal{R}^*$ projects onto that of Ω.*
4. *If $V \in C^\infty(J_m^{\ell-1}M)$ is a first integral of Ω (taken to $J_m^{\ell-1}M$ by the isomorphism $\mathcal{R} \simeq J_m^{\ell-1}M$), dV is a solution of $\mathcal{R}^*$, and conversely. Furthermore, $\{V = c\}$, where c is a constant, is a (local) fibration by intermediate integrals (of order $\ell - 1$) of $\mathcal{R}$.*

The theorem shows that the integration of the system $\mathcal{R}$ is equivalent to that of its associated first order system $\mathcal{R}^*$. As a matter of fact $\mathcal{R}^*$ is the *PDE* system whose solutions are the first integrals of Ω and the reduction of $\mathcal{R}$ to ODE's is made via $\mathcal{R}^*$.

Computation in local coordinates. Let us take local coordinates x_i, y_j $(1 \le i \le m,\ 1 \le j \le n-m)$; these and the derivatives $y_{j,\alpha}$ $(1 \le j \le n-m,\ 1 \le \alpha \le \ell)$ in $J_m^\ell M$. We can assume that the local equations of $\mathcal{R}$ are:

$$y_{j,\alpha} = F_{j,\alpha}(x_i, y_{r,\beta}), \qquad (1 \le j \le n-m, |\alpha| = \ell), \tag{15}$$

where $F_{j,\alpha}(x_i, y_{r,\beta}) \in C^\infty(J_m^{\ell-1}M)$.

The contact system restricted to $\mathcal{R}$ is spanned by

$$\omega_{j,\alpha} = dy_{j,\alpha} - \sum_{k=1}^m y_{j,\alpha+1_k}\, dx_k, \qquad (1 \le j \le n-m, |\alpha| \le \ell - 2)$$

$$\omega_{j,\alpha} = dy_{j,\alpha} - \sum_{k=1}^m F_{j,\alpha+1_k}\, dx_k, \qquad (1 \le j \le n-m, |\alpha| = \ell - 1)$$

and the distribution of vector fields tangent to $\mathcal{R}$ annihilating Ω is spanned by

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\substack{|\alpha|\le \ell-2\\ 1\le j\le n-m}} y_{j,\alpha+1_i}\frac{\partial}{\partial y_{j,\alpha}} + \sum_{\substack{|\alpha|=\ell-1\\ 1\le j\le n-m}} F_{j,\alpha+1_i}\frac{\partial}{\partial y_{j,\alpha}}, \quad (1\le i\le m) \tag{16}$$

Their Lie brackets are

$$[D_i, D_s] = \sum_{\substack{|\alpha|=\ell-1\\ 1\le j\le n-m}} \left(D_i(F_{j,\alpha+1_s}) - D_s(F_{j,\alpha+1_i})\right) \quad \frac{\partial}{\partial y_{j,\alpha}}, \qquad (1\le i,s\le m).$$

Hence, the conditions for the vector fields D_i $(1\le i\le m)$ to span an involutive distribution are the compatibility conditions for the prolongation of $\mathcal{R}$ to $J^{\ell+1}_m M$.

Maintaining the notation for the local coordinates in $J^{\ell-1}_m M$ and taking $x_i, y_{j,\alpha}$, $(1\le i\le m, 1\le j\le n-m, |\alpha|\le \ell-1)$ and the 'conjugated' ones $p_i, q_{j,\alpha}$, $(1\le i\le m, 1\le j\le n-m, |\alpha|\le \ell-1)$ as coordinates in T^*JMLLV, the local equations of the Lie correspondence $\bigwedge_{m,*}(J^{\ell-1}_m M)$ (restricted to $J^{\ell}_m M$) as submanifold of $J^{\ell}_m \times_{J^{\ell-1}_m M} T^*(J^{\ell-1}_m M)$ are (see §3):

$$p_i + \sum_{|\alpha|\le\ell-1}\sum_{k=1}^{n-m} q_{k,\alpha}\; y_{k,\alpha+1_i} = 0, \qquad (i=1,\dots,m) \tag{17}$$

The local equations of the first order associated system $\mathcal{R}^*\subseteq T^*J^{\ell-1}_m M$ are obtained by eliminating $y_{j,\alpha}$, $(1\le j\le n-m, |\alpha|=\ell)$ from (17) and (15). Thus, $\mathcal{R}^*$ is given by:

$$p_i + \sum_{\substack{|\alpha|\le\ell-2\\ 1\le k\le n-m}} q_{k,\alpha}\; y_{k,\alpha+1_i} + \sum_{\substack{|\alpha|=\ell-1\\ 1\le k\le n-m}} q_{k,\alpha}\; F_{k,\alpha+1_i} = 0, \quad (i=1,\dots,m) \tag{18}$$

