Journal of Lie Theory
Volume **15** (2005) 269–278

The classification of all simple Lie groups with surjective exponential map

Michael Wüstner

Communicated by K. H. Hofmann

Abstract. A Lie group is called exponential, if its exponential function is surjective. Here a complete classification of simple exponential Lie groups is given. Moreover, semisimple exponential Lie groups are characterized by special factor groups of products of simple Lie groups.
Mathematics Subject Classification: 22E15, 22E46
Key Words and Phrases: simple Lie groups, semisimple Lie groups, exponential map, surjectivity of exp

1. Introduction

It is an old problem to determine those Lie groups whose exponential function is surjective. Such Lie groups will be called *exponential.* In the last years much progress was made towards a solution of concerning this problem. See for instance [12] for a survey on this subject.

While D. Z. Djoković and T. Q. Nguyen have classified all *linear* exponential simple Lie groups ([2]), the aim of the present paper is to classify *all* exponential simple Lie groups. The first step was done in Theorem VII.3.2 of [12] (compare also Theorem 2.2 of [13]) where it was shown that $\widetilde{\mathrm{SU}}(m,1)$ is exponential for $m \geq 3$. The next step was done by A. L. Konstantinov and P. K. Rozanov [3] who classified all exponential Lie groups with Lie algebra $\mathfrak{su}(p,q)$, $p, q \in \mathbb{N}$.

Recall that a Lie group is called *weakly exponential* if the exponential image is dense. If G is a Lie group, we denote by $\mathfrak{g}$ its Lie algebra. For $X \in \mathfrak{g}$, the centralizer $\mathfrak{z}_{\mathfrak{g}}(X)$ is the subalgebra $\{Y \in \mathfrak{g} | [Y, X] = 0\}$. The centralizer $Z_G(X)$ is the subgroup $\{g \in G | \operatorname{Ad}(g)X = X\}$. By $\mathfrak{z}(\mathfrak{g})$, we denote the center of $\mathfrak{g}$, and by $Z(G)$ the center of G. The one-component of G is denoted by G_0. We will exploit Corollary 5.2 of [10]:

Theorem 1.1. *Let G be a real semisimple Lie group and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. Then G is exponential if and only if for each nilpotent $X \in \mathfrak{g}$ the centralizer $Z_G(X)$ is weakly exponential.*

ISSN 0949–5932 / \$2.50

In fact, one can even say "algebraic" instead of "semisimple" as an immediate consequence of Theorem 5.1 of [10]. The following is based on the results contained in these papers and on the famous tables of J. Tits ([9]). To the reader of the present paper we suggest that he consult the sources we just mentioned because we shall use their results by simply citing them.

2. On the reductive Levi decomposition

For our subsequent discussions we have to generalize the algebraic Levi decomposition of real and complex algebraic groups to locally isomorphic groups. For this purpose, let $\mathfrak{g}$ be a real or complex algebraic Lie algebra. We denote by $\mathfrak{u}$ the *nilpotent radical* of $\mathfrak{g}$, i.e., the maximal ideal consisting of nilpotent elements. Note that the nilpotent radical need not equal the nilradical, but is at least contained in it. An algebraic subalgebra of $\mathfrak{g}$ is called *reductive* if its intersection with $\mathfrak{u}$ is trivial. Recall that an algebraic Lie algebra decomposes into the semidirect product $\mathfrak{c} \ltimes \mathfrak{u}$, where $\mathfrak{c}$ is maximal reductive. This decomposition is called the *algebraic Levi decomposition* and $\mathfrak{c}$ is called a *reductive Levi factor.* Observe that all reductive Levi factors are conjugate to each other. Since any semisimple element generates a reductive subalgebra, every semisimple element is contained in some reductive Levi factor. Moreover, a reductive subalgebra is the direct sum of simple subalgebras and an algebraic abelian subalgebra consisting of semisimple elements.

Assume that G is a connected Lie group with algebraic centerfree real or complex Lie algebra $\mathfrak{g} \cong \mathfrak{c} \ltimes \mathfrak{u}$. In particular, Ad_G is the covering map from G onto $\mathrm{Ad}_G(G)$. Consider a closed connected subgroup M of G with algebraic subalgebra $\mathfrak{m}$. Now we take the algebraic Levi decomposition $\mathfrak{m} = \mathfrak{c}_\mathfrak{m} \ltimes \mathfrak{u}_\mathfrak{m}$, where $\mathfrak{c}_\mathfrak{m}$ is reductive in $\mathfrak{m}$. Since $\mathfrak{m} \cap \mathfrak{u} \subseteq \mathfrak{u}_\mathfrak{m}$, we deduce $\mathfrak{c}_\mathfrak{m} \cap \mathfrak{u} = \{0\}$, which implies that $\mathfrak{c}_\mathfrak{m}$ is also reductive in $\mathfrak{g}$. Now we define $C_M := \langle \exp_G \mathfrak{c}_\mathfrak{m} \rangle$ and $U_M := \langle \exp_G \mathfrak{u}_\mathfrak{m} \rangle = \exp_G \mathfrak{u}_\mathfrak{m}$.

