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Lifting Smooth Homotopies of Orbit Spaces of Proper Lie Group Actions

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Abstract. By a result of G. W. Schwarz, a smooth version of R. S. Palais' covering homotopy theorem holds for actions of compact Lie groups. This paper extends the result of Schwarz to the case of proper actions of non-compact Lie groups. An isotopy lifting theorem is obtained as a corollary.
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1. Introduction

This paper is concerned with lifting homotopies from orbit spaces of smooth (i.e., C^∞) Lie group actions. If G is a compact Lie group, there exists the covering homotopy theorem of R. S. Palais ([5]) for continuous actions. As pointed out by Palais in [6], his result also holds for proper actions of non-compact Lie groups. G. W. Schwarz has established a smooth analogue of Palais' theorem for compact Lie group actions. The purpose of this paper is to generalize Schwarz's result for proper actions of non-compact Lie groups.

The main result is the generalization of Schwarz's Theorem 2.3 in [8]:

Theorem 1.1. *Let G be a Lie group and let M and N be proper smooth G-manifolds without boundary. Let $f\colon M \to N$ be a G-equivariant smooth map and let $\bar{F}\colon M/G \times [0,1] \to N/G$ be a smooth map such that each $\bar{F}_t = \bar{F}(\cdot, t)\colon M/G \to N/G$ is normally transverse. Assume f induces $\bar{F}_0$. Then there is a G-equivariant smooth normally transverse homotopy $F\colon M \times [0,1] \to N$ inducing $\bar{F}$ and starting at f. Moreover, any two such liftings of $\bar{F}$ differ by composition with a G-equivariant isotopy of M starting at the identity map of M and inducing the trivial isotopy on M/G.*

While the construction of the map F is similar to that in [8], it isn't obvious that the construction actually works in the case of proper actions. In Sections 2. and 3., we prove results needed for the construction. As a corollary we obtain an isotopy lifting result, which generalizes Corollary 2.4 in [8]:

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Corollary 1.2. *Let G be a Lie group and let M be a proper smooth G-manifold without boundary. Let $\bar{F}\colon M/G \times [0,1] \to M/G$ be a smooth isotopy starting at the identity. Then there is a G-equivariant smooth isotopy $F\colon M \times [0,1] \to M$ starting at the identity and inducing $\bar{F}$.*

E. Bierstone ([1]) has proved Corollary 1.2 for smooth compact Lie group actions in the case where all the isotropy subgroups have the same dimension.

By the slice theorem, proper actions of non-compact Lie groups look locally like actions of compact Lie groups. Thus a standard method for proving results for proper Lie group actions is first to prove a local result by using the corresponding result for compact Lie groups, and then to somehow conclude the global result. This is also the idea behind many of the proofs of this paper. In particular, this is the way to prove the smooth lifting theorem for vector fields (Theorem 5.3), which is the main result needed for the proof of Theorem 1.1.

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2. Proper smooth actions

Let G be a Lie group and let M be a smooth (i.e., C^∞) manifold on which G acts. All manifolds are assumed to have at most countably many connected components and are allowed to have a non-empty boundary, unless otherwise stated. We say that the action is *proper*, if the map $G \times M \to M \times M$, $(g,x) \mapsto (gx,x)$, is a proper map, i.e., if the inverse images of all compact sets are compact. Thus all actions of compact groups are proper and the action of a discrete group is proper if and only if it is properly discontinuous. If the action map $G \times M \to M$ is smooth, we call M a smooth G-manifold. If the action is also proper, we call M a *proper smooth G-manifold*.

Let G be a Lie group and let M be a proper smooth G-manifold. We denote the *orbit space* of M by M/G and let $\pi_M\colon M \to M/G$ denote the natural projection. Let $x \in M$. Then the *isotropy subgroup* of x is $G_x = \{g \in G \mid gx = x\}$. Notice that G_x is compact, for every x. If H is a subgroup of G, then (H) denotes the family of subgroups of G which are conjugate to H. As usual,

$$M^H = \{x \in M \mid hx = x \text{ for every } h \in H\},$$

$$M_H = \{x \in M \mid G_x = H\} \text{ and}$$
$$M_{(H)} = \{x \in M \mid (G_x) = (H)\}.$$

A map $f\colon M \to N$ is said to *preserve the isotropy type* if $G_x = G_{f(x)}$, for every $x \in X$. An isotropy type preserving G-equivariant map is called *isovariant*.