The involutive distribution $\mathcal{L}$ of vector fields spanned by the vector fields D_i $(1\le i\le m)$ gives, by isomorphism, another one $\overline{\mathcal{L}}$ in $J^{\ell-1}_m M$. Its generators, $\overline{D}_i$ $(1\le i\le m)$, have in the coordinates $x_i, y_{j,\alpha}$, $(1\le i\le m, 1\le j\le n-m, |\alpha|\le \ell-1)$ the same expressions (Equations (16)) that the vector fields D_i. Note that the characteristic system of $\mathcal{R}^*$ is spanned by the lift to T^*JMLLV of the vector fields D_i $(1\le i\le m)$.

The simple inspection of (16) and (18) gives that $\mathcal{R}^*$ is the PDE system whose solutions $V\in C^\infty(J^{\ell-1}_m M)$ are the first integrals of $\mathcal{L}$: the integration of $\mathcal{R}$ and that of $\mathcal{R}^*$ are equivalent problems.

B. Systems of two second order partial differential equations with two independent variables and one unknown function.

These are ones of the best–studied systems in the classical literature. They have been studied by many mathematicians, among them Goursat [5], Darboux, Cartan [4], Lie [11] and more recently by Kakié [7, 8]. It is known that such an involutive system is integrable by a method that is a generalization of that of the Cauchy characteristics for a single partial differential equation of first order, due to the existence of characteristic vector fields (see [5] for instance). We shall resume some known results about this kind of systems (see [6, 17]), such as the existence of enough involutive distributions of vector fields tangent to them and annihilating the contact system to obtain all their solutions. Next, by applying the theory of the Lie correspondences we show that the intermediate integrals of such a system agree with the solutions of its associated first order system, which leads to the fundamental result: the integration of both systems is equivalent. The computations will be made in local coordinates and the statements and results are local.

A system of two second order partial differential equations in two independent variables and one unknown function is a 6–dimensional locally closed submanifold $\mathcal{R}$ of J_2^2M, where $\dim M = 3$. Moreover, the projection $\mathcal{R} \longrightarrow J_2^1 1M$ is assumed to have the highest rank at all the points of $\mathcal{R}$.

Let us take local coordinates x, y, z in M; x, y, z, p, q in $JJ21M$; and x, y, z, p, q, r, s, t in $JJ22M$. Thus, the local equations of the prolongation of the submanifold $z = Z(x,y)$ of M to $JJ22M$ are $z = Z(x,y)$, $p = \frac{\partial Z}{\partial x}$, $q = \frac{\partial Z}{\partial y}$, $r = \frac{\partial^2 Z}{\partial x^2}$, $s = \frac{\partial^2 Z}{\partial x \partial y}$, and $t = \frac{\partial^2 Z}{\partial y^2}$.

Without any loss of generality (see [4, 5, 11]) we can assume the local equations of $\mathcal{R}$ to be:

$$\begin{aligned} r + R(x,y,z,p,q,s) &= 0 \\ t + T(x,y,z,p,q,s) &= 0 \end{aligned} \tag{19}$$

where $R, T \in C^\infty(\mathcal{R})$. The functions x, y, z, p, q, s are local coordinates in $\mathcal{R}$.

The contact system, $\Omega(\mathcal{R})$, the specialization to (19) of $\Omega(JJ22M)$, is spanned by the 1–forms:

$$\begin{aligned} \omega_z &= dz - p\ dx - q\ dy \\ \omega_p &= dp + R\ dx - s\ dy \\ \omega_q &= dq - s\ dx + T\ dy \end{aligned} \tag{20}$$

A basis of the vector fields tangent to $\mathcal{R}$ annihilating the contact system is:

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - R\frac{\partial}{\partial p} + s\frac{\partial}{\partial q} \\ D_y &= \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + s\frac{\partial}{\partial p} - T\frac{\partial}{\partial q} \\ &\frac{\partial}{\partial s} \qquad \text{(which spans the symbol of } \mathcal{R}) \end{aligned} \tag{21}$$

Computations in local coordinates give that the necessary and sufficient condition for the contact system $\Omega(\mathcal{R})$ to have the characteristic system different from zero is that the equations

$$R_s\, T_s - 1 = 0 \tag{22}$$

$$D_y R - R_s\, D_x T = 0 \tag{23}$$

hold, where the subscript s denotes the partial derivative with respect to s. In this case, the characteristic system of $\Omega(\mathcal{R}_2^2)$ is spanned by

$$D = D_x + R_s D_y - D_y(R)\frac{\partial}{\partial s} \tag{24}$$

(see [17, page 104-105] for a detailed discussion).