Lemma 2.1. *Let G be a real or complex connected Lie group with algebraic centerfree Lie algebra. Then any connected subgroup $M \subseteq G$ with algebraic subalgebra $\mathfrak{m}$ is closed.*

Proof. Observe that $M = (\mathrm{Ad}_G^{-1}(\mathrm{Ad}_G(M)))_0$ because $\mathfrak{g}$ is centerfree. Moreover, since $\mathfrak{m}$ is algebraic, $\mathrm{Ad}_G(M)$ is the one-component of an algebraic subgroup of $\mathrm{Aut}(\mathfrak{g})$, therefore closed. Thus, $\mathrm{Ad}_G^{-1}(\mathrm{Ad}_G(M))$ is closed, hence also its one-component is closed. ■

In particular, M as well as C_M and U_M are closed. We will study intersections of certain subgroups.

Lemma 2.2. *Let G be a real or complex connected Lie group whose Lie algebra $\mathfrak{g}$ is linear and centerfree.*

(i) *If $X \in \mathfrak{g}$ is nilpotent and $\exp X \in Z(G)$, then $X = 0$.*

(ii) *If, in addition, $\mathfrak{g}$ is algebraic, $M \subseteq G$ closed connected with algebraic subalgebra $\mathfrak{m}$ and C_M, U_M are defined as above, the intersection $U_M \cap Z(G)$ is trivial, and $Z(G) \subseteq M$ implies $Z(G) \subseteq C_M$.*

Proof. (i) We get $e^{\mathrm{ad}\, X}Y = Y$ for all $Y \in \mathfrak{g}$, hence $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. Thus, we have $X \in \mathfrak{z}(\mathfrak{g}) = \{0\}$.

(ii) $U_M \cap Z(G) = \{1\}$ is an immediate consequence of (i). Now assume that $Z(G) \subseteq M$. If $g \in Z(G)$, then $g \in Z(M)$. Since every central element of a connected Lie group has a preimage, there is a $Z \in \mathfrak{m}$ with $g = \exp Z$. Now consider the Jordan decomposition $Z = Z_s + Z_n$, Z_s semisimple, Z_n nilpotent, and $[Z_s, Z_n] = 0$. Then in particular $\exp Z_n \in Z(G)$, thus $Z_n = 0$. On the other hand, Z_s is contained in a reductive Levi factor of $\mathfrak{m}$, thus $g = \exp Z_s$ is contained in one, hence all reductive Levi factors of M. ■

Lemma 2.3. *Let G be a real or complex connected Lie group whose Lie algebra is algebraic and centerfree. For M, C_M, and U_M defined as above, we have $C_M \cap U_M = \{1\}$.*

Proof. We observe that for $c \in C_M \cap U_M$ there is an $X \in \mathfrak{u}_\mathfrak{m}$ with $c = \exp X$. This implies that for every $Y \in \mathfrak{c}_\mathfrak{m}$ we have $e^{\mathrm{ad}\, X}Y \in ((Y + \mathfrak{u}_\mathfrak{m}) \cap \mathfrak{c}_\mathfrak{m}) = \{Y\}$. We deduce $c \in Z(C_M)$. Thus, there is an element $Z \in \mathfrak{c}_\mathfrak{m}$ such that $c = \exp Z$. Considering the Jordan decomposition $Z = Z_s + Z_n$, Z_s semisimple, Z_n nilpotent, $[Z_s, Z_n] = 0$, we get $\exp Z_n \in Z(C_M)$, and therefore $Z_n \in \mathfrak{z}(\mathfrak{c}_\mathfrak{m})$. But $\mathfrak{z}(\mathfrak{c}_\mathfrak{m})$ consists of semisimple elements, thus $c = \exp Z_s$. In particular, $\mathrm{Ad}_G(c)$ is both semisimple and unipotent, hence reduces to 1. We deduce $c \in Z(G)$. This implies $c \in Z(G) \cap U_M = \{1\}$. ■

So, we have established the following generalized algebraic Levi decomposition:

Theorem 2.4. *Let G be a connected Lie group with real or complex algebraic centerfree Lie algebra $\mathfrak{g}$. Assume that M is a connected subgroup with algebraic subalgebra $\mathfrak{m}$. Then M is closed and there is a closed connected subgroup C_M of M, whose Lie algebra $\mathfrak{c}_\mathfrak{m}$ is maximal reductive in $\mathfrak{m}$, such that $M \cong C_M \ltimes U_M$ where $\mathfrak{u}_M$ is the nilpotent radical of $\mathfrak{m}$ and U_M is closed. All subgroups of M with maximal reductive algebra in $\mathfrak{m}$ are conjugate to C_M. Moreover, if $Z(G) \subseteq M$, then $Z(G) \subseteq C_M$.*

We will use the expressions *reductive Levi factor* and *unipotent radical* also for C_M and U_M, respectively. We recall that by Corollary 2.1A of [4] and the fact that a factor group of a weakly exponential Lie group is weakly exponential, a Lie group G with algebraic centerfree Lie algebra is weakly exponential if and only if G factorized by its unipotent radical is weakly exponential. In particular, in order to show that such a group is weakly exponential, it is sufficient to show that one, hence all reductive Levi factors are weakly exponential. Obviously, our consideration holds in particular for semisimple Lie groups.