Assume H is a closed subgroup of G and let N be a smooth H-manifold. Let $G \times_H N$ denote the *twisted product* of G with N over H. Thus $G \times_H N$ is the orbit space of the smooth H-manifold $G \times N$ on which H acts by $H \times (G \times N) \to G \times N$, $(h,(g,x)) \mapsto (gh^{-1}, hx)$. We denote the orbit of (g,x) by $[g,x]$. The orbit space $G \times_H N$ is a smooth G-manifold with the boundary $\partial(G \times_H N) = G \times_H \partial N$. If N is a proper smooth H-manifold, then $G \times_H N$ is a proper smooth G-manifold. See Section 4 of [4], for a detailed discussion on twisted products.

Let M be a smooth manifold with a boundary ∂M and let S be a smooth submanifold of M. We call S a *nice* submanifold, if $\partial S = S \cap \partial M$.

Definition 2.1. Let G be a Lie group and let M be a proper smooth G-manifold. Let $x \in M$. Assume S is a nice G_x-invariant submanifold of M, $x \in S$ and GS is open in M. If there exists a smooth G-map $f\colon GS \to G/G_x$ such that $f^{-1}(eG_x) = S$, we call S a *smooth slice* (or just a *slice*) at x.

We call the following theorem the *differentiable slice theorem*:

Theorem 2.2. *Let G be a Lie group and let M be a proper smooth G-manifold. Then there exists a smooth slice at every point of M.*

Proof. If $x \in M \setminus \partial M$, the theorem follows from Proposition 2.2.2 in [6]. If $x \in \partial M$, the proof remains similar to that of Proposition 2.2.2 in [6]. ∎

If $x \in M$ and S is a smooth slice at x, then the map $G \times_{G_x} S \to GS$, $[g, y] \mapsto gy$, is a G-diffeomorphism. We will frequently identify the *tube* GS with the twisted product $G \times_{G_x} S$.

Let $\mathrm{T}_x(M)$ denote the tangent space of M at x. Then the normal space at x to the orbit Gx is $\mathrm{N}_x = \mathrm{T}_x(M)/\mathrm{T}_x(Gx)$. We call this representation of G_x the *slice representation* at x. Two G-orbits are said to have the same *normal type* if there are points in each which have the same isotropy group and whose slice representations are isomorphic up to trivial factors. If $x \in M \setminus \partial M$, we can choose the smooth slice at x to be G_x-diffeomorphic to N_x. A smooth slice G_x-diffeomorphic to a linear G_x-space (or to an open subset of such a space) is called *linear*.

Lemma 2.3. *Let G be a Lie group and let M and N be proper smooth G-manifolds. Assume $f\colon M \to N$ is a smooth isovariant map, $x \in M$, $y = f(x)$ and S_y is a smooth slice at y. Then $S_x = f^{-1}(S_y)$ is a smooth slice at x.*

Proof. Let $r\colon GS_y \to G/G_x$, $gs \mapsto gG_x$. Clearly, the restrictions of $r \circ f|_{GS_x}$ to each orbit in GS_x are submersions. It follows (see for example Theorem 1.4.1 in [3]), that $S_x = (r \circ f|_{GS_x})^{-1}(eG_x)$ is a smooth submanifold of M such that $\partial S_x = S_x \cap \partial(GS_x) = S_x \cap \partial M$. Thus S_x is a smooth slice at x. ∎

3. Smooth structure on the orbit space

In this section we follow the terminology in [8]. We present the definitions here for the sake of completeness.

Let G be a Lie group and let M be a proper smooth G-manifold. The orbit space M/G is equipped with the quotient topology and differentiable structure: Let U be an open subset of M/G. We say that the map $f\colon U \to \mathbb{R}$ is smooth, if the map $\pi_M^*(f) = f \circ \pi_M\colon \pi_M^{-1}(U) \to \mathbb{R}$ is smooth. We denote the set of smooth maps $U \to \mathbb{R}$ by $\mathrm{C}^\infty(U)$. Let N be another proper smooth G-manifold. Then we say that the map $\psi\colon M/G \to N/G$ is smooth, if $\psi^*\mathrm{C}^\infty(N/G) \subset \mathrm{C}^\infty(M/G)$. By a smooth diffeomorphism $M/G \to M/G$ we mean a smooth homeomorphism, whose inverse map is smooth. Let I denote the unit interval $[0, 1]$. A smooth isotopy $F\colon M/G \times I \to M/G$ is a smooth map such that $F_t\colon M/G \to M/G$ is a smooth diffeomorphism, for every $t \in I$.