The first prolongation $\mathcal{R}^{(1)}$ of $\mathcal{R}$ is defined by Equations (19) together with a system of four equations linear in the third order derivatives; the necessary and sufficient condition for this linear system to be undetermined is (22), and in this case (23) is its compatibility condition. Therefore the characteristic system of $\Omega(\mathcal{R})$ is different from zero if and only if the projection $\mathcal{R}^{(1)} \longrightarrow \mathcal{R}$ is an affine fibred bundle whose fibre has dimension 1.

Remark 5.2. When the prolongation to fourth order is made, the linear system that gives the fibre of $\mathcal{R}^{(2)} \longrightarrow \mathcal{R}^{(1)}$ has matrix of coefficients of rank 4. From this, it follows that the symbol of $\mathcal{R}^{(2)}$ (tangent space to fibres of the projection in $\mathcal{R}^{(1)}$) has dimension ≤ 1 in all its points. The same occurs for all the prolongations $\mathcal{R}^{(r)}$ ($r \geq 0$). Therefore, the above results can be formulated in terms of the terminology of the theory of formal integrability: If the system $\mathcal{R}$ written in the canonical form (19) satisfies (22), (23), then it is involutive in the sense of Spencer-Goldschmidt-Kuranishi.

Following some indications by Lie in [11], we wondered if there are enough involutive distributions of vector fields tangent to $\mathcal{R}$ annihilating the contact system to obtain all its solutions (see also [17]). When $\mathcal{R}$ is involutive, computations in local coordinates show that there are infinite involutive distributions of vector fields, that are determined by solving a first order single partial differential equation with 6 independent variables; the characteristic system of $\Omega(\mathcal{R})$ is the intersection of all these distributions. Otherwise, $\mathcal{R}^{(1)} \simeq \mathcal{R}$, that is to say, the symbol of $\mathcal{R}^{(1)}$ equals zero, like the systems studied in the former example.

From now on we assume that $\mathcal{R}$ is involutive. Let us take a section σ of $\mathcal{R} \longrightarrow J^1_2 M$ tangent to an involutive distribution $\mathcal{L}$ of vector fields as above; σ takes $\mathcal{L}$ to an involutive distribution $\overline{\mathcal{L}}$ of vector fields in $J^1_2 1M$ annihilating the contact system in $J^1_2 M$, whose solutions are (prolongations of) solutions of $\mathcal{R}$. Hence, any hypersurface $\mathcal{F} \subseteq J^1_2 M$ tangent to $\overline{\mathcal{L}}$ has a biparametric family of common solutions with $\mathcal{R}$: $\mathcal{F}$ is an intermediate integral of $\mathcal{R}$. Furthermore, each section σ determines infinite intermediate integrals of $\mathcal{R}$ in the following way: if $V \in C^\infty(J^1_2 M)$ is a first integral of $\overline{\mathcal{L}}$, for each constant c, $V = c$ is an intermediate integral (of first order) of $\mathcal{R}$. Moreover, $V = c$ is a (local) fibration of $J^1_2 M$ by intermediate integrals.

Since each solution X of $\mathcal{R}$ is a solution of an involutive distribution $\mathcal{L}$, its prolongation to $J^2_2 M$ is contained (locally) in the image of a section σ tangent to $\mathcal{L}$. Thus, from the above X can be deduced to be a solution of an intermediate integral of $\mathcal{R}$.

On the other hand, an easy computation in local coordinates allows one to characterize a section σ as above by the condition of tangency to the characteristic system of $\Omega(\mathcal{R})$. For later references, we summarize the above discussion in

Proposition 5.3. *Let $\mathcal{R} \subseteq J^2_2 M$ be an involutive system. The following assertions hold:*

1. *A section σ of $\mathcal{R} \longrightarrow J^1_2 1M$ is tangent to an involutive distribution of rank two of vector fields tangent to $\mathcal{R}$ annihilating the contact system precisely if* $\operatorname{Im} \sigma$ *is tangent to the characteristic system of the contact one in $\mathcal{R}$.*