Moreover, we observe that factor groups of an exponential Lie group are exponential. Furthermore, direct products of exponential Lie groups are exponential.

3. Results on semisimple Lie groups

We start with semisimple Lie groups because we will need a result on (not necessarily direct) products of exponential Lie groups later in the proof of the classification theorem of simple exponential Lie groups. We will get some results which reduce the problem of the classification of semisimple to those of simple Lie groups. We will see soon that in order to characterize semisimple exponential Lie groups, it is not enough to consider direct products of simple exponential Lie groups. At the end of this section, we give a generalization of Lemma 2.3 of [2]. We start with a useful theorem.

Theorem 3.1. *Let G be a connected Lie group, $A, B \subseteq G$ closed connected subgroups such that $\mathfrak{g} \cong \mathfrak{a} \times \mathfrak{b}$. Then G is isomorphic to $(A \times B)/D$ where $D := \{(z, z^{-1}) | z \in (A \cap B) \subseteq Z(G)\}$.*

Proof. Since $(X, Y) \mapsto X + Y : \mathfrak{a} \times \mathfrak{b} \to \mathfrak{g}$ is an isomorphism of Lie algebras, $\mathfrak{a}$ and $\mathfrak{b}$ centralize each other. Consequently, A and B centralize each other. Therefore, $\mu\colon A \times B \to G$, $\mu(a, b) = ab$ is a morphism of Lie groups. Its kernel $D = \ker \mu|_{A\times B}$ is discrete, its image is a neighborhood of the identity of G since $\mathfrak{a} + \mathfrak{b} = \mathfrak{g}$. Hence it is surjective because G is connected. By the Open Mapping Theorem it follows that μ is open and this implies that $(A \times B)/D \cong G$. ∎

We can obtain Lemma 2.3 of [2] as a corollary of this theorem, but we will give an additional, elementary, nevertheless useful lemma, which implies also Lemma 2.3 of [2].

Lemma 3.2. *Assume that A, B, G are connected Lie groups and $\varphi\colon G \to A \times B$ is a covering map such that $\ker \varphi \subseteq \bar{A} := \varphi^{-1}(A)_0 = \varphi^{-1}(A)$. Assume that B and $\bar{A}$ are exponential. Then G is exponential.*

Proof. We define $\bar{B} := \varphi^{-1}(B)_0$ and observe that $G = \bar{A}\bar{B}$. Since B is exponential, we get $\bar{B} \subseteq (\exp_G \mathfrak{b} \cdot \ker \varphi)$. Thus, $G = \bar{A} \exp_G \mathfrak{b} \cdot \ker \varphi = \bar{A} \exp_G \mathfrak{b} = \exp_G \mathfrak{a} \exp_G \mathfrak{b} = \exp_G(\mathfrak{a} + \mathfrak{b})$. ∎

Now we deduce Lemma 2.3 of [2].

Corollary 3.3. *Let $G = G_1 \times \cdots \times G_n$ where G_i are connected Lie groups, and let $z_i \in G_i$ be central elements of order 2. Assume that G_1 and $G_i/\langle z_i \rangle, i > 1$, are exponential. Then $\bar{G} := G/\langle z_1z_2, z_1z_3, \ldots, z_1z_n \rangle$ is exponential.*

Proof. Using the notation of the previous lemma, we set $A := G_1/\langle z_1 \rangle$ and $B := \prod^{\times}_{i\neq 1} G_i/\langle z_i \rangle$. Set $D := \langle z_1z_2, z_1z_3, \ldots, z_1z_n \rangle$. Now consider the covering map φ of $\bar{G}$ defined by $\ker \varphi = \langle z_1 \rangle D/D$. Then $\varphi(\bar{G}) = \prod^{\times}_i (G_i/\langle z_i \rangle)$ and $\bar{A} = G_1D/D$ as a factor group of an exponential Lie group is exponential itself. Moreover, the group B is a direct product of exponential Lie groups, hence also exponential. Lemma 3.2 implies the exponentiality of $\bar{G}$. ∎

By means of Theorem 3.1, we can reduce semisimple exponential Lie groups to simple exponential Lie groups:

Corollary 3.4. *Assume that G is a semisimple Lie group and $\mathfrak{g} \cong \bigoplus_{i=1}^n \mathfrak{g}_i$ is its Lie algebra, where the $\mathfrak{g}_i$ are simple. Denote by G_i the connected subgroup of G corresponding to $\mathfrak{g}_i$ and define $H_i := \prod_{j=i}^n G_j$, $1 \le i \le n$. Then G is exponential if and only if $(G_k \times H_{k+1})/D_k$ is exponential for all $k = 1, \ldots, n-1$, where $D_k := \{(z, z^{-1}) | z \in G_k \cap H_{k+1}\}$.*

Proof. We note that because of $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and the algebraity of the subalgebras $\mathfrak{g}_i$, Lemma 2.1 implies that the groups G_i and H_i are closed. Together with Theorem 3.1 we deduce the assertion. ■

Obviously, the Lie algebras of semisimple exponential Lie groups must be direct products of Lie algebras where at least the corresponding projective Lie groups are exponential. As example, observe that Lemma 3.2 implies that $(\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \widetilde{\mathrm{SU}}(3,1))/D$ with $D = \langle (z_1, z_2) \rangle$, z_1 a generator of $Z(\widetilde{\mathrm{SL}}(2,\mathbb{R}))$, and z_2 a generator of $Z(\widetilde{\mathrm{SU}}(3,1))$, is exponential, while $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \widetilde{\mathrm{SU}}(m,1)$ is not, because $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is not.