The subsets of M/G of given normal type form a stratification of M/G. Unless otherwise mentioned, by strata on M/G we mean *normal type strata.* The orbit space M/G has also another natural stratification. The strata of this stratification are the images $(M/G)_{(H)}$ of the sets $M_{(H)}$ in M/G, where H is a subgroup of G. We call these strata *isotropy type strata* on M/G. Each stratum $(M/G)_{(H)}$ has the smooth structure of the orbit space $M_{(H)}/G$.

By a *smooth Σ-manifold* we mean a topological sum of connected manifolds of possibly different dimensions. By a smooth Σ-submanifold of a smooth manifold M we mean a Σ-manifold which is a subspace of M.

The following proposition was proved for compact G in [8], Proposition 1.2.

Proposition 3.1. *Let G be a Lie group and let H be a compact subgroup of G. Let M be a proper smooth G-manifold. Then*

1) The isotropy type strata $\{(M/G)_{(H)}\}$ are smooth Σ-manifolds, and the inclusions $(M/G)_{(H)} \to M/G$ are smooth.

2) The connected components of the isotropy type strata are a locally finite collection of subsets of M/G.

3) Let $\{\sigma_\alpha\}$ be the set of the normal type strata of M/G. Then the connected components of the σ_α are the same as the connected components of the $(M/G)_{(H)}$.

Proof. By Corollary 4.2.8 of [7], the sets $M_{(H)}$ and M_H are smooth Σ-submanifolds of M. (This corollary and other results in [7] to which we refer in this proof are stated for manifolds without boundary, but they hold also for manifolds with boundary.) Let $N(H)$ denote the normalizer of H and let $\Gamma(H) = N(H)/H$. By Theorem 4.3.10 of [7], $M_{(H)} = G/H_{\Gamma(H)} \times M_H$. Since $\Gamma(H)$ acts properly and freely on M_H, it follows that $(M/G)_{(H)} = M_H/\Gamma(H)$ is a smooth Σ-manifold (see Proposition 4.2.10 in [7]). Moreover, $(M/G)_{(H)}$ has the correct C^∞-structure, i.e., $\mathrm{C}^\infty((M/G)_{(H)}) = \pi^*_{M_{(H)}}(\mathrm{C}^\infty(M_{(H)}))$. To show that the inclusions $(M/G)_{(H)} \to M/G$ are smooth, it is sufficient to show that the inclusions $(M/G)_{(H)} \cap GS/G \to GS/G$ are smooth, where S is a smooth slice at any $x \in M$. By identifying GS/G with S/G_x, this follows immediately from the case of compact G (Proposition 1.2.1 in [8]).

The second claim follows immediately from the corresponding result for compact G (Proposition 1.2.2 in [8]) by identifying GS/G with S/G_x. Let $\xi \in M/G$. Let $x \in M$ be such that $\xi = \pi_M(x)$ and let S be a smooth slice at x. By Proposition 1.2.3 in [8], we can assume that the connected components of the normal type strata on S/G_x are the same as those of the isotropy type strata. Identifying GS/G with S/G_x and covering a connected component of any $(M/G)_{(H)}$ by open sets of form GS/G proves the third claim. ∎

Let $\xi \in M/G$ and let $\mathcal{M}_\xi$ denote the ring of germs of smooth functions at ξ vanishing at ξ. Let $\mathcal{M}_\xi^2$ denote the ideal of $\mathcal{M}_\xi$ which is generated by products of elements of $\mathcal{M}_\xi$. Then the Zariski cotangent space $\mathrm{T}^*_\xi(M/G)$ of M/G at ξ is $\mathcal{M}_\xi/\mathcal{M}_\xi^2$. The Zariski tangent space $\mathrm{T}_\xi(M/G)$ of M/G at ξ is defined to be the dual space of $\mathrm{T}^*_\xi(M/G)$. By the differentiable slice theorem, M/G

looks locally like an orbit space of a compact group action. Since the Zariski tangent and cotangent spaces of orbit spaces of compact group actions are always finite dimensional vector spaces ([8], p. 44), it follows that the same is true for $\mathrm{T}_\xi(M/G)$ and $\mathrm{T}^*_\xi(M/G)$. A smooth map $\psi\colon M/G \to N/G$ induces a linear map $(d\psi)_\xi\colon \mathrm{T}_\xi(M/G) \to \mathrm{T}_{\psi(\xi)}(N/G)$.

