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Multicontact Vector Fields on Hessenberg Manifolds

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Abstract. In 1850, Liouville proved that any C^4 conformal map between domains in $\mathbb{R}^3$ is necessarily the restriction of the action of one element of $O(1,4)$. Cowling, De Mari, Koranyi and Reimann recently prove a Liouville-type result: they defined a generalized contact structure on homogeneous spaces of the type G/P, where G is a semisimple Lie group and P a minimal parabolic subgroup, and they show that the group of "contact" mappings coincides with G. In this paper, we consider the problem of characterizing the "contact" mappings on a natural class of submanifolds of G/P, namely the Hessenberg manifolds.
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1. Introduction

In 1850, Liouville proved that any C^4 conformal map between domains in $\mathbb{R}^3$ is necessarily a composition of translations, dilations and inversions in spheres. This amounts to saying that the group $\mathrm{O}(1,4)$ acts on the sphere S^3 by conformal transformations (and hence locally on $\mathbb{R}^3$, by stereographic projection), and then proving that any conformal map between two domains arises as the restriction of the action of some element of $\mathrm{O}(1,4)$. The same result also holds in $\mathbb{R}^n$ when $n > 3$ (see, for instance, [17]), and with metric rather than smoothness assumptions (see [12]).

A cornerstone in the extension process of Liouville's result is certainly the paper [16] by A. Korányi and H.M. Reimann, where the Heisenberg group $\mathbb{H}^n$ substitutes the Euclidean space and the sphere in $\mathbb{C}^n$ with its Cauchy-Riemann structure substitutes the real sphere. The authors study smooth maps whose differential preserves the contact ("*horizontal*") plane $\mathbb{R}^{2n} \subset \mathbb{H}^n$ and is in fact given by a multiple of a unitary map. These maps are called conformal by Korányi and Reimann. Their theorem states that all conformal maps belong to the group $\mathrm{SU}(1,n)$.

A second step was taken by P. Pansu [18], who proved that in the quaternionic and octonionic case (here the set-up is slightly different: the mappings are globally defined), a Liouville's theorem holds under the sole assumption that the map in question preserves a suitable contact structure of codimension greater than

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one. Similar phenomena have been studied in more general situations: see, e.g., [3], [4], [13], [14].

A remarkable piece of work concerning this circle of ideas is [21], by K. Yamaguchi. His approach is at the infinitesimal level and is based on the theory of G structures, as developed by N. Tanaka [19]. The crucial step in his analysis uses heavily Kostant's Lie algebra cohomology and classification arguments.

It is perhaps fair to say that the latest important contribution in this area is the point of view adopted by Cowling, De Mari, Korányi and Reimann in [6] and [7]. They introduce the notion of multicontact mapping in the context of the homogeneous spaces G/P. Roughly speaking, it refers to a collection of special sub-bundles of the tangent bundle with the property that their sections generate the whole tangent space by repeated brackets. The selection of the special directions is not only required to satisfy this Hörmander-type condition, but it is also dictated by the stratification of the tangent space T_x at each point $x \in$ G/P in terms of restricted root spaces. If for example P is minimal, then T_x can be identified with a nilpotent Iwasawa Lie algebra and therefore it may be viewed as the direct sum of all the root spaces associated to the positive restricted roots. Since a positive root is a sum of simple roots, it is natural to expect that the tangent directions along the simple roots will play a special role. Indeed, it is proved in [7] that, at least in rank greater than one, G acts on G/P by maps whose differential preserves each sub-bundle corresponding to a simple restricted root, or, at worst, it permutes them amongst themselves. It is thus natural to say that $g \in$ G induces a *multicontact mapping*. The main result in [7] is that the converse statement is also true: a locally defined C^2 multicontact mapping on G/P is the restriction of the action of a uniquely determined element $g \in$ G. Hence the boundaries G/P are (in most cases) *rigid*. Their results have a non-trivial overlap with those by Yamaguchi, but are independent of classification and rely on entirely elementary techniques.

In this paper, which is part of my Ph.D. thesis, that I have written under the scientific guidance of Filippo De Mari, we prove a Liouville-type result for a natural class of submanifolds of G/P, namely the Hessenberg manifolds (see [1], [2], [8], [9], [10], [11]). We show that it is possible to define a notion of multicontact mapping (Section 3.), hence of multicontact vector field, on every Hessenberg submanifold $\mathrm{Hess}_{\mathcal{R}}(H)$ of G/P associated to a regular element H in the Cartan subspace $\mathfrak{a}$ of the Lie algebra $\mathfrak{g}$ of G. The Hessenberg combinatorial data, namely the subset $\mathcal{R}$ of the positive restricted roots Σ_+ relative to $(\mathfrak{g}, \mathfrak{a})$ that defines the type of the manifold, single out an ideal $\mathfrak{n}_{\mathcal{C}}$ in the nilpotent Iwasawa subalgebra of $\mathfrak{g}$, labeled by the complement $\mathcal{C} = \Sigma_+ \setminus \mathcal{R}$. By means of a reduction theorem, it is shown that without loss of generality one can work under the assumption that $\mathcal{R}$ contains all the simple restricted roots (Section 4.). In order to avoid certain degeneracies, we assume further that $\mathcal{R}$ contains all height-two restricted roots as well. We prove that the normalizer of $\mathfrak{n}_{\mathcal{C}}$ in $\mathfrak{g}$ modulo $\mathfrak{n}_{\mathcal{C}}$ is naturally embedded in the Lie algebra of multicontact vector fields on $\mathrm{Hess}_{\mathcal{R}}(H)$ (Section 4.). In Section 5. the main result is proved (Theorem 5.2). It is shown that if the data $\mathcal{R}$ satisfy the property of encoding a finite number of positive root systems, each corresponding to an Iwasawa nilpotent algebra, then the above quotient actually coincides with the Lie algebra of multicontact vector fields on $\mathrm{Hess}_{\mathcal{R}}(H)$. This situation covers a wide variety of cases (for example all Hessenberg data in a root system of type

A_ℓ) but not all of them. Explicit exceptions are given in the C_ℓ case. One of the main motivations for the present study is the observation that $\mathrm{Hess}_{\mathcal{R}}(H)$ can be realized locally as a stratified nilpotent group that is *not* always of Iwasawa type. Hence our work is an extension of the theories of multicontact maps developed thus far.

2. Notation and preliminaries

We shall work with real simple Lie algebras, although most of what we do holds, *mutatis mutandis*, for semisimple Lie algebras. Let $\mathfrak{g}$ be a simple Lie algebra with Killing form B and Cartan involution θ. Let $\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and denote by Σ the set of restricted roots, a subset of the dual $\mathfrak{a}'$ of $\mathfrak{a}$. Choose an ordering on $\mathfrak{a}'$, this defining the subsets Σ_+ and $\Delta = \{\delta_1, \dots, \delta_r\}$ of positive and simple positive restricted roots. Since we shall always work with the restricted root spaces, we forget the adjective "restricted" when it is referred to roots. Every positive root α can be written as $\alpha = \sum_{i=1}^r n_i \delta_i$ for uniquely defined non-negative integers $n_1, \dots, n_r$, and the positive integer $\mathrm{ht}(\alpha) = \sum_{i=1}^r n_i$ is called the *height* of α. It is well-known that there is exactly one root ω, called the highest root, that satisfies $\omega \succ \alpha$ (strictly) for every other root α. The root space decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$, where $\mathfrak{m} = \{X \in \mathfrak{k} : [X, H] = 0,\, H \in \mathfrak{a}\}$. The nilpotent Iwasawa algebra $\mathfrak{n}$ is $\bigoplus_{\gamma \in \Sigma_+} \mathfrak{g}_\gamma$ and we denote with $\overline{\mathfrak{n}}$ its counterpart $\theta(\mathfrak{n})$. It is well known that $\mathfrak{n}$ is a stratified Lie algebra in the usual sense, that is $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$, where $\mathfrak{n}_i = \bigoplus_{\mathrm{ht}(\gamma)=i} \mathfrak{g}_\gamma$, $i = 1, \dots, \mathrm{ht}(\omega)$.

Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. Let $\mathrm{P} = \mathrm{MA\overline{N}}$ be a minimal parabolic subgroup of G. We may assume that the center of G is trivial. Indeed, if Z is the center of G, then $\mathrm{Z} \subset \mathrm{P}$, and so G/P and (G/Z)/(P/Z) may be identified. Moreover, the action of G on G/P factors to an action of G/Z. Among all groups with trivial centers and the same Lie algebra $\mathfrak{g}$, the largest is the group $\mathrm{Aut}(\mathfrak{g})$ of all automorphisms of $\mathfrak{g}$, and the smallest is the group $\mathrm{Int}(\mathfrak{g})$ of the inner automorphisms of $\mathfrak{g}$, the connected component of the identity of $\mathrm{Aut}(\mathfrak{g})$. Any group G_1 such that $\mathrm{Int}(\mathfrak{g}) \subseteq \mathrm{G}_1 \subseteq \mathrm{Aut}(\mathfrak{g})$, with corresponding minimal parabolic subgroup P_1, gives rise to the same space, meaning that $\mathrm{G}_1/\mathrm{P}_1$ may be identified with $\mathrm{Aut}(\mathfrak{g})/\mathrm{P}$ if P is a minimal parabolic subgroup of $\mathrm{Aut}(\mathfrak{g})$. For the purposes of this paper the correct assumption is that G is connected and centerless, and hence we can assume $\mathrm{G} = \mathrm{Int}(\mathfrak{g})$ and that P is a minimal parabolic subgroup of G.

By means of the Bruhat decomposition ([15],Ch.VII, Sec.4) the group N may be seen as open and dense in G/P. Indeed, if we denote by b the base point in G/P (that is, the identity coset), the Bruhat lemma states that the mapping $\psi : \mathrm{N} \to \mathrm{G/P}$ defined by $\psi(n) = nb$ is injective and its image is dense and open. The differential ψ_* then maps $\mathfrak{n}$, the tangent space to N at the identity e, onto T_b, the tangent space to G/P at the base point. When δ is a simple root, we denote by $S_{\delta,b}$ the subspace $\psi_*(\mathfrak{g}_\delta)$ of T_b. In Lemma 2.2 of [7] it is shown that the action of any element $p \in \mathrm{P}$ on G/P induces an action p_* on the tangent space T_b which in turn induces an action $\psi_*^{-1} p_* \psi_*$ on $\mathfrak{n}$. This last action preserves all the spaces $\mathfrak{g}_\delta$ for simple δ. This lemma allows us to identify $\mathfrak{n}$ with the tangent space T_x at any point x in G/P, and to identify the subspaces $\mathfrak{g}_\delta$ of $\mathfrak{n}$ with subspaces $S_{\delta,x}$

of T_x. Indeed we may write x as gb, where $g \in \mathrm{G}$; then the images $g_*\psi_*\mathfrak{g}_\delta$ are well defined, and independent of the representative g of the coset, although the identification $\mathfrak{g}_\delta \to S_{\delta,x}$ does depend on the representative. Since we never make use of the explicit identification, we shall always write $\mathfrak{g}_\delta$ in place of $S_{\delta,x}$. This interpretation of the tangent space to G/P allows the definition of multicontact mapping as it is given in [7].

3. Multicontact mappings on Hessenberg manifolds

Let $\mathcal{R}$ be some proper subset of the set of the positive roots Σ_+. We call it of Hessenberg type if it satisfies the following property:

if $\alpha \in \mathcal{R}$ and β is any negative root such that $\alpha+\beta \in \Sigma_+$, then $\alpha+\beta \in \mathcal{R}$.