4. The classification

We observe that complex Lie groups can be considered as real Lie groups and therefore we start by recalling the classification of the complex exponential simple Lie groups due to H. L. Lai ([5], [6], [7]).

Theorem 4.1. *The only complex exponential simple Lie groups are isomorphic to* $\mathrm{PSL}(n,\mathbb{C})$, $n \ge 2$.

For the next classes of Lie groups we need some preliminairies.

Proposition 4.2. *Assume that G is an exponential Lie group with real or complex algebraic centerfree Lie algebra. Then, for each nilpotent $X \in \mathfrak{g}$, the center $Z(G)$ must be contained in the center of one, hence all reductive Levi factors of $Z_G(X)_0$.*

Proof. Since G is exponential, the centralizer of each nilpotent X must be in particular connected. Moreover, the center $Z(G)$ is obviously contained in the center of $Z_G(X) = Z_G(X)_0$, which is a connected subgroup of G with algebraic subalgebra. Theorem 2.4 implies the assertion. ■

In the following, we will deal with factor groups of the center of $\widetilde{\mathrm{SO}}(2n-2,2)$, $n \ge 4$, and $\widetilde{\mathrm{SO}}^*(2n)$, $n \ge 4$ even. We observe that in both cases this center is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$. Now we determine all subgroups D of $\widetilde{Z} := Z(\widetilde{G})$ such that $\widetilde{Z}/D$ is isomorphic to $\mathbb{Z}_2$. We will say that such a D has *Property Z*.

We denote $\mathrm{Spin}(2n-2,2)$ resp. $\mathrm{Spin}^*(2n)$ by G, its universal covering group by $\widetilde{G}$, and denote by φ the covering map $\varphi\colon \widetilde{G} \to G$. Using the definitions introduced on page 277 of [2], we choose generators u and d of $\widetilde{Z}$ such that $d^2 = 1$, $\varphi(u) = z$, $\varphi(ud) = z'$. Then we have to consider the following cases:

(i) $D = \widetilde{Z}$,

(ii) $D = \langle d \rangle$,

(iii) $D = \langle u^k \rangle$, $k \in \mathbb{N}$,

(iv) $D = \langle du^k \rangle$, $k \in \mathbb{N}$,

(v) $D = \langle u^k, du^l \rangle = \langle u^{\gcd(k,2l)}, du^l \rangle$, $0 \le l < k$.

Case (v) can be reduced to $D = \langle u^2, d \rangle$: For odd k, it is obvious that $D = \widetilde{Z}$. If k is even and l is odd, we get $D = \langle u^2, du \rangle = \langle du \rangle$, hence we are in case (iv). If both, k and l, are even, we get $D = \langle u^2, d \rangle$.

Now we determine all D with Property Z: The cases (i) and (ii) can be excluded. In case (iii) and (iv), k must equal 1. Hence, only the following D have Property Z:

(a) $D = \langle u \rangle$,

(b) $D = \langle du \rangle$,

(c) $D = \langle u^2, d \rangle$.

Theorem 4.3. *The only exponential Lie groups with Lie algebra $\mathfrak{g} \cong \mathfrak{so}(2n-2, 2)$, $n \ge 4$ even, are isomorphic to* $\mathrm{Spin}(2n-2,2)/\langle z \rangle$ *or* $\mathrm{PSO}(2n-2,2)_0$. *For $n \ge 5$ odd, only* $\mathrm{PSO}(2n-2,2)_0$ *is exponential.*

Proof. We consider the reductive Levi factors of the centralizers of the nilpotent elements of $\mathfrak{g}$ (compare, e.g., Table 2 of [2] and note that this table fits for both, even and odd n). There exists a nilpotent element X where the reductive Levi factor $\mathfrak{c}$ of $\mathfrak{z}_{\mathfrak{g}}(X)$ is isomorphic to $\mathfrak{so}(2n-5)$.

Assume that a Lie group G with Lie algebra $\mathfrak{g}$ is exponential. By Proposition 4.2, this implies that the center $Z(G)$ of G must be contained in the center $Z(C)$ of the reductive Levi factor C of $Z_G(X) = Z_G(X)_0$ corresponding to $\mathfrak{c}$. Since C is connected, C is isomorphic to $\mathrm{Spin}(2n-5)$ or $\mathrm{SO}(2n-5)$. For $n \ge 4$, the center $Z(C)$ is isomorphic to $\mathbb{Z}_2$, or trivial. So, the center $Z(G)$ is either isomorphic to $\mathbb{Z}_2$, or trivial.