Let $\xi \in M/G$, and let σ_ξ denote the normal type stratum of M/G containing ξ. Let $\mathcal{N}_\xi(M/G)$ denote $\mathrm{T}_\xi(M/G)/\mathrm{T}_\xi(\sigma_\xi)$. Assume $\psi\colon M/G \to N/G$ is a strata preserving smooth map. Then $(d\psi)_\xi\colon \mathrm{T}_\xi(M/G) \to \mathrm{T}_{\psi(\xi)}(N/G)$ induces a linear map $(\delta\psi)_\xi\colon \mathcal{N}_\xi(M/G) \to \mathcal{N}_{\psi(\xi)}(N/G)$. We say that ψ is *normally transverse* if $(\delta\psi)_\xi$ is an isomorphism for all $\xi \in M/G$.

Let $x \in M$. Let $\mathcal{N}_x(M)$ denote $\mathrm{N}_x/\mathrm{N}_x^{G_x}$. We say that a smooth G-map $\psi\colon M \to N$ is *strata preserving* if it preserves the normal type of orbits. Then $(d\psi)_x$ induces $(\delta\psi)_x\colon \mathcal{N}_x(M) \to \mathcal{N}_{\psi(x)}(N)$, and we say that ψ is *normally transverse* if $(\delta\psi)_x$ is an isomorphism for all $x \in M$.

Propositions 3.2, 3.4 and 3.5 were proved for compact G in [8], Proposition 2.1.

Proposition 3.2. *Let M and N be proper smooth G-manifolds with $\partial N = \emptyset$. Let $f\colon M \to N$ be a strata preserving G-equivariant smooth map and let $\bar f\colon M/G \to N/G$ be the map induced by f. Then f is normally transverse if and only if $\bar f$ is normally transverse.*

Proof. Let $x \in M$ and let S_y be a smooth slice at $y = f(x)$. By Lemma 2.3, $S_x = f^{-1}(S_y)$ is a smooth slice at x. Since $\mathrm{T}_x(M) = \mathrm{T}_x(S_x) \oplus \mathrm{T}_x(Gx)$ and $\mathrm{T}_y(N) = \mathrm{T}_y(S_y) \oplus \mathrm{T}_y(Gy)$, it follows that $\mathcal{N}_x(M) = \mathcal{N}_x(S_x)$ and $\mathcal{N}_y(N) = \mathcal{N}_y(S_y)$. Let $f|\colon S_x \to S_y$ be the restriction of f. Clearly, $(\delta f)_x = (\delta f|)_x$. Thus f is normally transverse at x if and only if $f|$ is normally transverse at x. Similarly, identifying GS_x/G with S_x/G_x and GS_y/G with S_y/G_y, we see that $\bar f$ is normally transverse at $\pi_M(x)$ if and only if $\overline{f|}$ is normally transverse at $\pi_{S_x}(x)$. Since $f|$ is normally transverse if and only if $\overline{f|}$ is normally transverse (Proposition 2.1.3 of [8]), the proposition follows. ∎

Let $F\colon M/G \to N/G$ be a smooth map between the orbit spaces of two proper smooth G-manifolds M and N, where $\partial N = \emptyset$, and let $\pi_N\colon N \to N/G$ be the natural projection. As usual, we denote by F^*N the pullback of N by F. Since N is a smooth manifold and M/G has a smooth structure as well, F^*N has the induced smooth structure from $M/G \times N$. Thus $f\colon F^*N \to \mathbb{R}$ is smooth if and only if it is a restriction of a smooth map defined on a neighbourhood of F^*N in $M/G \times N$. Let P be another proper smooth G-manifold. Then we say that $f\colon F^*N \to P/G$ is smooth, if $f^*\mathrm{C}^\infty(P/G) \subset \mathrm{C}^\infty(F^*N)$.

Assume G is a Lie group and K is a compact subgroup of G. Let S and $\tilde S$ be smooth K-manifolds with $\partial S = \emptyset$, and let $F\colon \tilde S/K \to S/K$ be a smooth normally transverse map. By Proposition 2.1.1 in [8], the pullback F^*S of S by F, with the smooth structure inherited from $\tilde S/K \times S$, is a smooth K-Σ-manifold and $\partial(F^*S) = (F|_{\partial\tilde S})^*S$. As usual, let's identify $\tilde S/K$ and S/K with $(G\times_K \tilde S)/G$ and $(G\times_K S)/G$, respectively, and let's form the pullback $F^*(G\times_K S)$ of $G\times_K S$ by F. Then the following holds:

Lemma 3.3. *The canonical map $\theta\colon G\times_K F^*S \to F^*(G\times_K S)$, $[g,(\xi,x)]\mapsto (\xi,[g,x])$, is a G-equivariant smooth diffeomorphism.*

Proof. The restriction $\theta|$ to F^*S is clearly a smooth K-equivariant map and $\theta(F^*S)=F^*S$. Since G acts trivially on $\tilde{S}/K$ and smoothly on $G\times_K S$, it also acts smoothly on $F^*(G\times_K S)$. Thus the G-map $G\times_K F^*S\to F^*(G\times_K S)$ induced by $\theta|$ is smooth. But this map is precisely θ. Clearly, θ is a bijection.