2. *Each section σ as above gives infinite fibrations of J_2^1M by intermediate integrals of $\mathcal{R}$ in the following way: if σ is tangent to an involutive distribution $\mathcal{L}$, for each first integral V of $\mathcal{L}$ (transported to J_2^1M by σ), $\{V = c\}$, c runs through $\mathbb{R}$, is such a fibration.*

3. *Each solution X of $\mathcal{R}$ is a solution of an intermediate integral of $\mathcal{R}$.*

In the remainder of this section we shall apply the theory of the Lie correspondences to $\mathcal{R}$, which is assumed to be a Lie system. Let us consider $\mathcal{R}$ as a submanifold of $J_2^1(J_2^1M)$ and let us take J_2^1M as the base–manifold for the Lie correspondence. We thus have,

$$\begin{tikzcd} & \mathcal{R}_{2,*} \arrow[dl, "\approx"'] \arrow[dr] & \\ T^*J_2^1M \supseteq \mathcal{R}^* \arrow[rr, "\lambda"] \arrow[dr] & & \mathcal{R} \arrow[dl] \\ & J_2^1M & \end{tikzcd}$$

We shall denote by θ the canonical 1-form in $T^*J_2^1M$ specialized to $\mathcal{R}^*$. Note that $\mathcal{R}^*$ is a single PDE equation (see Proposition 3.13).

Theorem 5.4. *Let $\mathcal{R} \subseteq J_2 2M$ be an involutive Lie system. If $\{\mathcal{F}_c\}$ is a local fibration of J_2^1M by intermediate integrals of $\mathcal{R}$, whose local equation is $F = c$, with c constant and $F \in C^\infty(J_2^1M)$, then dF is a solution of $\mathcal{R}^*$, and, conversely, given dF a (local) solution of $\mathcal{R}^*$, $F = c$, with c constant, is such a fibration.*

Accordingly, solutions of $\mathcal{R}^*$ agree with (fibrations of J_2^1M by) intermediate integrals of $\mathcal{R}$.

Proof. The first assertion follows immediately from Theorem 4.4. Let us prove the converse. In the notation above, let dV be a solution of $\mathcal{R}^*$. By Propositions 4.1 and 5.3 it is sufficient to prove that $\sigma = \lambda \circ dV$ is a section of $\mathcal{R} \longrightarrow J_2^1M$ tangent to an involutive distribution of vector fields of rank 2 annihilating the contact system in $\mathcal{R}$. Since $\mathcal{R}^*$ is a single partial differential equation and since the dimension of the characteristic system of $\Omega(\mathcal{R})$ equals 1, from Corollary 3.15 it follows that $\mathcal{R}^*$ is involutive and that rad $d\theta$ (whose dimension equals 1) projects onto the characteristic system of $\Omega(\mathcal{R})$. Hence, dV being tangent to rad $d\theta$, Im σ is tangent to the characteristic system of $\Omega(\mathcal{R})$, which concludes the proof (see the proposition above). ■

Taking into account that each solution of $\mathcal{R}$ is a solution of an intermediate integral, we can formulate our main result for this kind of systems.

Theorem 5.5. *Let $\mathcal{R} \subseteq J_2^2M$ be an involutive Lie system. Each solution X of $\mathcal{R}$ is a solution of a solution of $\mathcal{R}^*$; conversely, each solution of $\mathcal{R}^*$ admits a complete integral formed by solutions common with $\mathcal{R}$. Moreover, each solution X of $\mathcal{R}$ is obtained as an intersection of solutions of $\mathcal{R}^*$.*

Remark 5.6. The results obtained for involutive systems $\mathcal{R}$ can be generalized to PDE involutive systems of arbitrary order in two independent variables and any number of unknown functions whose symbol has dimension equal to 1, which, in some way, shows that the above calculations and results essentially depend only on the number of the independent variables of the original system and the dimension of its symbol.

Calculus of the associated system in local coordinates.

The system (19) can be considered as a first order system $\mathcal{R} \subseteq J_2^1(J_2^1 M)$; let us take local coordinates x, y, z, p, q in $J_2^1 M$ (they will be denoted by x_1, x_2, y_1, y_2, y_3 from now on), and the corresponding ones $x_1, x_2, y_1, y_2, y_3, y_{1,1}, \ldots, y_{3,2}$ in $J_2^1(J_2^1 M)$. The equations of $\mathcal{R}$ are

$$\begin{aligned} y_{2,1} + R(x_1, x_2, y_1, y_2, y_3, y_{2,2}) &= 0 \\ y_{3,2} + T(x_1, x_2, y_1, y_2, y_3, y_{2,2}) &= 0 \\ y_{1,1} &= y_2 \\ y_{1,2} &= y_3 \\ y_{3,1} &= y_{2,2} \end{aligned} \tag{25}$$