A trivial center leads to $\mathrm{PSO}(2n-2,2)_0$. If the center is isomorphic to $\mathbb{Z}_2$, we have to consider the subgroups D of $\widetilde{Z}$ with Property Z. In case (a), we get $\widetilde{G}/D \cong \mathrm{Spin}(2n-2,2)/\langle z \rangle$. Next consider case (b). Here we have $\widetilde{G}/D \cong \mathrm{Spin}(2n-2,2)/\langle z' \rangle \cong \mathrm{Spin}(2n-2,2)/\langle z \rangle$. In case (c), we get $\mathrm{SO}(2n-2,2)_0$. Together with Proposition 4.2 of [2] we get the assertion for even $n \ge 4$. Proposition 4.3 of [2] implies the assertion for odd $n \ge 5$. ∎

Theorem 4.4. *The only exponential Lie groups with Lie algebra $\mathfrak{g} \cong \mathfrak{so}^*(2n)$, $n \ge 4$ even, are isomorphic to* $\mathrm{SO}^*(2n)$, $\mathrm{Spin}^*(2n)/\langle z' \rangle$, *or* $\mathrm{PSO}^*(2n)$.

Proof. We consider the reductive Levi factors of the centralizers of the nilpotent elements of $\mathfrak{g}$ (compare, e.g., Proposition 5.2 of [2]). There exists a nilpotent element X where the reductive Levi factor $\mathfrak{c}$ of $\mathfrak{z}_{\mathfrak{g}}(X)$ is isomorphic to $\mathfrak{sp}(\frac{n}{2}, \frac{n}{2})$. Thus, C is isomorphic to $\mathrm{Sp}(\frac{n}{2}, \frac{n}{2})$ or to $\mathrm{PSp}(\frac{n}{2}, \frac{n}{2})$. Thus, the center $Z(G)$ of G is either trivial, or isomorphic to $\mathbb{Z}_2$. Hence, by Proposition 4.2, the center $Z(G)$ is either trivial, or isomorphic to $\mathbb{Z}_2$.

A trivial center leads to $\mathrm{PSO}^*(2n)$. Now we examine the subgroups D with Property Z of $\widetilde{Z}$. In case (a), we have $\widetilde{G}/D \cong \mathrm{Spin}^*(2n)/\langle z\rangle$. In case (b), we have $\widetilde{G}/D \cong \mathrm{Spin}^*(2n)/\langle z'\rangle$. Case (c) leads to $\mathrm{SO}^*(2n)$. Together with Proposition 7.2 of [2] we get the assertion. ■

Next, we will prove that the universal covering group $\widetilde{\mathrm{SO}}^*(2n)$ with odd $n \geq 3$ is exponential. Recall that $\mathfrak{so}^*(2n) \cong \mathfrak{u}_\alpha(n,\mathbb{H})$, where α denotes the anti-hermitian form J given by iI_n with respect to the corresponding basis B of $\mathbb{H}^n$. We refer to Paragraph 5 of [2]. There, the nilpotent elements X of $\mathfrak{u}_\alpha(n,\mathbb{H})$ are explicitly described. Moreover, observe that on $\mathrm{Mat}(n,\mathbb{H})$ there are given two traces, the sum of all diagonal entries of $M \in \mathrm{Mat}(n,\mathbb{H})$, denoted by $\mathrm{tr}(M)$, and the sum of the diagonal entries of the matrix in $\mathrm{Mat}(2n,\mathbb{C})$ which one gets by replacing every entry $a+bi+cj+dk$ of M by the corresponding 2×2-matrix $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$. This trace is denoted by $\mathrm{tr}_\mathbb{C}$. We observe that for every element $M \in \mathfrak{u}_\alpha(n,\mathbb{H})$ we have $\mathrm{tr}_\mathbb{C}(M) = 0$, while $\mathrm{tr}(M)$ is in general unequal to 0. Let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{u}_\alpha(n,\mathbb{H})$. Then $M \in \mathfrak{k}$ lies in $\mathfrak{k}'$, a maximal simple compact subalgebra, if and only if $\mathrm{tr}(M) = 0$. Recall that $\mathfrak{k}' \cong \mathfrak{su}(n)$. Like in [2], we will use only the denotation $\mathfrak{so}^*(2n)$, $\mathrm{SO}^*(2n)$ etc. though we actually deal with the isomorphic representation $\mathfrak{u}_\alpha(n,\mathbb{H})$.

Proposition 4.5. *A generator z of $Z(\widetilde{\mathrm{SO}}^*(2n))$ is given by $z := \exp_{\widetilde{\mathrm{SO}}^*(2n)}(u+w)$, where $u = \frac{-1}{n}\pi i I_n$ and $w = \frac{n+1}{n}\pi i I_n - \pi i(n+1)\xi$, $\xi_{11} = 1$, and all other entries equal to 0. If $\varphi\colon \widetilde{\mathrm{SO}}^*(2n) \to \mathrm{SO}^*(2n)$ denotes the covering map, then any generator of $\ker\varphi$ equals $\exp_{\widetilde{\mathrm{SO}}^*(2n)}\Delta$, where Δ is diagonal with any entries in $2\pi i\mathbb{Z}$ satisfying $\mathrm{tr}(\Delta) = \pm 2\pi i$.*