It remains to show that the inverse map θ^{-1} is smooth. By the equivariance of θ^{-1}, it is sufficient to show that the restriction $\theta^{-1}|F^*S$ is a restriction of some smooth map defined on a neighbourhood of F^*S in $\tilde{S}/K\times(G\times_K S)$. Let U be an open neighbourhood of eK in G/K, and let $\delta\colon U\to G$ be a smooth local cross section of the projection $G\to G/K$. We may assume that $\delta(eK)=e$. Let $f_S\colon G\times_K S\to G/K$ be the slice map. Then the map

$$\delta_0\colon \tilde{S}/K\times f_S^{-1}(U)\to \tilde{S}/K\times S,\ \ (\xi,[g,x])\mapsto(\xi,\delta(f_S[g,x])^{-1}[g,x]),$$

is smooth as a composition of smooth maps. Since $\theta^{-1}|F^*S=\delta_0|F^*S$, the claim follows. ∎

Proposition 3.4. *Let G be a Lie group and let M and N be proper smooth G-manifolds with $\partial N=\emptyset$. Let $F\colon M/G\to N/G$ be a smooth normally transverse map. Then F^*N is a proper smooth G-Σ-manifold whose boundary is $(F|_{\partial M})^*N$.*

Proof. Since G acts smoothly on M/G and N, it also acts smoothly on F^*N. Since N is a proper G-space, also $M/G\times N$ is a proper G-space. Thus F^*N is a proper G-space.

To prove that F^*N is a smooth Σ-manifold, it suffices to show that each point $(\xi,x)\in F^*N$ has a neighbourhood which the smooth structure inherited from $M/G\times N$ makes a smooth manifold. Thus, without loss of generality, we may assume that $M=G\times_K\tilde{S}$ an $N=G\times_K S$, where K is a compact Lie group and $\tilde{S}$ and S are smooth K-manifolds. But then the claim follows from Lemma 3.3.

By Proposition 2.1.1 in [8], $\partial(F^*S)=(F|_{\partial\tilde{S}})^*S$. Consequently,

$$\partial(F^*(G\times_K S))=(F|_{G\times_K\partial\tilde{S}})^*(G\times_K S)=(F|_{\partial(G\times_K\tilde{S})})^*(G\times_K S).$$

Covering M and N by tubes shows that $\partial(F^*N)=(F|_{\partial M})^*N$. ∎

Proposition 3.5. *Let G be a Lie group and let M and N be proper smooth G-manifolds with $\partial N=\emptyset$. Let $f\colon M\to N$ be a G-equivariant smooth normally transverse map and let $\bar{f}\colon M/G\to N/G$ be the map induced by f. Then the canonical map $\varphi\colon M\to\bar{f}^*N$, $x\mapsto(\pi_M(x),f(x))$, is a G-equivariant smooth diffeomorphism.*

Proof. Since π_M and f are smooth and f is isovariant, it follows that φ is a smooth bijection. Let $x\in M$ and let S be a smooth slice at $f(x)$. By Lemma 2.3, $f^{-1}(S)$ is a smooth slice at x. Let $f|\colon f^{-1}(S)\to S$ be the restriction of f and let $F\colon f^{-1}(S)/G_x\to S/G_x$ be the map induced by $f|$. By Proposition 2.1.2 in [8], the map

$$\varphi_S\colon f^{-1}(S)\to F^*S,\ s\mapsto(\pi_{f^{-1}(S)}(s),f(s)),$$

is a smooth G_x-diffeomorphism. The map φ_S induces a smooth G-diffeomorphism $\tilde{\varphi}_S\colon Gf^{-1}(S) \to G \times_{G_x} F^*S$. Let θ be the diffeomorphism of Lemma 3.3. Then the restriction $\varphi|_{Gf^{-1}(S)} = \theta \circ \tilde{\varphi}_S$ is a diffeomorphism onto $F^*(G \times_{G_x} S)$. It follows that also φ is a diffeomorphism. ∎

The following two propositions will be used in the proof of Corollary 1.2.