Write $\mathfrak{b}_\mathcal{R} = \mathfrak{a} \oplus \overline{\mathfrak{n}} \oplus \bigoplus_{\gamma\in\mathcal{R}} \mathfrak{g}_\gamma$ and fix a regular element H in the Cartan subspace $\mathfrak{a}$. Then

$$\mathrm{Hess}_\mathcal{R}(H) = \{\langle g\rangle_\mathrm{P} \in \mathrm{G/P} : \mathrm{Ad}g^{-1}H \in \mathfrak{b}_\mathcal{R}\}.$$

Denote with m_α the multiplicity of the root α, that is the dimension of the root space $\mathfrak{g}_\alpha$.

Proposition 3.1. [9] $\mathrm{Hess}_\mathcal{R}(H)$ *is a smooth submanifold of* G/P *of dimension* $\sum_{\alpha\in\mathcal{R}} m_\alpha$.

Denote by $\mathcal{C}$ the complement in Σ_+ of $\mathcal{R}$. Any Hessenberg manifold can be locally viewed as an algebraic submanifold of N. More precisely, the intersection of N with a Hessenberg manifold is defined by a set of linear equations of the form

$$p_{\alpha,j}(x) = 0, \qquad \alpha \in \mathcal{C}, j = 1, \dots, m_\alpha, \tag{1}$$

where

$$p_{\alpha,j} = \alpha(H)x_{\alpha,j} + (\text{terms containing } x_{\beta,i}, \text{ with } \mathrm{ht}(\beta) < \mathrm{ht}(\alpha)).$$

It is rather easy to check that

$$\mathfrak{n}_\mathcal{C} = \bigoplus_{\alpha\in\mathcal{C}} \mathfrak{g}_\alpha \tag{2}$$

is an ideal in $\mathfrak{n}$.

We ask ourselves how to relate with $\mathfrak{n}$ the tangent space to some point of $\mathrm{Hess}_\mathcal{R}(H)$. The coefficients of the polynomials (1) depend on H and more is true: those that are not zero are in fact given by functions that never vanish on the set of regular elements in $\mathfrak{a}$. Thus, the slice S of N obtained by setting $x_{\alpha,j} = 0$ if $\alpha \in \mathcal{C}$ is diffeomorphic to $\mathrm{Hess}_\mathcal{R}(H) \cap \mathrm{N}$ for every regular element H. The graph mapping $\phi : (\{x_{\beta,k}\}_{\beta\in\mathcal{R}}, 0) \longmapsto (\{x_{\beta,k}\}_{\beta\in\mathcal{R}}, \{p_{\alpha,j}(x_{\beta,k})\}_{\alpha\in\mathcal{C}})$ gives the diffeomorphism. Consider the basis $\{X_{\alpha,j} : \alpha \in \Sigma_+, 1 \le j \le m_\alpha\}$ of left-invariant vector fields on N, where

$$X_{\alpha,j}(n) = (l_n)_{*e} \frac{\partial}{\partial x_{\alpha,j}}\bigg|_e,$$

and write

$$X_{\alpha,j} = \sum_{\gamma\in\Sigma_+}\sum_{k=1}^{m_\gamma} a_{\gamma,k}^{\alpha,j}\frac{\partial}{\partial x_{\gamma,k}},$$

where $a_{\gamma,k}^{\alpha,j}$ are some smooth functions on N. If $\alpha = \sum_{\delta\in\Delta} a_\delta\delta$ and $\beta = \sum_{\delta\in\Delta} b_\delta\delta$ are two positive roots, we write $\alpha \preceq \beta$ if $a_\delta \leq b_\delta$ for all $\delta \in \Delta$. We say that $\alpha_1 + \cdots + \alpha_n$ is a *chain* if each α_j and each partial sum $\alpha_1 + \cdots + \alpha_j$ is a root for all $j = 1, \ldots, n$. Ordered pairs of roots can be joined by chains:

Lemma 3.2. [7] *Let α and β be distinct positive roots and suppose that $\alpha \succeq \beta$. Then there exist simple roots $\delta_1, \ldots, \delta_p$ such that $\alpha = \beta + \delta_1 + \cdots + \delta_p$ is a chain.*

Lemma 3.3. *For every root $\alpha \in \Sigma_+$ and $j = 1, \ldots, m_\alpha$ we have*

$$a_{\gamma,k}^{\alpha,j} = \begin{cases} 0 & \text{if } \mathrm{ht}(\alpha) \geq \mathrm{ht}(\gamma) \text{ and } \alpha \neq \gamma \\ 0 & \text{if } \alpha = \gamma \text{ and } k \neq j \\ 1 & \text{if } \alpha = \gamma \text{ and } k = j \\ P & \text{if } \mathrm{ht}(\alpha) < \mathrm{ht}(\gamma), \end{cases} \tag{3}$$

where P is a polynomial that does not vanish only if $\alpha \preceq \gamma$. In this case, it depends only on those variables labeled by those roots $\alpha_1, \cdots, \alpha_q$ for which $\alpha + \alpha_1 + \cdots + \alpha_q = \gamma$ is a chain. This implies that

$$X_{\alpha,j} = \sum_{\gamma\in\mathcal{C}}\sum_{k=1}^{m_\gamma} a_{\gamma,k}^{\alpha,j}\frac{\partial}{\partial x_{\gamma,k}}, \tag{4}$$

for every $\alpha \in \mathcal{C}$.

Proof. The proof of the above statements follows from a direct calculation that arises from the left-invariance and that involves the Baker-Campbell-Hausdorff formula. ∎

For every $\alpha \in \Sigma_+$, and $1 \leq j \leq m_\alpha$, consider the vector field $\overline{X}_{\alpha,j}$ whose (γ, k) component is

$$r_{\gamma,k}^{\alpha,j} = \begin{cases} a_{\gamma,k}^{\alpha,j} & \text{if } \gamma \in \mathcal{R} \text{ and } k = 1, \cdots, m_\gamma \\ 0 & \text{otherwise.} \end{cases}$$

The $\overline{X}_{\alpha,j}$ are vector fields on S, and from (4) $\overline{X}_{\alpha,j} = 0$ for every $\alpha \in \mathcal{C}$. Moreover, (3) implies that the set $\{\overline{X}_{\alpha,j} : \alpha \in \mathcal{R}, j = 1, \cdots, m_\alpha\}$ is a basis of the tangent space at any point of S. Indeed, writing the matrix of the coefficients of $\{\overline{X}_{\alpha,j} : \alpha \in \mathcal{R}, j = 1, \cdots, m_\alpha\}$, ordering the roots according to any lexicographic order, we obtain a triangular matrix with ones along the diagonal. Hence $\{\phi_*(\overline{X}_{\alpha,j}) : \alpha \in \mathcal{R}, j = 1, \cdots, m_\alpha\}$ is a basis of the tangent space at all points of an open set of $\mathrm{Hess}_{\mathcal{R}}(H)$.

Denote by $\mathfrak{X}(\mathrm{N})$ the Lie algebra of all smooth vector fields on N.

Proposition 3.4. *Given X and $Y \in \mathfrak{X}(\mathrm{N})$, the following formula holds at every point $n \in \mathrm{S}$*

$$[\overline{X}, \overline{Y}](n) = \overline{[X, Y]}(n).$$

Proof. Let X and $Y \in \mathfrak{X}(\mathrm{N})$ and write $X = \overline{X} + \underline{X}$, where

$$\overline{X} := \sum_{\beta\in\mathcal{R}} \sum_{i=1}^{m_\beta} r_{\beta,i} \frac{\partial}{\partial x_{\beta,i}}, \qquad \underline{X} := \sum_{\gamma\in\mathcal{C}} \sum_{k=1}^{m_\gamma} c_{\gamma,k} \frac{\partial}{\partial x_{\gamma,k}},$$

and similarly $Y = \overline{Y} + \underline{Y}$. Then

$$\overline{[X,Y]}(n) = \overline{[\overline{X},\overline{Y}]}(n) + \overline{[\overline{X},\underline{Y}]}(n) + \overline{[\underline{X},\overline{Y}]}(n) + \overline{[\underline{X},\underline{Y}]}(n).$$

Clearly $\overline{[\overline{X},\overline{Y}]} = [\overline{X},\overline{Y}]$. Moreover, $\overline{[\overline{X},\underline{Y}]} = \overline{[\underline{X},\overline{Y}]} = 0$, because when expanded in terms of partial derivatives, each of the above brackets contains only coefficients of the form $(\partial/\partial x_{\gamma,k}) r_{\beta,i}$, which vanish whenever $\gamma \in \mathcal{C}$ and $\beta \in \mathcal{R}$ because of (3). Finally, $\overline{[\underline{X},\underline{Y}]} = 0$, because in $[\underline{X},\underline{Y}]$ only the coefficients of components labeled by $\mathcal{C}$ will appear, but they become zero once we project them on S. ■

Let $\bar{\mathfrak{g}}_\delta = \operatorname{span}\{\overline{X}_{\delta,i} : i = 1, \cdots, m_\delta\}$. The proposition above implies that the vector fields in the family $\{\bar{\mathfrak{g}}_\delta\}_{\delta\in\Delta_\mathcal{R}}$, $\Delta_\mathcal{R} = \Delta \cap \mathcal{R}$ generate at each point the tangent space of S by the Lie brackets. Let $\mathcal{A}, \mathcal{B}$ be some open subsets of $\mathrm{Hess}_\mathcal{R}(H)$. Without loss of generality, we can assume $\mathcal{A}, \mathcal{B} \subset (\mathrm{N} \cap \mathrm{Hess}_\mathcal{R}(H))$. Let $f : \mathcal{A} \to \mathcal{B}$ be a diffeomorphism. We say that f is a *multicontact* map if

$$f_*(\phi_*(\bar{\mathfrak{g}}_\delta)) \subseteq \phi_*(\bar{\mathfrak{g}}_\delta), \qquad \text{for every simple root } \delta \text{ in } \mathcal{R}.$$

4. Multicontact vector fields

Lifting the multicontact conditions to the infinitesimal level. Since all Hessenberg manifolds corresponding to different choices of regular H give rise to the same slice S, the group of multicontact maps does not depend on H. Therefore, from now on we focus our attention on the slice S of N. Fix an open set $\mathcal{A}$ of S. We lift the problem to the Lie algebra level, by considering multicontact vector fields, that is, vector fields F on $\mathcal{A}$ whose local flow $\{\psi_t^F\}$ consists of multicontact maps. If $\delta \in \Delta_\mathcal{R}$, then

$$\frac{d}{dt}(\psi_t^F)_*(\overline{X}_\delta)\Big|_{t=0} = -\mathcal{L}_F(\overline{X}_\delta) = [\overline{X}_\delta, F],$$

where $\mathcal{L}$ denotes the Lie derivative. Hence a smooth vector field F on $\mathcal{A}$ is a multicontact vector field if and only if

$$[F, \bar{\mathfrak{g}}_\delta] \subseteq \bar{\mathfrak{g}}_\delta \text{ for every } \delta \in \Delta_\mathcal{R}. \tag{5}$$