Proof. Observe that $Z(\widetilde{\mathrm{SO}}^*(2n))$ is cyclic. Note that w is contained in a maximal compact simple Lie algebra $\mathfrak{k}'$, which is necessarily isomorphic to $\mathfrak{su}(n)$, and u centralizes $\mathfrak{k}'$. We observe that the projection of z on $\exp_{\widetilde{\mathrm{SO}}^*(2n)}\mathfrak{k}'$ is the $\frac{n+1}{2}$-th multiple of a natural generator of $Z(\exp_{\widetilde{\mathrm{SO}}^*(2n)}\mathfrak{k}')$ and itself a generator. Denote by ψ the covering map from $\widetilde{\mathrm{SO}}^*(2n)$ onto $\mathrm{Spin}^*(2n)$. We observe that diagonal elements Γ_{Spin^*} in the preimage of 1 under $\exp_{\mathrm{Spin}^*(2n)}$ have entries inside $2\pi i\mathbb{Z}$ with $\mathrm{tr}(\Gamma_{\mathrm{Spin}^*})/2\pi i$ even (in opposite to $\mathrm{SO}^*(2n)$ where also those diagonal elements Γ_{SO^*} with $\mathrm{tr}(\Gamma_{\mathrm{SO}^*})/2\pi i$ odd are contained in the preimage of 1 under $\exp_{\mathrm{SO}^*(2n)}$). So, we obtain $\psi(z)^4 = \exp_{\mathrm{Spin}^*(2n)} 4(u+w) = \exp_{\mathrm{Spin}^*(2n)} 4(\pi i I_n - \pi i(n+1)\xi) = 1$, while $\exp_{\mathrm{Spin}^*(2n)} l(u+w) \neq 1$ for $l = 1,2,3$. Due to [9], we have shown, that z is a generator of $Z(\widetilde{\mathrm{SO}}^*(2n))$.

We note that z^2 generates $\ker\varphi$ and observe that any diagonal preimage $\Gamma_{\widetilde{\mathrm{SO}}^*}$ of 1 under $\exp_{\widetilde{\mathrm{SO}}^*(2n)}$ has entries in $2\pi i\mathbb{Z}$ and $\mathrm{tr}(\Gamma_{\widetilde{\mathrm{SO}}^*}) = 0$ because $\Gamma_{\widetilde{\mathrm{SO}}^*}$ must be contained in $\mathfrak{k}'$. So, we calculate $z^2 = \exp_{\widetilde{\mathrm{SO}}^*(2n)} 2(\pi i I_n - \pi i(n+1)\xi) = \exp_{\widetilde{\mathrm{SO}}^*(2n)}(-2\pi i\xi)$. Adding any $\Gamma_{\widetilde{\mathrm{SO}}^*}$ and regarding the fact that z^{-2} is the only other generator of $\ker\varphi$ implies the assertion. ■

Proposition 4.6. *Let $X \in \mathfrak{so}^*(2n)$ be a nilpotent element and φ the covering map from $\widetilde{\mathrm{SO}}^*(2n)$ onto $\mathrm{SO}^*(2n)$. Assume that g is a generator of $\ker\varphi$. Denote* *by* $\mathfrak{c} = \bigoplus \mathfrak{c}_i$ *a reductive Levi complement of* $\mathfrak{z}_\mathfrak{g}(X)$ *and assume w.l.o.g. that* $\mathfrak{c}_1 \cong \mathfrak{so}^*(2p)$, p *odd. Then* $\exp^{-1}(g) \cap \mathfrak{c}_1 \neq \emptyset$.

Proof. For every nilpotent element X, there is given a basis B' of the natural $\mathfrak{so}^*(2n)$-module V ($\dim_\mathbb{H} V = n$) such that X is given in canonical form (upper secondary diagonal with entries 0 or 1). With respect to this basis, the skew-hermitian form J is given by diagonal blocks $J_{m_j}, K_{m_k}, L_{m_l}$, m_j odd, m_k, m_l even, defined as follows: Let E_{ij} be the standard basis of $\mathrm{Mat}_{B'}(t, \mathbb{H})$. A block J_{m_j} is defined as $m_j \times m_j$-matrix $\sum_{r=1}^{m_j}((-1)^{\frac{m_j+1}{2}+r} i E_{r,m_j+1-r})$. A block K_{m_k} is defined as $m_k \times m_k$-matrix $\sum_{r=1}^{m_k}((-1)^r E_{r,m_k+1-r})$, while $L_{m_l} = -K_{m_l}$. As was pointed out in [2], there is at least one odd p such that there is an odd number k of blocks J_{m_s}, $s = 1, \ldots, k$, with $m_1 = \cdots = m_k = p$. Let $X = \sum_j X_{m_j} + \sum_k X_{m_k} + \sum_l X_{m_l}$, such that X_{m_j} fits to J_{m_j} while X_{m_k} fits to K_{m_k} and X_{m_l} to L_{m_l}, respectively. Let W_p be the subspace of V belonging to the form $\begin{pmatrix} J_{m_1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & J_{m_k} \end{pmatrix}$. Recall that $\mathfrak{c}_1$ leaves W_p invariant and acts trivially on the invariant complement of W_p. Furthermore, we have $\dim W_p = p \cdot m_p$. Moreover, $\mathfrak{c}_1$ centralizes X. We observe that the diagonal element $\Delta_{B'}$ with $(\Delta_{B'})_{jj} = 2\pi i$ for $j = 1, \ldots, m_1$ and all other entries 0 is contained in $\mathfrak{c}_1$. Applying basis transformation from B' to the basis B such that J is given in canonical form, we get $(\Delta_B)_{jj} = (-1)^{j+1} 2\pi i$ for $j = 1, \ldots, m_1$ and all the other entries 0. By Proposition 4.5, Δ_B is contained in the preimage of z^{-2} and $-\Delta_B$ is in the preimage of z^2. Since z^2 and z^{-2} are the only generators of $\ker \varphi$, we deduce the assertion. ∎