Proposition 3.6. *Let G be a Lie group and let M and N be proper smooth G-manifolds without boundary. Assume $f\colon M \to N$ is a G-equivariant smooth normally transverse map such that the induced map $\bar{f}\colon M/G \to N/G$ is a smooth diffeomorphism. Then f is a G-equivariant smooth diffeomorphism.*

Proof. Let $\varphi\colon M \to M/G \times N$, $x \mapsto (\pi_M(x), f(x))$. By Proposition 3.5, φ maps M diffeomorphically onto $\bar{f}^*N$. Let $\mathrm{pr}\colon N/G \times N \to N$ denote the projection. Then $(\bar{f}, \mathrm{id}_N)\colon M/G \times N \to N/G \times N$ is a diffeomorphism and $\mathrm{pr} \circ (\bar{f}, \mathrm{id}_N)$ maps $\bar{f}^*N$ diffeomorphically onto N. Since $f = \mathrm{pr} \circ (\bar{f}, \mathrm{id}_N) \circ \varphi$, the proposition follows. ∎

Proposition 3.7. *Let G be a Lie group and let M be a proper smooth G-manifold without boundary. Assume $F\colon M/G \times I \to M/G$ is a smooth isotopy starting at the identity. Then F_t is normally transverse, for every $t \in I$.*

Proof. Let $\xi \in M/G$ and let $x \in M$ be such that $\pi_M(x) = \xi$. Let S be a linear slice at x. We identify the open neighbourhood GS/G of ξ with S/G_x. We may assume that $F_t(S/G_x)$ is of form S_0/G_y, where S_0 is a linear slice at some $y \in M$. Since the restriction $F_t|\colon S/G_x \to F_t(S/G_x)$ is a smooth diffeomorphism, it follows that $F_t(S/G_x)$ has the same *primary stratification* (see [1], or p. 43 in [8]) as S/G_x. It follows, by using Theorem 1.5 in [8], that $F_t|$ must map the connected components of the normal type strata of S/G_x in a one-to-one manner onto the connected components of the normal type strata of $F_t(S/G_x)$.

Covering M/G by sets of form S/G_x shows that F_t must permute the connected components of the normal type strata of M/G. Since F_t is isotopic to the identity, it follows that it must preserve the normal type strata. As a normal type strata preserving diffeomorphism, F_t is normally transverse. ∎

4. Equivariant vector fields

Let G be a Lie group and let M be a proper smooth G-manifold. We denote the set of real-linear derivations of $\mathrm{C}^\infty(M/G)$ by $\mathrm{Der}(\mathrm{C}^\infty(M/G))$ and call the elements of $\mathrm{Der}(\mathrm{C}^\infty(M/G))$ *smooth vector fields* on M/G, following the terminology in [8]. An element of $\mathrm{Der}(\mathrm{C}^\infty(M/G))$ is called *strata preserving* if it preserves the ideals of $\mathrm{C}^\infty(M/G)$ vanishing on the various strata of M/G. We denote the collection of strata preserving smooth vector fields on M/G by $\mathcal{X}^\infty(M/G)$, and we denote the smooth vector fields on M by $\mathcal{X}^\infty(M)$.

The group G acts on $\mathrm{C}^\infty(M)$ by

$$G \times \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M), \ \ (g, f) \mapsto gf = f \circ g^{-1}.$$

Moreover, G acts on $\mathcal{X}^\infty(M)$ as follows: Let $D \in \mathcal{X}^\infty(M)$, $f \in \mathrm{C}^\infty(M)$, $g \in G$ and $x \in M$. Then

$$(gD)(f)(x) = D(f \circ g^{-1})(gx).$$

Thus D can be considered as a G-equivariant map $\mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$ if and only if it is a G-fixed point under the action on $\mathcal{X}^\infty(M)$. We denote the fixed point set by $\mathcal{X}^\infty(M)^G$ and call its elements the *G-equivariant smooth vector fields* on M. If $D \in \mathcal{X}^\infty(M)^G$ and $f \in \mathrm{C}^\infty(M)$ is a G-invariant map, then also the map $D(f)$ is G-invariant. The set of G-invariant smooth maps $M \to \mathbb{R}$ is denoted by $\mathrm{C}^\infty_G(M)$.