We take now local coordinates $x_1, x_2, y_1, y_2, y_3, p_1, p_2, q_1, q_2, q_3$ in T^*M; the equations of the Lie correspondence in $J_2^1(J_2^1 M) \times_{J_2^1 M} T^* J_2^1 M$ are:

$$\begin{aligned} p_1 + q_1 y_{1,1} + q_2 y_{2,1} + q_3 y_{3,1} &= 0 \\ p_2 + q_1 y_{1,2} + q_2 y_{2,2} + q_3 y_{3,2} &= 0 \end{aligned} \tag{26}$$

whose restriction to $\mathcal{R} \times_{J_2^1 M} T^* J_2^1 M$ is defined by the equations (25) and (26); using (25), the last can be replaced by

$$\begin{aligned} p_1 + q_1 y_2 - q_2 R + q_3 y_{2,2} &= 0 \\ p_2 + q_1 y_3 + q_2 y_{2,2} - q_3 T &= 0 \end{aligned} \tag{27}$$

Eliminating $y_{2,2}$ of these two equations we obtain the system $\mathcal{R}^*$ associated to $\mathcal{R}$ by means of the Lie correspondence; this system is defined by one (when the system is a Lie one) or two first order equations with x_1, x_2, y_1, y_2, y_3 as independent variables and only one unknown function that does not appear explicitly.

Example 5.7. Let us consider the PDE system $\mathcal{R} \subseteq J_2^2 \mathbb{R}^3$ given by

$$r + s = 0, \qquad t + s = 0;$$

its associated first order system $\mathcal{R}^* \subseteq T^* J_2^1 \mathbb{R}^3$ is the single PDE equation

$$p_1 + p_2 + (p + q) q_1 = 0, \tag{28}$$

whose characteristic system is spanned by the hamiltonian vector field

$$D^* = -\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - (p + q)\frac{\partial}{\partial z} + q_1 \frac{\partial}{\partial q_2} + q_1 \frac{\partial}{\partial q_3}.$$

Because of the linearity of $\mathcal{R}^*$, D^* is the lift to $T^* J_2^1 \mathbb{R}^3$ of the vector field $\overline{D} = -\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - (p + q)\frac{\partial}{\partial z}$ in $J_2^1 \mathbb{R}^4$.

From above we know that the set of intermediate integrals of $\mathcal{R}$ agrees with that of the (classical) solutions of $\mathcal{R}^*$. These are first integrals of $\overline{D}$; since the functions $u_1 = x - y$, $u_2 = p$, $u_3 = q$, $u_4 = z - (p+q)x$ are independent first integrals of this vector field, the intermediate integrals of $\mathcal{R}$ are of the form $V(x-y, p, q, z-(p+q)x)$, V being an arbitrary function.

On the other hand, the local equation of the projection $\lambda : \mathcal{R}^* \longrightarrow \mathcal{R}$ is

$$\lambda : (x, y, z, p, q, p_1, p_2, q_1, q_2, q_3) \longmapsto (x, y, z, p, q, r = -s, s = \frac{p_1 + q_1 z}{q_2 - q_3}, t = -s),$$

defined whenever $q_2 - q_3 \neq 0$. By means of λ we can compute the characteristic vector fields for $\mathcal{R}$: the candidate is the projection by λ of the vector field D^*. It easy to check that $D = \lambda_*(D^*) = -\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - (p+q)\frac{\partial}{\partial z}$ spans the characteristic system of $\mathcal{R}$.

Next we search for the solutions of $\mathcal{R}$. The (classical) solutions of $\mathcal{R}$ are among the solutions of $\mathcal{R}^*$ whose projection over $J_2^1\mathbb{R}^3$ has dimension 2. Since $\dim J_2^1\mathbb{R}^3 = 5$ these latter are obtained by establishing 3 relations among u_1, u_2, u_3, u_4:

$$z = (p+q)x + f_1(x-y),\ p = f_2(x-y),\ q = f_3(x-y),$$

where f_1, f_2, f_3 are arbitrary smooth functions of one variable. In order to obtain also a solution of $\mathcal{R}$ we must impose the additional condition $f_2' + f_3' = 0$. Hence, the solutions of $\mathcal{R}$ are $z = cx + f_1(x-y)$ with c an arbitrary constant and f_1 an arbitrary function of one variable.

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Received May 7, 2004
and in final form July 20, 2004