Now we are able to prove the following:

Theorem 4.7. *The universal covering groups* $\widetilde{\mathrm{SO}}^*(2n)$, $n \geq 3$ *odd, are exponential.*

Proof. This proof is a variation of the proof of Proposition 7.1 of [2]. Note that $\widetilde{\mathrm{SO}}^*(2) \cong \mathbb{R}$ is exponential. Assume that $n \geq 3$ and that the claim is true for all odd $p < n$. For $X = 0$, the centralizer equals the group itself, which is weakly exponential by Theorem IV.6 of [8]. Let $X \neq 0$ be a nilpotent element in the Lie algebra of $\widetilde{\mathrm{SO}}^*(2n)$. Again, denote by φ the covering map of $\widetilde{\mathrm{SO}}^*(2n)$ onto $\mathrm{SO}^*(2n)$. By Proposition 5.2 of [2], the reductive Levi factor $\mathfrak{c}$ of $\mathfrak{z}_\mathfrak{g}(X)$ is isomorphic to the direct sum of Lie algebras $\mathfrak{so}^*(2p)$ (where $\mathfrak{so}^*(2)$ is isomorphic to $\mathbb{R}$) and $\mathfrak{sp}(p,q)$. There is at least one simple factor $\mathfrak{c}_1$ isomorphic to $\mathfrak{so}^*(2p)$, $p < n$ odd. By Proposition 4.6, it contains a preimage of a generator of $\ker \varphi$. Moreover, $C \cong (\prod_i^\times C_i)/D$, C_i connected, $\mathfrak{c}_i$ isomorphic to $\mathfrak{so}^*(2k)$ or to $\mathfrak{sp}(p,q)$, $D \subseteq Z(\prod_i^\times C_i)$. Note that C_1 is covered by $\widetilde{\mathrm{SO}}^*(2p)$ which is exponential by induction assumption. Thus, C_1 is exponential, hence $C_1 D/D$ is exponential. Moreover, the group $C_1 D/D = \exp_{\widetilde{\mathrm{SO}}^*(2n)} \mathfrak{c}_1$ contains $\ker \varphi$. Note that $\varphi((\prod_{i\neq 1} C_i)D/D)$ is exponential as a product of some copies of $\mathrm{SO}^*(2l)$ and $\mathrm{Sp}(p,q)$ by [2]. Thus, Lemma 3.2 implies that C is exponential, hence $Z_G(X)$ is weakly exponential. This implies the assertion. ∎

Now we gather the results on Lie groups with Lie algebra $\mathfrak{su}(p,q)$. Recall that $\widetilde{\mathrm{SU}}(m,1)$ is exponential for $m \geq 3$. In their paper [3] in this issue, pp. 51–61, A. L. Konstantinov and P. K. Rozanov have determined all simple exponential Lie groups with Lie algebra $\mathfrak{su}(p,q)$ with $p,q \in \mathbb{N}$, a generalization of the corresponding criterion of [2] (Proposition 3.1). The subject at hand causes their result to be rather technical; it read as follows (Theorem 3.5 and Theorem 1.7 of [3]):

Theorem 4.8. *Denote by* $\varphi\colon \widetilde{\mathrm{SU}}(p,q) \to \mathrm{SU}(p,q)$ *the canonical covering map. Observe that by* [9], *the center* $\widetilde{Z} = Z(\widetilde{\mathrm{SU}}(p,q))$ *is isomorphic to* $\mathbb{Z} \times \mathbb{Z}_d$, $d = \gcd(p,q)$. *Moreover, we may assume* $\ker\varphi = \langle(1,n')\rangle$ *and denote by* ν *the projection from* $\widetilde{Z}$ *to* $\mathbb{Z}$.

(i) *A Lie group* G *locally isomorphic to* $\mathrm{SU}(p,p)$ *is exponential if and only if* $G = \mathrm{PSU}(p,p)$.

(ii) *Let* $p \neq q$ *and* $D = \langle x_1, x_2\rangle$ *be a nontrivial central subgroup of* $\widetilde{\mathrm{SU}}(p,q)$, $\varphi(x_1) = y^{an'}$, $a|d$, $\nu(x_2) = b$, $\varphi(x_2) = y^c$, $0 \leq c < an', b + cq' = ln'$.