Let next G be a Lie group, H a closed subgroup of G and M a proper smooth H-manifold. Then $G \times H$ acts on $G \times M$ by

$$(G \times H) \times (G \times M) \to G \times M,\ \ ((g,h),(g_0,x)) \mapsto (gg_0h^{-1}, hx).$$

Moreover, $G \times H$ acts on $\mathrm{C}^\infty(G \times M)$ by

$$(G \times H) \times \mathrm{C}^\infty(G \times M) \to \mathrm{C}^\infty(G \times M),\ \ ((g,h)f)(g_0,x) = f(g^{-1}g_0h, h^{-1}x).$$

Let $D \in \mathcal{X}^\infty(M)^H$. Define

$$D^G \colon \mathrm{C}^\infty(G \times M) \to \mathrm{C}^\infty(G \times M),\ \ D^G(f)(g_0,x) = D(f_{g_0})(x),$$

where $f_{g_0}(x) = f(g_0,x)$, for every $x \in M$. Then $D^G \in \mathcal{X}^\infty(G \times M)^{G\times H}$. Let $\pi \colon G \times M \to G \times_H M$ be the natural projection. Then D^G induces the vector field $\tilde{D} \in \mathcal{X}^\infty(G \times_H M)^G$, where

$$\tilde{D}(f)[g,x] = D^G(f \circ \pi)(g,x),$$

for every $f \in \mathrm{C}^\infty(G \times_H M)$ and for every $(g,x) \in G \times M$.

5. The Results

We begin this section by generalizing the lifting theorem for smooth vector fields ([8], Theorem 0.2) for proper group actions. In [8], the lifting problem was reduced to a local problem ([8], Theorem 3.7). Therefore our generalization can be done by using the differentiable slice theorem and G-invariant partitions of unity, just like in the case of compact G.

Let G be a Lie group and let M be a proper smooth G-manifold. The natural projection $\pi_M \colon M \to M/G$ induces the map

$(\pi_M)_* \colon \mathcal{X}^\infty(M)^G \to \mathrm{Der}(\mathrm{C}^\infty(M/G))$, $D \mapsto D_M$, where $D_M(f)(\xi) = D(f \circ \pi_M)(x)$, for every $f \in \mathrm{C}^\infty(M/G)$ and for every $\xi \in M/G$, where $\xi = \pi_M(x)$.

Lemma 5.1. *Let G be a Lie group and let M be a proper smooth G-manifold. Then $(\pi_M)_*(\mathcal{X}^\infty(M)^G) \subset \mathcal{X}^\infty(M/G)$.*

Proof. The claim is the same as that of Corollary 1.3 in [8] for compact G. The proof of Corollary 1.3 in [8] also works for proper actions. ■

Lemma 5.2. *Let G be a Lie group and let K be a compact subgroup of G. Let M be a smooth K-manifold and $N = G \times_K M$. Then the map $(\pi_N)_*\colon \mathcal{X}^\infty(N)^G \to \mathrm{Der}(\mathrm{C}^\infty(N/G))$ has image $\mathcal{X}^\infty(N/G)$.*

Proof. By Theorem 0.2 in [8], the map

$$(\pi_M)_*\colon \mathcal{X}^\infty(M)^K \to \mathrm{Der}(\mathrm{C}^\infty(M/K))$$

has image $\mathcal{X}^\infty(M/K)$. We identify N/G with M/K and $\mathrm{C}^\infty_G(N)$ with $\mathrm{C}^\infty_K(M)$ in the standard way. Let $D \in \mathcal{X}^\infty(M)^K$ and let $\tilde{D} \in \mathcal{X}^\infty(N)^G$ be the vector field induced by D. Then $(\pi_M)_*(D) = (\pi_N)_*(\tilde{D})$. Since $(\pi_M)_*$ has image $\mathcal{X}^\infty(M/K)$, it follows that $\mathcal{X}^\infty(N/G) \subset (\pi_N)_*(\mathcal{X}^\infty(N)^G)$. Since $(\pi_N)_*(\mathcal{X}^\infty(N)^G) \subset \mathcal{X}^\infty(N/G)$, by Lemma 5.1, the lemma follows. ∎

Theorem 5.3. *Let G be a Lie group and let M be a proper smooth G-manifold. Then the map $(\pi_M)_*\colon \mathcal{X}^\infty(M)^G \to \mathrm{Der}(\mathrm{C}^\infty(M/G))$ has image $\mathcal{X}^\infty(M/G)$.*

Proof. The manifold M has a locally finite cover by smooth tubes $X_i = G \times_{K_i} S_i$, $i \in \mathbb{N}$. By Lemma 5.2, each map

$$(\pi_{X_i})_*\colon \mathcal{X}^\infty(X_i)^G \to \mathrm{Der}(\mathrm{C}^\infty(X_i/G))$$

has image $\mathcal{X}^\infty(X_i/G)$. By Theorem 4.2.4.(4) in [7], there exists a G-invariant smooth partition of unity $\{f_i\colon M \to \mathbb{R}\}_{i\in\mathbb{N}}$ subordinate to $\{X_i\}_{i\in\mathbb{N}}$. (The theorem is stated for manifolds without boundary, but it holds also for manifolds with boundary.) For each i, let $D_i \in \mathcal{X}^\infty(X_i)^G$. Define $f_iD_i \in \mathcal{X}^\infty(M)^G$ by