We write a vector field on $\mathcal{A}$ as

$$F = \sum_{\gamma\in\mathcal{R}} \sum_{j=1}^{m_\gamma} f_{\gamma,j} \overline{X}_{\gamma,j}, \tag{6}$$

where $f_{\gamma,j}$ are smooth functions on $\mathcal{A}$. Condition (5) becomes

$$[F, \overline{X}_{\delta,i}] = \sum_{k=1}^{m_\delta} \lambda_{\delta,k}^i \overline{X}_{\delta,k}, \qquad \delta \in \Delta_\mathcal{R},\ i = 1, \ldots, m_\delta,$$

where $\{\lambda^i_{\delta,k}\}$ is a set of smooth functions. We can write the multicontact conditions as the system of equations

$$\sum_{\gamma\in\mathcal{R}}\sum_{j=1}^{m_\gamma}\overline{X}_{\delta,i}(f_{\gamma,j})\overline{X}_{\gamma,j}+\sum_{\gamma\in\mathcal{R}}\sum_{j=1}^{m_\gamma}\left(\sum_{l=1}^{m_{\gamma-\delta}}c^{ilj}_{\delta,\gamma-\delta}f_{\gamma-\delta,l}\right)\overline{X}_{\gamma,j}=-\sum_{j=1}^{m_\delta}\lambda^i_{\delta,j}\overline{X}_{\delta,j},$$

as δ varies in $\Delta_{\mathcal{R}}$ and $i=1,\ldots,m_\delta$. Equivalently, F is a multicontact vector field on $\mathcal{A}$ if and only if for all $\gamma\in\mathcal{R}$ and some functions $\{\lambda^i_{\delta,j}\}$ the following equations are satisfied on $\mathcal{A}$:

$$\begin{cases}\overline{X}_{\delta,i}(f_{\delta,j})=-\lambda^j_{\delta,i} & \\ \overline{X}_{\delta,i}(f_{\gamma,j})=0 & \text{if } \gamma-\delta\notin\Sigma_+\cup\{0\}\\ \overline{X}_{\delta,i}(f_{\gamma,j})+\sum_{l=1}^{m_{\gamma-\delta}}c^{ilj}_{\delta,\gamma-\delta}f_{\gamma-\delta,l}=0 & \text{if } \gamma-\delta\in\Sigma_+\end{cases}\tag{7}$$

for all the simple roots δ in $\Delta_{\mathcal{R}}$ and $1\le i,j\le m_\delta$. We may clearly forget the equation $\overline{X}_{\delta,i}(f_{\delta,j})=-\lambda^j_{\delta,i}$ because $\lambda^j_{\delta,i}$ is arbitrary.

We write $MC(\mathrm{N})$ and $MC(\mathrm{S})$ for the Lie algebra of multicontact vector fields on some open subset of N and S respectively. If $F\in MC(\mathrm{S})$ is as in (6), then $\overline{X}_{\delta,i}f_{\gamma,j}=X_{\delta,i}f_{\gamma,j}$. Thus, from now on we shall write $X_{\delta,i}$ in place of $\overline{X}_{\delta,i}$ whenever treating multicontact vector fields, if no ambiguity arises.

Let $\mathcal{C}$ be the complement in Σ_+ of some Hessenberg type set. We say that a function f on N is *$\mathcal{C}$-independent* if it does not depend on the coordinates labeled by $\mathcal{C}$. From (4) it follows that if $\mathcal{R}$ is a Hessenberg type set of roots and $\gamma\in\mathcal{C}$, then a (basis) left invariant vector field $X_{\gamma,k}$ on N does not depend on the partial derivative vector fields that are labeled by the positive roots in $\mathcal{C}$. This implies in particular that the system of equations

$$X_{\gamma,k}f=0\qquad\text{for every }\gamma\in\mathcal{C}\text{ and }k=1,\ldots,m_\gamma\tag{8}$$

is equivalent to the $\mathcal{C}$-independence, namely to

$$\frac{\partial}{\partial x_{\gamma,k}}f=0\qquad\text{for every }\gamma\in\mathcal{C}\text{ and }k=1,\ldots,m_\gamma.\tag{9}$$

Dark zones. We split (7) into suitable independent subsystems, each defining multicontact vector fields on some Hessenberg manifold of lower dimension, and we show that we can focus our attention to only one of them at a time. Call a positive root μ in $\mathcal{R}$ *maximal* if $\mu+\alpha\notin\mathcal{R}$ for any other root $\alpha\in\Sigma_+$. Since, by definition of $\mathcal{R}$, $\mu+\alpha\notin\mathcal{R}$ if $\alpha\in\mathcal{C}$, it suffices to check maximality for all $\alpha\in\mathcal{R}$. Denote by $\mathcal{R}_M$ the set of maximal roots. For a fixed $\mu\in\mathcal{R}_M$, we call *shadow* of μ the set

$$S_\mu=\{\alpha\in\mathcal{R}:\alpha\preceq\mu\}.$$

It is not difficult to show that the union $\bigcup_{\mu\in\mathcal{R}_M}S_\mu$ covers $\mathcal{R}$.

We partition $\mathcal{R}$ into the disjoint union of *dark zones*, a dark zone being a connected component of $\mathcal{R}$ in a loose sense, that is, a maximal union of shadows $\mathcal{Z}=\cup_{i=1}^k\mathcal{S}_{\mu_i}$ with the property that either $k=1$ or any $\mathcal{S}_{\mu_i}$ intersects at least another $\mathcal{S}_{\mu_j}$ in the same dark zone. By their very definition, dark zones are disjoint. This will allows us to reduce the problem of solving (7) to the problem of solving several simpler systems, each naturally associated to a dark zone.

Suppose that $\mathcal{Z}_1, \ldots, \mathcal{Z}_p$ is a numbering of the dark zones of $\mathcal{R}$. Given $F \in \mathfrak{X}(\mathrm{S})$ as in (6), we write $F = \sum_{i=1}^{p} F_i$, where $F_i = \sum_{\gamma \in \mathcal{Z}_i} \sum_{j=1}^{m_\gamma} f_{\gamma,j} \overline{X}_{\gamma,j}$. Clearly, each F_i is itself a vector field in $\mathfrak{X}(\mathrm{S})$. Since F_i picks the components of F along the directions labeled by $\mathcal{Z}_i$, it is natural to consider the sub-slice of S that corresponds to it, as we now explain.

Fix a dark zone $\mathcal{Z}$. The set of roots contained in $\mathcal{Z}$ generate the positive set of an irreducible root system, say $\Sigma_+(\mathcal{Z})$, and the corresponding Lie algebra

$$\mathfrak{n}(\mathcal{Z}) = \bigoplus_{\beta \in \Sigma_+(\mathcal{Z})} \mathfrak{g}_\beta$$

is a nilpotent Iwasawa algebra. The roots in $\mathcal{Z}$ play, within $\Sigma_+(\mathcal{Z})$, the role of a Hessenberg set of roots. Also, $\mathfrak{n}(\mathcal{Z})$ is a subalgebra of $\mathfrak{n}$ and we may consider the (connected, simply connected, nilpotent) Lie subgroup $\mathrm{N}(\mathcal{Z})$ of N whose Lie algebra is $\mathfrak{n}(\mathcal{Z})$. Thus, if $\mathcal{Z}$ is a dark zone we write

$$S_{\mathcal{Z}} = \{n \in \mathrm{N} : x_{\gamma,k} = 0 \ \text{ if } \ \gamma \notin \mathcal{Z}\}.$$

Coming back to the decomposition $\mathcal{R} = \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_p$, we write for simplicity S_i in place of $S_{\mathcal{Z}_i}$. We prove the following reduction result.

Theorem 4.1. *If $F \in MC(\mathrm{S})$, then $F_i \in MC(\mathrm{S}_i)$ for all $i = 1, \ldots, p$. Conversely, given $G_i \in MC(\mathrm{S}_i)$ with $i = 1, \ldots, p$, then $\sum_i G_i \in MC(\mathrm{S})$.*

The proof requires some remarks, that we state in the next lemmas.

Lemma 4.2. *Let $\mathcal{Z} \subset \mathcal{R}$ be a dark zone and let $\alpha \in \mathcal{Z}$. The (γ, k) component of the vector field $X_{\alpha,j}$ is zero for every $\gamma \in \mathcal{R} \setminus \mathcal{Z}$.*

Proof. Suppose $\alpha \in \mathcal{Z}$, $\gamma \in \mathcal{R} \setminus \mathcal{Z}$ and suppose the (γ, k)-component of the vector field $X_{\alpha,j}$ is not zero. By Lemma 3.3, there exist roots $\alpha_1, \ldots, \alpha_q$ such that $\alpha + \alpha_1 + \cdots + \alpha_q = \gamma$ is a chain, so that in particular $\gamma - \alpha_q - \cdots - \alpha_j$ is also a root for $j = 1, \ldots, q-1$. Now, since $\gamma \in \mathcal{R}$, then $\gamma \in S_\mu$ for some maximal root μ. Therefore $\alpha = \gamma - \alpha_q - \cdots - \alpha_1 \in S_\mu$. This implies that both α and γ belong to the same shadow, and hence to the same dark zone, that is a contradiction. ∎

Lemma 4.3. *The coefficients of a multicontact vector field F are determined by its $\mathfrak{g}_\mu$ components, as μ varies in $\mathcal{R}_M$.*

Proof. The proof of this statement is analogous to the proof of Proposition 3.3 of [7]. ∎

Lemma 4.4. [7] *Let $\alpha, \beta \in \Sigma$ such that $\alpha + \beta$ is a root, then*

$$\{[X, Y] : X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta\} = \mathfrak{g}_{\alpha+\beta},$$

and $\{Z \in \mathfrak{g}_\beta : [\mathfrak{g}_\alpha, Z] = \{0\}\} = \{0\}$.

Lemma 4.3 suggests a hierarchic structure of the equations (7). In particular, if $\gamma + \delta_1 + \cdots + \delta_s = \alpha$ is a chain, there exist vector fields $X_1 \in \mathfrak{g}_{\delta_1}, \ldots, X_s \in \mathfrak{g}_{\delta_s}$ such that the differential monomial $X_1 \cdots X_s$ maps a α-component to a γ-component of a vector field whose coefficients solve (7). The next result follows from Lemma 4.4.

Lemma 4.5. *Let $F \in MC(\mathrm{S})$ be as in (6). Then $Xf_{\gamma,j} = 0$ for every $\gamma \in \mathcal{S}_\mu$, every $j = 1, \ldots, m_\gamma$ and every $X \in \mathfrak{g}_\alpha$ with $\alpha \notin \mathcal{S}_\mu$.*

Proof. If $\alpha \notin \mathcal{S}_\mu$, then it is either out of $\mathcal{R}$ or it is in some other shadow. If $\alpha \in \mathcal{C}$, then $Xf_{\gamma,j} = 0$ by (8).