The group $G = \widetilde{\mathrm{SU}}(p,q)/D$ *is exponential if and only if for all* $j = 0, \ldots, [\frac{q}{p-q}]$ *the following conditions are fulfilled:*

(a) $\gcd(b, q' - j(p' - q')) = 1$;

(b) $\gcd(a, l(2j+1) - cj) = 1$.

(iii) $\widetilde{\mathrm{SU}}(p,q)$ *is not exponential for* $p,q > 1$.

We get the **classification of the simple exponential real Lie groups**:

Theorem 4.9. *Every simple exponential real Lie group is isomorphic to one of the following Lie groups:*

(i) *A compact simple connected Lie group.*

(ii) $\mathrm{PSL}_n(\mathbb{C})$, $n \geq 2$.

(iii) *A Lie group described by* Theorem 4.8 *(including* $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{PSU}(1,1)$ *and* $\mathrm{PSO}(4,2)_0 \cong \mathrm{PSU}(2,2)$*) or a Lie group covered by* $\widetilde{\mathrm{SU}}(m,1)$, $m \geq 3$.

(iv) *A Lie group covered by* $\mathrm{SL}(n,\mathbb{H})$, $n \geq 2$.

(v) *A Lie group covered by* $\mathrm{Spin}(2n,1)$, $n \geq 2$.

(vi) *A Lie group covered by* $\mathrm{Sp}(p,q)$, $p \geq q \geq 1$.

(vii) *A Lie group covered by* $\mathrm{Spin}(2n-1,1)$, $n \geq 3$.

(viii) $\mathrm{Spin}(2n-2,2)/\langle z\rangle$ *or* $\mathrm{PSO}(2n-2,2)_0$, $n \geq 4$ *even.*

(ix) $\mathrm{PSO}(2n-2,2)_0$, $n \geq 5$ *odd.*

(x) $\mathrm{SO}^*(2n)$, $\mathrm{Spin}^*(2n)/\langle z'\rangle$, *or* $\mathrm{PSO}^*(2n)$, $n \geq 4$ *even.*

(xi) *A Lie group covered by* $\widetilde{\mathrm{SO}}^*(2n)$, $n \geq 3$ *odd.*

(xii) E IV$= E_{6,-26}$.

Proof. (i) is well-known, and (ii) is Theorem 4.1. For the noncompact non-complex exponential Lie groups, we only have to check Lie groups with Lie algebra isomorphic to the Lie algebra of those groups described in the list of [2]. (iii) is due to Theorem 4.8 and [12], Theorem 1.7. Since $\mathrm{SL}(n,\mathbb{H})$ is simply connected, we get (iv). By the same argument, we get (v), (vi), and (vii). (viii) is Theorem 4.3, (ix) and (x) is Theorem 4.4, and (xi) is Theorem 4.7. (xii) is clear because its universal covering group is centerfree, hence equals E IV. ∎

Corollary 4.10. *If G is a simple exponential nonlinear simply connected Lie group, then G is isomorphic to $\widetilde{\mathrm{SO}}^*(2n)$, $n \geq 3$ odd, or to $\widetilde{\mathrm{SU}}(m,1)$, $m \geq 3$.*

References

[1] Collingwood, D. H., and W. M. McGovern, "Nilpotent Orbits in Semisimple Lie Algebras," Van Nostrand Reinhold, New York, 1993.

[2] Djoković, D. Ž., and T. Q. Nguyen, *The exponential group of almost simple algebraic groups*, J. Lie Theory **5** (1996), 275–291.

[3] Konstantinov, A. L., and P. K. Rozanov, *On the Exponential Map of the Lie Groups Locally Isomorphic to* $\mathrm{SU}(p,q)$, J. Lie Theory **15** (2005), 51–61.

[4] Hofmann, K. H. and A. Mukherjea, *On the density of the image of the exponential function*, Math. Ann. **234**, (1978), 263–273.

[5] Lai, H. L., *Exponential map of a center-free complex simple Lie group*, Osaka J. Math. **15** (1978), 553–560.

[6] —, *Exponential map of a complex simple Lie group*, Osaka J. Math. **15** (1978), 561–567.

[7] —, *Index of the exponential map in a complex simple Lie group*, Osaka J. Math. **17** (1980), 525–530.

[8] Neeb, K.-H., *Weakly Exponential Lie Groups*, J. Alg. **179** (1996), 331–361.

[9] Tits, J., "Tabellen zu den einfachen Lie-Gruppen und ihre Darstellun gen," Lecture Notes in Mathematics **40**, Springer-Verlag, Berlin, 1967.

[10] Wüstner, M., *On the exponential function of real splittable and real semisimple Lie groups*, B. Algebra Geom. **39** (1998), 37–46.

[11] —, *Splittable Lie groups and Lie algebras*, J. Alg. **226** (2000), 202–215.

[12] —, "Lie groups with surjective exponential function," TU Darmstadt 2000 and Shaker-Verlag Aachen 2001.

[13] —, *Supplements on the theory of exponential Lie groups*, J. Alg. **265** (2003), 148–170.

Michael Wüstner
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstr. 7
64289 Darmstadt
Germany
michael.wuestner@gmx.de

Received April 7, 2004
and in final form September 24, 2004