$$(f_iD_i)(h)(x) = \begin{cases} f_i(x)(D_ih|_{X_i})(x), & \text{if } x \in X_i \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{f}_i\colon M/G \to \mathbb{R}$ be the map induced by f_i, for every i. Let $D' \in \mathcal{X}^\infty(M/G)$. Then there are $D_i \in \mathcal{X}^\infty(X_i)^G$ which induce the restriction of D' to $X_i/G \subset M/G$. For each i, f_iD_i induces $\bar{f}_iD'$. But then $\Sigma_i f_iD_i$ induces $\Sigma_i\bar{f}_iD' = D'$. ∎

Proof of Theorem 1.1. Let's form the pullback $\bar{F}^*N$ of N by $\bar{F}$. Then $(\bar{F}^*N)/G = M/G \times I$. The vector field $(0, \frac{d}{dt})$ on $M/G \times I$ is strata preserving. Thus it follows from Theorem 5.3, that there is a G-equivariant smooth vector field A on $\bar{F}^*N$ such that $\pi_*(A) = (0, \frac{d}{dt})$. Let $(Gx, 0, y) \in \bar{F}_0^*N$. Since $A(Gx, 0, y)$ points into, the maximal integral curve $\delta_{(Gx,0,y)}$ of A at $(Gx, 0, y)$ is defined on $[0, \varepsilon)$ (or $[0, 1]$), for some $\varepsilon > 0$. If $g \in G$, then the maximal integral curve $\delta_{(Gx,0,gy)}$ of A at $(Gx, 0, gy)$ is $g \circ \delta_{(Gx,0,y)}$. Moreover, $\delta_{(Gx,0,y)}$ must have the property that $\delta_{(Gx,0,y)}(t) = (Gx, t, \tilde{y}(t))$, for every $t \in [0, \varepsilon)$, where $\bar{F}_t(Gx) = \pi_N(\tilde{y}(t))$. Assume $\delta_{(Gx,0,y)}$ is defined on $[0, \varepsilon)$, where $\varepsilon \leq 1$. Considering the integral curves of A at points of form (Gx, ε, y) and using the uniqueness of the maximal integral curves yields a contradiction with the maximality of $\delta_{(Gx,0,y)}$. Thus $\delta_{(Gx,0,y)}$ must be defined on $[0, 1]$. It follows that the integral curves at points of $\bar{F}_0^*$ define a G-equivariant smooth map $\phi_A\colon \bar{F}_0^*N \times I \to \bar{F}^*N$, $((Gx, 0, y), t) \mapsto \delta_{(Gx,0,y)}(t)$, whose induced map is the identity map of $M/G \times I$. By Proposition 3.5, there is a canonical smooth G-diffeomorphism $\varphi\colon M \to \bar{F}_0^* N$. Let $\bar{F}^*\colon \bar{F}^*N \to N$ be the projection. Then the diagram

$$\begin{array}{ccccccc} M \times I & \stackrel{\varphi\times\mathrm{id}}{\longrightarrow} & \bar{F}_0^*N \times I & \stackrel{\phi_A}{\longrightarrow} & \bar{F}^*N & \stackrel{\bar{F}^*}{\longrightarrow} & N \\ \pi_M \times \mathrm{id}\downarrow & & \pi_{\bar{F}_0^*N} \times \mathrm{id}\downarrow & & \pi\downarrow & & \downarrow \pi_N \\ M/G \times I & \stackrel{\mathrm{id}}{\longrightarrow} & M/G \times I & \stackrel{\mathrm{id}}{\longrightarrow} & M/G \times I & \stackrel{\bar{F}}{\longrightarrow} & N/G \end{array}$$

commutes and $F = \bar{F}^* \circ \phi_A \circ (\varphi \times \mathrm{id})$ is a smooth lift of $\bar{F}$. Each F_t is normally transverse, by Proposition 3.2.

The second part of the proof is similar to that of Theorem 2.3 of [8]. The reference to Proposition 2.1.3 in [8] should be replaced by a reference to Proposition 3.2. ∎

Proof of Corollary 1.2. By Proposition 3.7, each $\bar{F}_t$ is normally transverse. Thus it follows from Theorem 1.1, that there exists a G-equivariant smooth normally transverse homotopy $F\colon M \times I \to M$ starting at the identity and inducing $\bar{F}$. By Proposition 3.6, each F_t is a smooth G-diffeomorphism. ∎

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