Assume $\alpha \in \mathcal{R}$. It is enough to prove the statement for $\gamma = \mu$. Indeed, suppose the result true for all $f_{\mu,j}$'s. Then, by the equivalence of (8) and (9), these functions are $(\Sigma_+ \setminus \mathcal{S}_\mu)$-independent, because $\mathcal{S}_\mu$ is a Hessenberg type subset. If $\gamma + \delta_1 + \cdots + \delta_p = \mu$ is a chain, then by Lemma 4.3 there exist vector fields $X_1, \ldots, X_p$ in $\mathfrak{g}_{\delta_1}, \ldots, \mathfrak{g}_{\delta_p}$ such that $X_1 \cdots X_p f_{\mu,j} = f_{\gamma,k}$. Each X_i, $i = 1, \ldots, p$, has the form calculated in Lemma 3.3, that is

$$X_i = \sum_{\alpha \in \Sigma_+} \sum_{j=1}^{m_\alpha} a^i_{\alpha,j} \frac{\partial}{\partial x_{\alpha,j}},$$

where $a_{\alpha,j}$ is a nonzero polynomial only if there exists a chain of roots going from δ_i to α. In this case $a^i_{\alpha,j}$ is a polynomial in the variables $\{x_{\beta,l}\}$ with $\beta \prec \alpha$. In particular this holds for $i = p$ and we show next that this forces $X_p f_{\mu,j}$ to be $(\Sigma_+ \setminus \mathcal{S}_\mu)$-independent. Indeed, if $a^p_{\alpha,j}$ depends on some variable in $(\Sigma_+ \setminus \mathcal{S}_\mu)$, then $\alpha \in (\Sigma_+ \setminus \mathcal{S}_\mu)$ and therefore $\partial f_{\mu,j}/\partial x_{\alpha,k} = 0$ for all $k = 1, \ldots, m =_\alpha$. Hence all coefficients $f_{\gamma,j}$ with $\mathrm{ht}(\gamma) = \mathrm{ht}(\mu) - 1$ are $(\Sigma_+ \setminus \mathcal{S}_\mu)$-independent. By iteration, the conclusion holds for every possible height, thus for every γ.

It remains to be proved that the lemma is true for $f_{\mu,i}$. If α is simple, then it is clear by (7) that $Xf_{\mu,j} = 0$. Let now $\alpha = \delta_1 + \cdots + \delta_p$ be a non simple root in $\mathcal{R} \setminus \mathcal{S}_\mu$. Then there exists $\delta \in \{\delta_1, \ldots, \delta_p\}$ such that $\delta \notin \mathcal{S}_\mu$, for otherwise $\alpha \succ \mu$ and μ would not be maximal. By Lemma 4.4 there exist vector fields $X_1, \ldots, X_p$ in $\mathfrak{g}_{\delta_1}, \ldots, \mathfrak{g}_{\delta_p}$, respectively, such that $X = [X_p, [\ldots, [X_2, X_1]] \ldots]$. Then there exists a set Λ of permutations of p elements such that

$$[X_p, [\ldots, [X_2, X_1]] \ldots] f_{\mu,j} = (\sum_{\lambda \in \Lambda} c_\lambda X_{\lambda(1)} \cdot \cdots \cdot X_{\lambda(p)}) f_{\mu,j},$$

for some costants c_λ. Let $h \in \{1, \ldots, p\}$ be the largest index such that $\delta_{\lambda(h-1)} \notin \mathcal{S}_\mu$, so that clearly $\delta_{\lambda(k)}$ is in $\mathcal{S}_\mu$ for all $k \geq h$. We show that each differential monomial that appears in the sum of the right hand side is zero on $f_{\mu,j}$. Consider $X_{\lambda(i)} \ldots X_{\lambda(p)}$, with $i \geq h$. Three possible cases arise.

(i) $\mu - \delta_{\lambda(p)} - \cdots - \delta_{\lambda(i)} = 0$, so that $\mu = \delta_{\lambda(p)} + \cdots + \delta_{\lambda(i)}$. In this case α is the sum of μ and some other simple roots. Hence α is a root in $\mathcal{R}$ greater than μ, a contradiction.

(ii) There exists $i \geq h$ such that $\mu - \delta_{\lambda(p)} - \cdots - \delta_{\lambda(i+1)}$ is a positive root and $\mu - \delta_{\lambda(p)} - \cdots - \delta_{\lambda(i)}$ is not a root. In this case, from Lemma 4.3 and the remark thereafter, the differential monomial $X_{\lambda(i+1)} \cdot \cdots \cdot X_{\lambda(p)}$ maps $f_{\mu,j}$ into a component that belongs to the root space associated to $\mu - \delta_{\lambda(p)} - \cdots - \delta_{\lambda(i+1)}$, say g. Since $\mu - \delta_{\lambda(p)} - \cdots - -\delta_{\lambda(i+1)} - \delta_{\lambda(i)}$ is not a root, $X_{\lambda(i)} g = 0$ by (7).

(iii) $\mu - \delta_{\lambda(p)} - \cdots - \delta_{\lambda(i)}$ is a root for all $i \geq h$. Again the differential monomial $X_{\lambda(h)} \ldots X_{\lambda(p)}$ maps $f_{\mu,j}$ into a component along the root space labeled by

$\mu - \delta_{\lambda(p)} - \cdots - \delta_{\lambda(h)}$. But $\mu - \delta_{\lambda(p)} - \cdots - \delta_{\lambda(h)} - \delta_{\lambda(h-1)}$ is not a root, for otherwise $\delta_{\lambda(h-1)}$ would lie in $\mathcal{S}_\mu$. Therefore we can conclude as in the previous case. Thus $X_{\lambda(h-1)} \dots X_{\lambda(p)}$ maps the function $f_{\mu,j}$ to zero.

■

Proof of Theorem 4.1.

"⇒". Lemma 4.5 applies in particular to each dark zone, in the sense that a coefficient $f_{\gamma,k}$ of a multicontact vector field on S is annihilated by those left invariant vector fields corresponding to the roots that do not belong to the dark zone where γ lies. Since each dark zone plays the rôle of a Hessenberg set of roots,= its complement defines an ideal in $\mathfrak{n}$, namely

$$\mathfrak{n}_{\mathcal{Z}^c} = \bigoplus_{\alpha \in \Sigma_+ \setminus \mathcal{Z}} \mathfrak{g}_\alpha,$$

where $\mathcal{Z}^c = \Sigma_+ \setminus \mathcal{Z}$. The corresponding nilpotent Lie group admits the set $\{X_{\alpha,j} : \alpha \in \Sigma_+ \setminus \mathcal{Z}\}$ as a basis for its tangent space at each point. From (4) in Lemma 3.3, all these vector fields depend on the coordinate vector fields labeled by the positive roots in $\Sigma_+ \setminus \mathcal{Z}$. Recall in particular that from (8) and (9)

$$X_{\gamma,k} f = 0 \text{ for all } \gamma \notin \mathcal{Z} \iff \frac{\partial}{\partial x_{\gamma,k}} f = 0 \text{ for all } \gamma \notin \mathcal{Z}.$$

This fact, toghether with Lemma 4.5, tells us that the coefficients of the vector field F_i are functions on S_i, that is, they are $(\mathcal{R} \setminus \mathcal{Z})$-independent. Moreover, by Lemma 4.2, the projections $\overline{X}_\delta$ onto the tangent space at each point of S are in fact projections on the tangent space of S_i. Therefore $F_i \in \mathfrak{X}(\mathrm{S}_i)$. Hence F_i is in $MC(\mathrm{S}_i)$ if and only if

$$\begin{cases} \overline{X}_{\delta,i}(f_{\gamma,j}) = 0 & \gamma - \delta \notin \Sigma_+ \cup \{0\} \\ \overline{X}_{\delta,i}(f_{\gamma,j}) + \sum_{l=1}^{m_{\gamma-\delta}} c_{\delta,\gamma-\delta}^{ilj} f_{\gamma-\delta,l} = 0 & \gamma - \delta \in \Sigma_+, \end{cases} \tag{10}$$

with $\delta \in \Delta \cap \mathcal{Z}_i$ and $\gamma \in \mathcal{Z}_i$. We conclude by observing that these equations are satisfied by assumption.

"⇐". Each vector field G_i can be naturally viewed as a vector field on S. Furthermore, since each G_i satisfies the system of equations (10), then the vector field $\sum_i G_i$ satisfies the system (7). Thus, it defines a multicontact vector field on S. This concludes the proof of the theorem. ■

Theorem 4.1 allows us to assume that $\mathcal{R}$ contains all simple roots, and that it consists of exactly one dark zone .

A set of solutions. In [7], the authors determine the multicontact vector fields on the Iwasawa group N, by solving a system of differential equations similar to (7). In particular, if $V = \sum_{\gamma \in \Sigma_+} \sum_{j=1}^{m_\gamma} v_{\gamma,j} X_{\gamma,j}$ is a vector field on N, then V is of multicontact type if it satisfies the following system of equations

$$\begin{cases} X_{\delta,i}(v_{\gamma,j}) = 0 & \text{if } \gamma - \delta \notin \Sigma_+ \cup \{0\} \\ X_{\delta,i}(v_{\gamma,j}) + \sum_{l=1}^{m_{\gamma-\delta}} c_{\delta,\gamma-\delta}^{ilj} v_{\gamma-\delta,l} = 0 & \text{if } \gamma - \delta \in \Sigma_+, \end{cases} \tag{11}$$

where γ varies in Σ_+, δ in Δ, and the $v_{\gamma,j}$ are smooth functions on N. Write $\overline{V} = \sum_{\gamma \in \Sigma_+} \sum_{j=1}^{m_\gamma} v_{\gamma,j} \overline{X}_{\gamma,j}$. If V solves (11), then the projection $\overline{V}$ satisfies (7).

Moreover, if the coefficients $v_{\gamma,j}$ are $\mathcal{C}$-independent for every $\gamma \in \mathcal{R}$, then the vector field $\overline{V}$ is tangent at each point to S. Summarizing, in this case $\overline{V}$ is a multicontact vector field on S. In [7] it is proved that the multicontact vector fields on N are all of the form $\tau(E)$ for some $E \in \mathfrak{g}$, where

$$\tau(E)h(n) = \frac{d}{dt}h([\exp(-tE)n])\Big|_{t=0}, \tag{12}$$

where $[gn]$ denotes the N-component of gn in the Bruhat decomposition of G/P. We ask ourselves for which $E \in \mathfrak{g}$ the coefficients of $\overline{\tau(E)}$ are $\mathcal{C}$-independent. Denote by $\mathfrak{q}$ the parabolic subalgebra of $\mathfrak{g}$ defined as the normalizer in $\mathfrak{g}$ of $\mathfrak{n}_{\mathcal{C}}$

$$\mathfrak{q} := N_{\mathfrak{g}}\mathfrak{n}_{\mathcal{C}} = \{X \in \mathfrak{g} : [X, Y] \in \mathfrak{n}_{\mathcal{C}}, \forall Y \in \mathfrak{n}_{\mathcal{C}}\}.$$

Clearly $\mathfrak{q} \supset \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, so that $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$.

Theorem 4.6. *Let $\mathcal{R} \subseteq \Sigma_+$ a Hessenberg type set, $\mathcal{C}$ the complement of $\mathcal{R}$, and $\mathfrak{q} = N_{\mathfrak{g}}\mathfrak{n}_{\mathcal{C}}$. For every $E \in \mathfrak{q}$, $\overline{\tau(E)}$ is a multicontact vector field on* S. *In particular, the map*

$$\nu : \mathfrak{q} \longrightarrow \mathfrak{X}(\mathrm{S}) \tag{13}$$

defined by $\nu(E) = \overline{\tau(E)}$ is a Lie algebra homomorphism. If $\Delta \subset \mathcal{R}$, then the kernel of ν is $\mathfrak{n}_{\mathcal{C}}$. Thus $\nu(\mathfrak{q})$ is isomorphic to $\mathfrak{q}/\mathfrak{n}_{\mathcal{C}}$.

Proof. We show first that the coefficients of $\overline{\tau(E)}$ are $\mathcal{C}$-independent for every $E \in =q$. Let $E' \in \mathfrak{n}_{\mathcal{C}}$. Then

$$[\tau(E), \tau(E')] = [\sum_{\alpha\in\mathcal{R}}\sum_{i=1}^{m_\alpha} f_{\alpha,i}X_{\alpha,i} + \sum_{\beta\in\mathcal{C}}\sum_{j=1}^{m_\beta} f_{\beta,j}X_{\beta,j}, \sum_{\gamma\in\mathcal{C}}\sum_{k=1}^{m_\gamma} g_{\gamma,k}X_{\gamma,k}]$$

must lie in $\tau(\mathfrak{n}_{\mathcal{C}})$. By direct calculation, this happens if and only if $X_{\gamma,k}(f_{\alpha,i}) = 0$, or equivalently if and only if $\frac{\partial}{\partial x_{\gamma,k}}(f_{\alpha,i}) = 0$ for every $\alpha \in \mathcal{R}$ and $\gamma \in \mathcal{C}$.

The map ν is a homomorphism because τ and the projection operator are such. Hence $\nu(\mathfrak{q})$ is a Lie algebra of multicontact vector fields on S.

We now investigate the kernel of ν in the case $\Delta \subset \mathcal{R}$. Since $\tau(E) = \sum_{\gamma\in\mathcal{C}}\sum_{k=1}^{m_\gamma} g_{\gamma,k}X_{\gamma,k}$ for every $E \in \mathfrak{n}_{\mathcal{C}}$, the inclusion $\mathfrak{n}_{\mathcal{C}} \subseteq \ker\nu$ follows. We prove the opposite inclusion by treating separetely each component of $E \in \ker\nu$, written according to the decomposition $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus (\overline{\mathfrak{n}} \cap \mathfrak{q})$. Write $n = \exp(W) = \exp(\sum_{\alpha\in\Sigma_+} W_\alpha)$, where $W_\alpha \in \mathfrak{g}_\alpha$.

If $E \in \mathfrak{n} \cap \ker\nu$, then $\overline{\tau(E)} = 0$. Write $E = \sum_{\gamma\in\Sigma_+}\sum_{k=1}^{m_\gamma} a_{\gamma,k}E_{\gamma,k}$ and compute

$$\begin{aligned} \tau(E)f &= \frac{d}{dt}f(\exp(-tE)n)\Big|_{t=0} \\ &= \frac{d}{dt}f(\exp(-tE + W - \frac{t}{2}[E, W] + \dots))\Big|_{t=0}. \end{aligned}$$

If E were not in $\mathfrak{n}_{\mathcal{C}}$, there would exist $\beta \in \mathcal{R}$ and $j = 1, \dots, m_\beta$ such that $a_{\beta,j} \neq 0$. If $f : n \mapsto x_{\beta,j}$ then we have that $\tau(E)f$ is a polynomial in $\{x_{\alpha,i}\}_{\alpha\in\Sigma_+}$ whose term of degree zero is $a_{\beta,j}$. On the other hand

$$\tau(E)x_{\beta,j} = 0 \;\forall\, \beta \in \mathcal{R},$$

because its decomposition on the basis of left invariant vector fields involves only components corresponding to the roots in $\mathcal{C}$. This is a contradiction.

Let $E \in \mathfrak{a} \cap \ker\nu$. Recalling that we view N as a dense subset of G/P and that $\exp(tE) \in \mathrm{P}$, we have

$$\begin{aligned}
\tau(E)f(n) &= \frac{d}{dt} f(\exp(-tE)n)\Big|_{t=0} \\
&= \frac{d}{dt} f(\exp(-tE)n\exp(tE))\Big|_{t=0} \\
&= \frac{d}{dt} f(\exp(\sum_{\alpha\in\Sigma_+} e^{-t\alpha(E)} W_\alpha))\Big|_{t=0}.
\end{aligned}$$

Choose now $f : n \mapsto x_{\gamma,j}$, so that

$$\tau(E)f(n) = \frac{d}{dt}(e^{-t\gamma(E)} x_{\gamma,j})\Big|_{t=0} f(n) = -\gamma(E) x_{\gamma,j}.$$

This is zero for every $\gamma \in \mathcal{R}$ because E is in the kernel of ν, so that $\gamma(E) = 0$ for every $\gamma \in \mathcal{R}$. Since $\mathcal{R} \supset \Delta$ and Δ is a basis of $\mathfrak{a}^*$, the dual space of $\mathfrak{a}$, it follows that $E = 0$.

Let $E \in \mathfrak{m} \cap \ker\nu$. Since $\mathfrak{m}$ normalizes every root space, if $f : n \mapsto x_{\gamma,j}$, then

$$\begin{aligned}
\tau(E)f(n) &= \frac{d}{dt} f(\exp(e^{-\mathrm{ad}tE} W))\Big|_{t=0} \\
&= \frac{d}{dt} f(\exp(\sum_{\alpha\in\Sigma_+}\sum_{n=1}^{\infty} (-1)^n t^n \frac{(\mathrm{ad}E)^n}{n!} W_\alpha))\Big|_{t=0} \\
&= ((-\mathrm{ad}E)W_\gamma)_j.
\end{aligned}$$

Whenever $\gamma \in \mathcal{R}$ we have $((-\mathrm{ad}E)W_\gamma)_j = 0$ for every j. Thus $(\mathrm{ad}E)\mathfrak{g}_\gamma = 0$ for every $\gamma \in \mathcal{R}$. In particular $(\mathrm{ad}E)\mathfrak{g}_\delta = 0$ for every simple root δ, and Jacobi identity implies $(\mathrm{ad}E)\mathfrak{n} = 0$. Since $\theta E = E$, it follows that $(\mathrm{ad}E)\mathfrak{g}_{-\delta} = (\mathrm{ad}\theta E)\mathfrak{g}_\delta = (\mathrm{ad}E)\mathfrak{g}_\delta = 0$. Hence $(\mathrm{ad}E)\mathfrak{g} = 0$. Thus $E \in Z(\mathfrak{g}) = \{0\}$.

Let now $E \in \mathfrak{g}_\beta \cap \mathfrak{q} \cap \ker\nu$ for some negative root β, so that $\overline{\tau(E)} = 0$.

For every $E' \in \mathfrak{n}$ we have

$$[\tau(E), \tau(E')] = [\sum_{\alpha\in\mathcal{C}}\sum_{i=1}^{m_\alpha} f_{\alpha,i} X_{\alpha,i}, \sum_{\beta\in\mathcal{R}}\sum_{j=1}^{m_\beta} g_{\beta,j} X_{\beta,j} + \sum_{\gamma\in\mathcal{C}}\sum_{k=1}^{m_\gamma} g_{\gamma,k} X_{\gamma,k}]$$

All terms of the bracket above lie on $\mathfrak{n}_{\mathcal{C}}$, except for summands of the form

$$f_{\alpha,i} X_{\alpha,i}(g_{\beta,j}) X_{\beta,j},$$

but $X_{\alpha,i}(g_{\beta,j}) = 0$, for every $\alpha \in \mathcal{C}$ and $\beta \in \mathcal{R}$, because the coefficients $g_{\beta,j}$ are $\mathcal{C}$-independent. It follows in particular that

$$\overline{[\tau(E), \tau(E')]} = 0,$$

thus $[E, E'] \in \ker\nu$ for every $E' \in \mathfrak{n}$. Therefore one can chose E' such that $[E, E'] \in \mathfrak{m} \oplus \mathfrak{a}$. But this is a contradiction, because no elements of $\mathfrak{m} \oplus \mathfrak{a}$ lie in the kernel of ν. ∎

5. Iwasawa sub-models

The converse of Theorem 4.6 is true under the hypothesis (I) and (II) of the Theorem 5.2 below.

Lemma 5.1. *If the vector space $\mathfrak{n}^\mu = \bigoplus_{\alpha\in\mathcal{S}_\mu} \mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{n}$, then it is an Iwasawa nilpotent Lie algebra.*

Proof. The algebra $\mathfrak{n}^\mu$ coincides with the nilpotent algebra generated by the root spaces corresponding to the simple roots in $\mathcal{S}_\mu$. Hence it is the Iwasawa Lie algebra canonically associated to a connected Dynkin diagram, toghether with admissible multiplicity data [20]. ■

Theorem 5.2. *Let $\mathfrak{g}$ be a simple Lie algebra of real rank strictly greater than two and $\mathcal{R} \subset \Sigma_+$ a subset of Hessenberg type satisfying*

(I) each shadow in the Hessenberg set defines a subalgebra of $\mathfrak{n}$,

(II) each shadow contains at least two simple roots.

Then the Lie algebra of multicontact vector fields on $\mathrm{Hess}_{\mathcal{R}}(H)$ is isomorphic to $\mathfrak{q}/\mathfrak{n}_{\mathcal{C}}$, for every regular element $H \in \mathfrak{a}$ and where $\mathfrak{q} = N_{\mathfrak{g}}\mathfrak{n}_{\mathcal{C}}$.

If (II) is not true, then $\mathcal{R}$ defines a rank one Iwasawa subalgebra. In this case, the finite dimensionality of the Lie algebra $MC(\mathrm{S})$ is no longer guaranteed[1].

Proof of Theorem 5.2.

We must show that if $F \in MC(\mathrm{S})$, then $F = \overline{\tau(E)}$ for some $E \in \mathfrak{q}$. We look again at the system of differential equation (7). If $F \in MC(\mathrm{S})$, then its coefficients solve all the subsystems that we can extract from (7). In particular we consider a subsystem for each shadow, namely:

$$\begin{cases} X_{\delta,i}(f_{\gamma,j}) = 0 & \text{if } \gamma-\delta \notin \Sigma_+ \cup \{0\} \\ X_{\delta,i}(f_{\gamma,j}) + \sum_{l=1}^{m_{\gamma-\delta}} c^{ilj}_{\delta,\gamma-\delta} f_{\gamma-\delta,l} = 0 & \text{if } \gamma-\delta \in \Sigma_+ \\ X_{\delta,i}(f_{\gamma,j}) = 0 & \delta \notin S_\mu, \end{cases} \tag{14}$$

for every root γ in $\mathcal{S}_\mu$. We want to interpret (14) like the system of differential equations that defines the multicontact vector fields on nilpotent Iwasawa Lie groups. Indeed, Lemma 4.5 tells us that the functions $f_{\gamma,j}$, as γ varies in $\mathcal{S}_\mu$, are $(\Sigma_+ \setminus \mathcal{S}_\mu)$-independent. Hence

$$X_{\delta,i}(f_{\gamma,j}) = X^\mu_{\delta,i}(f_{\gamma,j}), \text{ for every } \gamma,\delta \in \mathcal{S}_\mu,$$

where $X^\mu_{\delta,i}$ is the vector field that is obtained from $X_{\delta,i}$ by setting all the components that are labeled by roots that are not in S_μ equal to zero . We then consider, in place of (14), the equivalent system

$$\begin{cases} X^\mu_{\delta,i}(f_{\gamma,j}) = 0 & \text{if } \gamma-\delta \notin \Sigma_+ \cup \{0\} = A8 \\ X^\mu_{\delta,i}(f_{\gamma,j}) + \sum_{l=1}^{m_{\gamma-\delta}} c^{ilj}_{\delta,\gamma-\delta} f_{\gamma-\delta,l} = 0 & \text{if } \gamma-\delta \in \Sigma_+, \end{cases} \tag{15}$$

[1] Personal comunication by the authors of [7], who intend to clarify this matter in full detail in a forthcoming paper.

where $\gamma, \delta \in S_\mu$. Define $\mathfrak{n}^\mu$ as in Lemma 5.1. Using hypothesis (I) of Theorem 5.2, Lemma 5.1 implies that the Lie algebra $\mathfrak{n}^\mu$ is an Iwasawa nilpotent Lie algebra. The system of differential equations above coincides with the multicontact conditions for a vector field on $\mathrm{N}^\mu = \exp \mathfrak{n}^\mu$, because the vector fields $X^\mu_{\delta,i}$ are exactly the left–invariant vector fields on N^μ. This latter assertion is a consequence of a direct calculation .

Lemma 5.3. *Let $F \in \mathfrak{X}(\mathrm{S})$. Then $F \in MC(\mathrm{S})$ if and only if its projection $F^\mu = \sum_{\alpha\in\mathcal{S}_\mu}\sum_{i=1}^{m_\alpha} f_{\alpha,i}\overline{X}_{\alpha,i}$ is a multicontact vector field on N^μ for every maximal root μ.*

Proof. "$\Rightarrow$". By Lemma 4.5, any multicontact vector field on S can be naturally viewed as a vector field on N^μ for every maximal root μ. If the coefficients of F solve the system of differential equations (7), then in particular they solve all subsystems (15), that is any projected vector field F^μ is in $MC(\mathrm{N}^\mu)$.

"$\Leftarrow$". If F has the property that each F^μ solves (15), then F solves all the equations in (7), so that it is in $MC(\mathrm{S})$. ∎

Write $\mathfrak{g}^\mu = \mathfrak{n}^\mu + \theta\mathfrak{n}^\mu + \mathfrak{m}^\mu + \mathfrak{a}^\mu$, where $\mathfrak{m}^\mu = \mathfrak{m} \cap [\mathfrak{n}^\mu, \theta\mathfrak{n}^\mu]$, and $\mathfrak{a}^\mu = \mathfrak{a} \cap [\mathfrak{n}^\mu, \theta\mathfrak{n}^\mu]$. From [7] it follows that the multicontact vector fields on N^μ are all of the form $\tau_\mu(E)$, where

$$\tau_\mu(E)f(n) = \frac{d}{dt}f([\exp(-tE)n])\Big|_{t=0},$$

with $E \in \mathfrak{g}^\mu$, $n \in \mathrm{N}^\mu$ and some function f on N^μ.

Lemma 5.4. *The set of vector fields $\{\tau(E)^\mu, E \in \mathfrak{g}^\mu\}$ generates the Lie algebra $MC(\mathrm{N}^\mu)$, where*

$$\tau(E)^\mu = \sum_{\gamma\in\mathcal{S}_\mu}\sum_{j=1}^{m_\gamma} f_{\gamma,j}\overline{X}_{\gamma,j},$$

whenever $\tau(E) = \sum_{\gamma\in\Sigma_+}\sum_{j=1}^{m_\gamma} f_{\gamma,j}X_{\gamma,j}$. In particular, if $E \in \mathfrak{q}$, it follows that $\tau(E)^\mu \neq 0$ if and only if $E \in \mathfrak{g}^\mu \setminus \{0\}$.

Proof. Let $E \in \mathfrak{g}^\mu$. We show that $E \in \mathfrak{b}$, the normalizer in $\mathfrak{g}$ of the nilpotent ideal consisting of all the root spaces labeled by $\mathcal{S}^c_\mu = \Sigma_+ \setminus \mathcal{S}_\mu$, namely $\mathfrak{b} = N_\mathfrak{g}\mathfrak{n}_{\mathcal{S}^c_\mu}$. Since $\mathfrak{g}^\mu = \mathfrak{m}^\mu + \mathfrak{a}^\mu + \mathfrak{n}^\mu + \theta\mathfrak{n}^\mu$ and $\mathfrak{m}^\mu + \mathfrak{a}^\mu + \mathfrak{n}^\mu \subseteq \mathfrak{b}$, we can suppose that $E \in \theta\mathfrak{n}^\mu$. Write $E = \sum E_\beta$ (here β varies in a subset of negative roots) . If $E \notin \mathfrak{b}$, then there exists $\beta \in \Sigma_-$ such that $E_\beta \notin \mathfrak{b}$. Since $\mathfrak{b}$ normalizes, there would exists $\alpha \in \mathcal{S}^c_\mu$ such that $\alpha + \beta \notin \mathcal{S}^C_\mu$. Hence (I) implies $\alpha = (\alpha+\beta) + (-\beta) \in \mathcal{S}_\mu$, a contradiction. Theorem 4.6 applied to $\mathcal{S}_\mu$ implies that $\tau(E)^\mu \in MC(\mathrm{N}^\mu)$.

We now show that $\tau(E)^\mu \neq 0$ for every $E \in \mathfrak{g}^\mu \setminus \{0\}$. Suppose that there exists $E \in \mathfrak{g}^\mu$ such that $\tau(E)^\mu = 0$. Write $E = H + K + \sum E_\alpha$, with $H \in \mathfrak{a}^\mu$ and $K \in \mathfrak{m}^\mu$. Since $Y \mapsto Y^\mu$ preserves (homomorphic images of) root spaces, the hypothesis $\tau(E)^\mu = 0$ is equivalent to assuming $\tau(H)^\mu = \tau(K)^\mu = 0$ and $\tau(E_\alpha)^\mu = 0$ for every α. Proceeding as in the second part of the proof of Theorem 4.6, we get $H = K = E_\alpha = 0$.

Finally, let $E \in \mathfrak{q}$. Then $\overline{\tau(E)} \in MC(\mathrm{S})$,and $\tau(E)^\mu \in MC(\mathrm{N}^\mu)$. If $E \notin \mathfrak{g}^\mu$, then the latter assertion is true only if $\tau(E)^\mu = 0$. ∎

Corollary 5.5. *Let $\mathcal{I} = \bigcap_{\mu\in\mathcal{E}} \mathcal{S}_\mu$ with $\mathcal{E}$ a subset of maximal roots in $\mathcal{R}$. Then:*

(i) the nilpotent Lie algebra $\mathfrak{n}^{\mathcal{I}} = \bigoplus_{\alpha\in\mathcal{I}} \mathfrak{g}_\alpha$ is an Iwasawa Lie algebra.

(ii) Let $\mathfrak{g}^{\mathcal{I}}$ denote the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}^{\mathcal{I}}$ and $\theta\mathfrak{n}^{\mathcal{I}}$, and let $\mathrm{N}^{\mathcal{I}} = \exp \mathfrak{n}^{\mathcal{I}}$. The vector fields of the type

$$\tau(E)^{\mathcal{I}} = \sum_{\alpha\in\mathcal{I}} \sum_{j=1}^{m_\gamma} f_{\gamma,j} \overline{X}_{\gamma,j},$$

with $E \in \mathfrak{g}^{\mathcal{I}}$, are in $MC(\mathrm{N}^{\mathcal{I}})$.

(iii) If $E \in \mathfrak{q}$, then $E \in \mathfrak{g}^{\mathcal{I}} \setminus \{0\}$ implies that $\tau(E)^{\mathcal{I}} \neq 0$.

Proof. (i) Let α and β two roots in $\mathcal{I}$ such that $\alpha+\beta$ is a root. Then by (I) follows that $\alpha+\beta \in \mathcal{S}_\mu$ for every $\mu \in \mathcal{E}$. Hence $\alpha+\beta \in \mathcal{I}$, and $\mathfrak{n}^{\mathcal{I}}$ is a subalgebra in $\mathfrak{n}$. By Lemma 5.1, $\mathfrak{n}^{\mathcal{I}}$ is an Iwasawa nilpotent Lie algebra. ■

Fix a numbering $\mu_1, \dots, \mu_p$ of the maximal roots and write $\mathfrak{g}^i$ for $\mathfrak{g}^{\mu_i}$. By Lemma 5.3, we can associate to each $F \in MC(\mathrm{S})$ a vector $(F^1, \dots, F^p)$, where each $F^i = F^{\mu_i} \in MC(\mathrm{N}^{\mu_i})$ is the natural projection. Moreover, Lemma 5.4 implies $F^i = \tau(E^i)^i$ for some $E^i \in \mathfrak{g}^i$, so that $(F^1, \dots, F^p) = (\tau(E^1)^1, \dots, \tau(E^p)^p)$. If we prove

$$E^1 = \cdots = E^p = E, \qquad \text{for some } E \in \mathfrak{q}, \tag{16}$$

then the theorem follows.

The proof of (16) needs a technical result (Lemma 5.7) that characterizes $\mathfrak{q}$ in terms of roots, and in particular its "negative" part $\mathfrak{q} \cap \overline{\mathfrak{n}} = \sum_{\alpha\in\mathcal{D}} \mathfrak{g}_\alpha$, with $\mathcal{D} \subset \Sigma_-$.

Given a root $\alpha = \sum_{\delta\in\Delta} n_\delta(\alpha)\delta$ we denote by $\mathcal{Y}(\alpha)$ the subset of Δ consisting of those δ for which $n_\delta(\alpha) \neq 0$, and we call it the *simple support* of α.

Proposition 5.6. [5] (i) *Let $\alpha \in \Sigma$. Then $\mathcal{Y}(\alpha)$ is a connected subset of the Dynkin diagram associated to Σ.*

(ii) *Let $\mathcal{Y}$ be any connected non empty subset of a Dynkin diagram. Then $\sum_{\beta\in\mathcal{Y}} \beta$ is a root.*

We say that a simple root δ is a *boundary* simple root if there exists a maximal root ν in $\mathcal{R}$ whose simple support is a connected diagram that does not contain δ but to which δ is adjacent, i.e. such that there exists $\delta' \in \mathcal{Y}(\nu)$ with the property that $\delta + \delta'$ is a root. The set of all the boundary simple roots will be denoted by $\mathcal{B}$.

Lemma 5.7. *Let $\mathfrak{q} \cap \overline{\mathfrak{n}} = \sum_{\alpha\in\mathcal{D}} \mathfrak{g}_\alpha$.*

(i) If δ is a simple root, then $-\delta \notin \mathcal{D}$ if and only if $\delta \in \mathcal{B}$.

(ii) If α is any positive root, then $-\alpha \notin \mathcal{D}$ if and only if the simple support of α contains a simple root in $\mathcal{B}$.

Proof. We prove (i) first.

"$\Leftarrow$". Let $\delta \in \mathcal{B}$ and let ν be a maximal root to whose shadow δ is adjacent. Proposition 5.6 implies that $\sum_{\epsilon\in\mathcal{Y}(\nu)} \varepsilon + \delta = \sigma + \delta$ is a root. Moreover, it does not lie in $\mathcal{R}$. Indeed, if $\sigma + \delta \in \mathcal{R}$, then it would belong to a shadow containing $\mathcal{S}_\nu$, contradicting the maximality of ν. On the other hand, σ itself is a root, again by Proposition 5.6, and it lies in $\mathcal{R}$, because it is sum of simple roots in a same shadow $\mathcal{S}_\nu$. Thus, we found a root in $\mathcal{C}$, namely $\sigma+\delta$, such that $(\sigma+\delta)-\delta \notin \mathcal{C}$. Therefore $-\delta \notin \mathcal{D}$.

"$\Rightarrow$". Suppose $\delta \notin \mathcal{B}$. Let $\alpha \in \mathcal{C}$ with $\delta \prec \alpha$ and consider its simple support $\mathcal{Y}(\alpha)$. We shall show that $\alpha - \delta \in \mathcal{C}$ whenever $\alpha - \delta \in \Sigma$. Take a maximal connected set $\mathcal{F}$ of simple roots in $\mathcal{Y}(\alpha)$ with the following properties:

- $\diamond$ $\delta \in \mathcal{F}$;
- $\diamond$ there exists a shadow containing $\mathcal{F}$.

This means that $\delta \in \mathcal{F} \subset \mathcal{S}_\nu$ for some ν, but no larger connected subset of $\mathcal{Y}(\alpha)$ containing δ is contained in any other single shadow. Necessarily $\mathcal{F}$ is a proper subset of $\mathcal{Y}(\alpha)$, for otherwise α would lie in $\mathcal{R}$. Take $\varepsilon \in \mathcal{Y}(\alpha)$ adjacent to $\mathcal{F}$. Then two cases arise.

(a) $\mathcal{Y}(\alpha-\delta)$ does contain δ. In this case $\mathcal{Y}(\alpha-\delta)$ contains both $\mathcal{F}$ and ε. Thus $\alpha - \delta \notin \mathcal{R}$, for otherwise $\mathcal{F}\cup\{\varepsilon\}$ would be a connected set contained in a single shadow (namely any shadow containing $\alpha-\delta$) and it would be larger than $\mathcal{F}$.

(b) $\mathcal{Y}(\alpha-\delta)$ does not contain δ. Then $\mathcal{Y}(\alpha-\delta)$ is connected and δ is adjacent to it. If $\alpha-\delta \in \mathcal{R}$ then δ would be a boundary root because $\mathcal{Y}(\alpha-\delta) \subset \mathcal{S}_\nu$ for some maximal root ν, and $\delta \notin \mathcal{S}_\nu$ (for otherwise $\alpha = (\alpha-\delta)+\delta \in \mathcal{S}_\nu$, which is impossible). Hence δ would be adjacent to the simple support of $\mathcal{S}_\nu$, contradicting $\delta \notin \mathcal{B}$. Therefore $\alpha - \delta \notin \mathcal{R}$.

We have seen that in all cases $-\delta \in \mathcal{D}$. This concludes the proof of (i).

As for (ii), take a non simple root $-\alpha \notin \mathcal{D}$. Then $\mathcal{Y}(\alpha)$ contains at least one simple root $\delta \notin -\mathcal{D}$. Indeed, since $\mathfrak{q}$ is a subalgebra, if $\mathcal{Y}(\alpha)$ were contained in $-\mathcal{D}$, then α itself would lie in $\mathfrak{q}$. Thus $\mathcal{Y}(\alpha)$ contains a boundary simple root. Conversely, if $\alpha \in \Sigma_+$ is such that $\mathcal{Y}(\alpha)$ contains a simple root in $\mathcal{B}$, then it contains a simple root that is not in $-\mathcal{D}$, so that $-\alpha$ is not in $\mathcal{D}$. ∎

We can now prove (16). Write $E^i = \sum_{\alpha\in\Sigma^i\cup\{0\}} E^i_\alpha$, with $\Sigma^i = \mathcal{S}_{\mu_i} \cup (-\mathcal{S}_{\mu_i})$. By definition, $\tau(E^i)^i \in MC(\mathrm{N}^{\mu_i})$ if and only if $\tau(E^i_\alpha)^i \in MC(\mathrm{N}^{\mu_i})$ for every $\alpha \in \Sigma^i \cup \{0\}$.

Recall that $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus (\bar{\mathfrak{n}} \cap \mathfrak{q})$, and write $\mathfrak{q} = \bigoplus_{\alpha\in\mathcal{G}} \mathfrak{g}_\alpha$, where $\mathcal{G} = \Sigma_+ \cup \{0\} \cup \mathcal{D}$. We shall prove the following two claims:

(a) $\alpha \in \mathcal{G} \Rightarrow E^i_\alpha = E^j_\alpha$, for every i, j;

(b) $\alpha \notin \mathcal{G} \Rightarrow E^i_\alpha = 0$.

These two facts allows us to define an element $E = \sum_{\alpha\in\Sigma\cup\{0\}} E_\alpha$ by

$$E_\alpha = \begin{cases} E^i_\alpha & \text{if } \alpha \in \mathcal{G} \\ 0 & \text{if } \alpha \notin \mathcal{G}, \end{cases}$$

for all $i = 1, \dots, p$. In particular, $E \in \mathfrak{q}$ and (16) follows.

(a) If $\alpha \in \mathcal{G}$, then $E^i_\alpha \in \mathfrak{q}$ for every $i = 1, \dots, p$. By Theorem 4.6, $\tau(E^i_\alpha) \in MC(\mathrm{S})$ and, by Lemma 5.4, $\tau(E^i_\alpha) \in MC(\mathrm{N}^\mu)$ for every maximal root μ. Moreover, Lemma 5.4 also implies that $\tau(E^i_\alpha)^j \neq 0$ if and only if $E^i_\alpha \in \mathfrak{g}^{\mu_j}$. Suppose that E^i_α belongs to $\mathfrak{g}^{\mu_j}$ with $j \neq i$ and let $\mathcal{I} = \mathcal{S}_{\mu_i} \cap \mathcal{S}_{\mu_j}$ ($\mathcal{I}$ is not empty, otherwise $\mathfrak{g}^{\mu_i}$ and $\mathfrak{g}^{\mu_j}$ would not have a common element). Then statement (iii) of Corollary 5.5 implies that the components of $\tau(E^i_\alpha)^i$ labeled by $\mathcal{I}$ do not vanish identically. This forces $F^j \neq 0$, because

$$\tau(E^j_\alpha)^{\mathcal{I}} = \tau(E^i_\alpha)^{\mathcal{I}} \neq 0.$$

Moreover, since $\mathfrak{g}_\beta \subset \mathfrak{q}$, the identity $\tau(E^j_\alpha - E^i_\alpha)^{\mathcal{I}} = 0$ holds only if $E^i_\alpha = E^j_\alpha$, again by (iii) in Corollary 5.5. This proves (a).

(b) Let $\alpha \notin \mathcal{G}$, and suppose that $E^i_\alpha \neq 0$. We show that this hypothesis takes us to a contradiction. In particular, we shall show that in the vector $(F^1, \dots, F^p)$ appears one component that is not of multicontact type, there implying that F itself is not a multicontact vector field.

By definition of $\mathcal{G}$, the root α must be negative. Furthermore, by (ii) of Lemma 5.7, there exists $\delta \in \mathcal{B}$ such that $\delta + \delta_1 + \dots + \delta_q = -\alpha$. Let $\mathcal{S}_{\mu_j}$ be a shadow to which δ is adjacent. Then there exists at least a shadow to which δ belongs that intersects $\mathcal{S}_{\mu_j}$. Indeed, if this does not happen, then δ would belong to a dark zone disjoint from $\mathcal{S}_{\mu_j}$, which is impossible. Call $\mathcal{S}_{\mu_k}$ such a shadow and $\mathcal{J} = \mathcal{S}_{\mu_j} \cap \mathcal{S}_{\mu_k} \neq \emptyset$. We show that a multicontact vector field corresponding to the root α cannot be identically zero in its components labeled by the intersection $\mathcal{J}$ ($\mathcal{S}_{\mu_j} \cap \mathcal{S}_{\mu_k}$). In short, we prove that

$$\tau(E^i_\alpha)^{\mathcal{J}} \neq 0. \tag{17}$$

If the equation above holds, then the relation $\tau(E^i_\alpha)^{\mathcal{J}} = \tau(E^j_\alpha)^{\mathcal{J}}$ forces $F^j = \tau(E^j)^j$ to be non-zero because $\tau(E^j_\alpha)^j \neq 0$. On the other hand $-\alpha \notin \mathcal{S}_{\mu_j}$, for otherwise δ would lie in $\mathcal{S}_{\mu_j}$. This implies that $\tau(E^j_\alpha)^j$ is not in $MC(\mathrm{N}^{\mu_j})$ by Lemma 5.4. This, in turn, implies that F^{μ_j}, hence F, is not a multicontact vector field, that is the contradiction we expected.

It remains to prove equation (17). Suppose $\tau(E^i_\alpha)^{\mathcal{J}} = 0$. This will give that $\tau(E^i_{-\delta})^{\mathcal{J}} = 0$ which, in turn, implies that δ is not a boundary root, a contradiction. First, by $\tau(E^i_\alpha)^{\mathcal{J}} = 0$, it follows that for every $E' \in \mathfrak{n}$ is

$$[\tau(E^i_\beta), \tau(E')] = [\sum_{\gamma_1 \in \mathcal{J}^c} \sum_{i=1}^{m_{\gamma_1}} f_{\gamma_1,i} X_{\gamma_1,i}, \sum_{\gamma_2 \in \mathcal{J}} \sum_{j=1}^{m_{\gamma_2}} g_{\gamma_2,j} X_{\gamma_2,j} + \sum_{\gamma_3 \in \mathcal{J}^c} \sum_{k=1}^{m_{\gamma_3}} g_{\gamma_3,k} X_{\gamma_3,k}].$$

All terms of the bracket above lie in $\mathfrak{X}(\mathrm{N}^{\mathcal{J}^c})$, except

$$f_{\gamma_1,i} X_{\gamma_1,i}(g_{\gamma_2,j}) X_{\gamma_2,j},$$

but $X_{\gamma_1,i}(g_{\gamma_2,j}) = 0$, for every $\gamma_1 \in \mathcal{J}^c$ and $\gamma_2 \in \mathcal{J}$, because the coefficients $g_{\gamma_2,j}$ are $(\Sigma_+ \setminus \mathcal{J})$-independent. Indeed, $\mathcal{J}$ is a Hessenberg set and its complement $\mathcal{J}^c$ defines an ideal $\mathfrak{n}_{\mathcal{J}^c}$ in $\mathfrak{n}$ whose normalizer contains $\mathfrak{n}$. Therefore, since $E' \in \mathfrak{n}$, it also lies in $N_{\mathfrak{g}}\mathfrak{n}_{\mathcal{J}^c}$, so that the coefficients of $\tau(E')^{\mathcal{J}}$ are $(\Sigma_+ \setminus \mathcal{J})$-independent. Hence $[\tau(E^i_\alpha), \tau(E')] \in \mathfrak{X}(\mathrm{N}^{\mathcal{J}^c})$, that implies

$$\tau([E^i_\alpha, E'])^{\mathcal{J}} = [\tau(E^i_\alpha), \tau(E')]^{\mathcal{J}} = 0$$

for every $E' \in \mathfrak{n}$. The same argument can be iterated for showing that

$$\tau([[E^i_\alpha, E'], \ldots, E^{(n)}])^{\mathcal{J}} = [[\tau(E^i_\alpha), \tau(E')], \ldots, \tau(E^{(n)})]^{\mathcal{J}} = 0 \tag{18}$$

for every collection of elements $E', \ldots, E^{(n)}$ in $\mathfrak{n}$.

Let $\delta_1, \ldots, \delta_q$ simple roots such that $\alpha + \delta_1 + \cdots + \delta_q = -\delta$ is a chain, and $E_1 \in \mathfrak{g}_{\delta_1}, \ldots, E_q \in \mathfrak{g}_{\delta_q}$ such that

$$[[E^i_\alpha, E_1], \ldots, E_q] = E_{-\delta} \in \mathfrak{g}_{-\delta} \setminus \{0\}.$$

We apply (18) to the bracket above, there obtaining $\tau(E_{-\delta})^{\mathcal{J}} = 0$. Since $\theta E_{-\delta} \in \mathfrak{n}$, (18) together with Prop 6.52 in [15] gives

$$0 = [\tau(E_{-\delta}), \tau(\theta E_{-\delta})]^{\mathcal{J}} = B(E_{-\delta}, \theta E_{-\delta}) \tau(H_\delta)^{\mathcal{J}}. \tag{19}$$

By Lemma 5.7, since $\delta \in \mathcal{B}$, there exists a simple root $\delta' \in \mathcal{S}_{\mu_j}$ such that $\delta + \delta'$ is a root. This implies that $\langle \delta, \delta' \rangle \neq 0$, because $\delta - \delta'$ is never a root. Hence $\delta'(H_\delta) \neq 0$, so that $H_\delta \in \mathfrak{g}^{\mu_j} \cap \mathfrak{g}^{\mu_k}$. By Corollary 5.5, it follows that $\tau(H_\delta)^{\mathcal{J}} \neq 0$, contradicting (19). This concludes our proof. ∎

Remarks. One can easily see that if the root system corresponding to $\mathfrak{g}$ is A_r, then hypothesis (I) holds for every subset $\mathcal{R} \subseteq A_r$ of Hessenberg type. Nevertheless, it is also easy to build counter-examples to Theorem 5.2. Indeed, it is enough to consider $\mathfrak{g} = \mathrm{Sp}(2, \mathbb{R})$ with corresponding positive roots $\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$ and Hessenberg structure given by $\mathcal{R} = \{\alpha, \beta, \alpha + \beta\}$. Here hypothesis (I) does not hold and a direct calculation of the Lie algebra of multicontact vector fields gives a larger space than $\mathfrak{q}/\mathfrak{n}_{\mathcal{C}}$.

Finally, we note that by [7] it follows that all the results we proved are true under the assumption that the multicontact mappings are C^2.

6. Multiconformal mappings

Consider $\mathrm{N}_{\mathcal{C}} = \exp \mathfrak{n}_{\mathcal{C}}$. Since $\mathrm{N}_{\mathcal{C}}$ is a normal subgroup of N, the quotient $\mathrm{N}/\mathrm{N}_{\mathcal{C}}$ is a nilpotent Lie group. We identify the Lie algebra of $\mathrm{N}/\mathrm{N}_{\mathcal{C}}$ with $\mathfrak{n}/\mathfrak{n}_{\mathcal{C}}$, and we define a natural multicontact structure on this quotient simply considering the subbundles $\{\langle \mathfrak{g}_\delta \rangle_{\mathfrak{n}_{\mathcal{C}}} : \delta \in \Delta\}$, where $\langle E \rangle_{\mathfrak{n}_{\mathcal{C}}}$ denotes the coset of E in $\mathfrak{n}/\mathfrak{n}_{\mathcal{C}}$. Let f be a diffeomorphism between open subsets of $\mathrm{N}/\mathrm{N}_{\mathcal{C}}$. Then f is a multicontact mapping if for every simple root δ

$$f_*(\langle \mathfrak{g}_\delta \rangle_{\mathfrak{n}_{\mathcal{C}}}) \subset \langle \mathfrak{g}_\delta \rangle_{\mathfrak{n}_{\mathcal{C}}}.$$

The coordinates system on the slice S define the analytic structure on $\mathrm{N}/\mathrm{N}_{\mathcal{C}}$. Thus, $\mathrm{N}/\mathrm{N}_{\mathcal{C}}$ and S are diffeomorphic by the assignment

$$\chi : \langle (\{x_{\alpha,i}\}_{\alpha \in \Sigma_+}) \rangle_{\mathrm{N}_{\mathcal{C}}} \mapsto (\{x_{\alpha,i}\}_{\alpha \in \mathcal{R}}, 0),$$

where $\langle n \rangle_{\mathrm{N}_{\mathcal{C}}}$ denotes the coset of $n \in \mathrm{N}$ in the quotient group. The differential χ_* maps the left–invariant vector field $\langle X_{\alpha,i} \rangle_{\mathfrak{n}_{\mathcal{C}}}$ to $\overline{X}_{\alpha,i}$, therefore the multicontact structure on $\mathrm{N}/\mathrm{N}_{\mathcal{C}}$ is mapped onto the multicontact structure on S. Let $\mathrm{Q} = \mathrm{Int}(\mathfrak{q})$. We have $\mathrm{Int}(\mathfrak{q}) \subset \mathrm{Int}(\mathfrak{g})$, because $\mathrm{Int}(\mathfrak{q}) = e^{\mathrm{ad}\,\mathfrak{q}}$, $\mathrm{Int}(\mathfrak{g}) = e^{\mathrm{ad}\,\mathfrak{g}}$ and $\mathfrak{q} \subset \mathfrak{g}$.

Lemma 6.1. *The action of every element $q \in \mathrm{Q}$ on N induces a well-posed action on the quotient $\mathrm{N}/\mathrm{N}_{\mathcal{C}}$, namely*

$$\hat{q}(\langle n\rangle_{\mathrm{N}_{\mathcal{C}}}) = \langle [qn]\rangle_{\mathrm{N}_{\mathcal{C}}},$$

where $[qn]$ is the N-component of qn in the Bruhat decomposition.

Proof. Let $n \in \mathrm{N}$ and $n_{\mathcal{C}} \in \mathrm{N}_{\mathcal{C}}$. Then n and $nn_{\mathcal{C}}$ both represent $\langle n\rangle_{\mathrm{N}_{\mathcal{C}}} \in \mathrm{N}/\mathrm{N}_{\mathcal{C}}$. We show that $[qn]$ and $[qnn_{\mathcal{C}}]$ represent the same element in $\mathrm{N}/\mathrm{N}_{\mathcal{C}}$, that is $[qn][qnn_{\mathcal{C}}]^{-1} \in \mathrm{N}_{\mathcal{C}}$. Let $p \in \mathrm{P}$ such that $[qn] = qnp$. Since $\mathrm{N}_{\mathcal{C}}$ is a normal subgroup of Q, there exists $n'_{\mathcal{C}} \in \mathrm{N}_{\mathcal{C}}$ such that $[qnn_{\mathcal{C}}] = [n'_{\mathcal{C}}qn] = n'_{\mathcal{C}}[qn] = n'_{\mathcal{C}}qnp$. Then $[qn][qnn_{\mathcal{C}}]^{-1} = qnp(n'_{\mathcal{C}}qnp)^{-1} = (n'_{\mathcal{C}})^{-1} \in \mathrm{N}_{\mathcal{C}}$, as required. ∎

Proposition 6.2. *Let Q be as above, and $\mathcal{A}$ an open subset of $\mathrm{N}/\mathrm{N}_{\mathcal{C}}$. For every $q \in \mathrm{Q}$, the map*

$$\hat{q} : \mathcal{A} \subset \mathrm{N}/\mathrm{N}_{\mathcal{C}} \to \mathrm{N}/\mathrm{N}_{\mathcal{C}}$$

is a multicontact mapping on $\mathcal{A}$. Furthermore $\hat{q} = \mathrm{id}_{\mathcal{A}}$ for every $q \in \mathrm{N}_{\mathcal{C}}$.

Proof. Since $q \in \mathrm{G} = \mathrm{Int}(\mathfrak{g})$, it is a multicontact mapping on G/P. Thus, $q_*(\mathfrak{g}_\delta) \subseteq \mathfrak{g}_\delta$ for every simple root δ ([7]). Let $E \in \mathfrak{g}_\delta$, for some $\delta \in \Delta$, and consider a representative in $\mathfrak{n}/\mathfrak{n}_{\mathcal{C}}$ of $\langle E\rangle_{\mathfrak{n}_{\mathcal{C}}}$, say $E + E'$, with $E' \in \mathfrak{n}_{\mathcal{C}}$. Then

$$\hat{q}_*(\langle E\rangle_{\mathfrak{n}_{\mathcal{C}}}) = \langle (l_q)_*(E+E')\rangle_{\mathfrak{n}_{\mathcal{C}}}.$$

By definition

$$(l_q)_*(E') = \frac{d}{dt}(q\exp(tE'))\Big|_{t=0}.$$

Since $[\mathfrak{q}, \mathfrak{n}_{\mathcal{C}}] \subset \mathfrak{n}_{\mathcal{C}}$, a straightforward calculation implies that $(l_q)_*(E') \in \mathfrak{n}_{\mathcal{C}}$. Therefore there exists $E'' \in \mathfrak{n}_{\mathcal{C}}$ such that

$$\hat{q}_*(\langle E\rangle_{\mathfrak{n}_{\mathcal{C}}}) = \langle (l_q)_{*e}(E+E')\rangle_{\mathfrak{n}_{\mathcal{C}}} = \langle q_*(E) + E''\rangle_{\mathfrak{n}_{\mathcal{C}}} \subset \langle \mathfrak{g}_\delta + \mathfrak{n}_{\mathcal{C}}\rangle_{\mathfrak{n}_{\mathcal{C}}} \subset \langle \mathfrak{g}_\delta\rangle_{\mathfrak{n}_{\mathcal{C}}}.$$

Since $\langle n_{\mathcal{C}}n\rangle_{\mathrm{N}_{\mathcal{C}}} = \langle n\rangle_{\mathrm{N}_{\mathcal{C}}}$, it follows that $\hat{n}_{\mathcal{C}}$ maps $\langle n\rangle_{\mathrm{N}_{\mathcal{C}}}$ in itself, for every $n \in \mathrm{N}$. Hence the proposition holds. ∎

Now, the inner product on $\mathfrak{n}$ derived from the Killing form may be propagated to the tangent space of any point of G/P using the G action: the result is determined up to the action of an element of A ([7]). Let $\langle X\rangle_{\mathfrak{n}_{\mathcal{C}}}$ and $\langle Y\rangle_{\mathfrak{n}_{\mathcal{C}}} \in \langle \mathfrak{g}_\delta\rangle_{\mathfrak{n}_{\mathcal{C}}}$ and let X and $Y \in \mathfrak{g}_\delta$ be left invariant vector fields on N representing $\langle X\rangle_{\mathfrak{n}_{\mathcal{C}}}$ and $\langle Y\rangle_{\mathfrak{n}_{\mathcal{C}}}$. Such representatives are unique. Then $\hat{B}_\theta(\langle X\rangle_{\mathfrak{n}_{\mathcal{C}}}, \langle Y\rangle_{\mathfrak{n}_{\mathcal{C}}}) := B_\theta(X, Y)$ defines a well-posed conformal structure on each stratus $\langle \mathfrak{g}_\delta\rangle_{\mathfrak{n}_{\mathcal{C}}}$, whenever δ is a simple root. We then say that a multicontact map f is *multiconformal* if the restriction of f_* to $\langle \mathfrak{g}_\delta\rangle_{\mathfrak{n}_{\mathcal{C}}}$ is a multiple of an isometry, for every simple root δ. A direct calculation shows that the left action of Q is multiconformal. Since the action of $\mathrm{N}_{\mathcal{C}}$ induces the identity map, we then have that $\mathrm{Q}/\mathrm{N}_{\mathcal{C}}$ is a group of multiconformal mappings.

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