# On the exceptional set of Lagrange's equation with three prime and one almost–prime variables

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RÉSUMÉ. Nous considérons une version affaiblie de la conjecture sur la représentation des entiers comme somme de quatre carrés de nombres premiers.

ABSTRACT. We consider an approximation to the popular conjecture about representations of integers as sums of four squares of prime numbers.

## 1. Introduction and statement of the result

The famous theorem of Lagrange states that every non–negative integer $n$ can be represented as

$$(1.1) \qquad x_1^2 + x_2^2 + x_3^2 + x_4^2 = n,$$

where $x_1, \ldots, x_4$ are integers. There is a conjecture, which asserts that every sufficiently large integer $n$, such that $n \equiv 4 \pmod{24}$, can be represented in the form (1.1) with prime variables $x_1, \ldots, x_4$. This conjecture has not been proved so far, but there are various approximations to it established.

We have to mention first that in 1938 Hua [9] proved the solvability of the corresponding equation with five prime variables. In 1976 Greaves [3] and later Shields [19] and Plaksin [18] proved the solvability of (1.1) with two prime and two integer variables (in [18] and [19] an asymptotic formula for the number of solutions was found).

In 1994 Brüdern and Fouvry [2] considered (1.1) with almost–prime variables and proved that if $n$ is large enough and satisfies $n \equiv 4 \pmod{24}$ then (1.1) has solutions in integers of type $P_{34}$. Here and later we denote by $P_r$ any integer with no more than $r$ prime factors, counted according to multiplicity. Recently Heath–Brown and the author [8] proved that, under the same conditions on $n$, the equation (1.1) has solutions in one prime and three $P_{101}$ – almost–prime variables and also in four $P_{25}$ – almost–prime variables. These results were sharpened slightly by the author [21], who

established the solvability of (1.1) in one prime and three $P_{80}$ – almost–primes and, respectively, in four $P_{21}$ – almost–primes.

There are several papers, published during the last years, devoted to the study of the exceptional set of the equation (1.1) with prime variables. Suppose that $Y$ is a large real number and denote by $E_1(Y)$ the number of positive integers $n \leq Y$ satisfying $n \equiv 4 \pmod{24}$ and which cannot be represented in the form (1.1) with prime variables $x_1, \ldots, x_4$. In 2000 J.Liu and M.-C. Liu [16] proved that $E_1(Y) \ll Y^{13/15+\varepsilon}$, where $\varepsilon > 0$ is arbitrarily small. This result was improved considerably by Wooley [22], who established that $E_1(Y) \ll Y^{13/30+\varepsilon}$. Recently L. Liu [15] established that $E_1(Y) \ll Y^{2/5+\varepsilon}$.

In the paper [22] Wooley obtained other interesting results, concerning the equation (1.1). We shall state one of them. Denote by $R(n)$ the number of solutions of (1.1) in three prime and one integer variables. It is expected that $R(n)$ can be approximated by the expression $\frac{1}{2}\pi^2\,\mathfrak{S}(n)\,n\,(\log n)^{-3}$, where $\mathfrak{S}(n)$ is the corresponding singular series (see [22] for the definition). Wooley proved that the set of integers $n$, for which $R(n)$ fails to be close to the expected value, is remarkably thin. More precisely, let $\psi(t)$ be any monotonically increasing and tending to infinity function of the positive variable $t$, such that $\psi(t) \ll (\log t)^B$ for some constant $B > 0$. Let $Y$ be a large real number and denote by $E^*(Y, \psi)$ the number of positive integers $n \leq Y$ such that

$$\left| R(n) - \frac{1}{2}\pi^2\,\mathfrak{S}(n)\,n\,(\log n)^{-3} \right| > n\,(\log n)^{-3}\,\psi(n)^{-1}.$$

Theorem 1.2 of [22] asserts that

$$E^*(Y, \psi) \ll \psi(Y)^4\,(\log Y)^6.$$

We note that if the integer $n$ satisfies

(1.2)                    $n \equiv 3, 4, 7, 12, 15$ or $19 \pmod{24}$,

then

$$1 \ll \mathfrak{S}(n) \ll \log\log n.$$

Therefore, if $E_2(Y)$ denotes the number of positive integers $n \leq Y$, satisfying (1.2) and which cannot be represented in the form (1.1) with three prime and one integer variables, then

(1.3)                    $E_2(Y) \ll (\log Y)^{6+\varepsilon}$

for any $\varepsilon > 0$.

The purpose of the present paper is to obtain an estimate of almost the same strength as (1.3) for the exceptional set of the equation (1.1) with three prime and one almost–prime variables. We shall prove the following

**Theorem.** *Let $Y$ be a large real number and denote by $E(Y)$ the number of positive integers $n \leq Y$ satisfying $n \equiv 4 \pmod{24}$ and which cannot be represented in the form*

$$(1.4) \qquad p_1^2 + p_2^2 + p_3^2 + x^2 = n \, ,$$

*where $p_1, p_2, p_3$ are primes and $x = \mathrm{P}_{11}$. Then we have*

$$(1.5) \qquad E(Y) \ll (\log Y)^{1053} \, .$$

As one may expect, the proof of this result is technically more complicated than the proof of Theorem 1.2 of [22]. We use a combination of the circle method and the sieve methods.

In the circle method part we apply the approach of Wooley [22], adapted for our needs. On the set of minor arcs we apply the method of Kloosterman, introduced in the classical paper [14]. This technique was, actually, applied also by Wooley in the estimation of the sums $T_1$ and $T_2$ in section 3 of [22]. In his analysis, however, only Ramanujan's sums appear, whiles in our situation we have to deal with much more complicated sums, defined by (8.19). Fortunately, these sums differ very slightly from sums considered by Brüdern and Fouvry [2], so we can, in fact, borrow their result for our needs.

The sieve method part is rather standard. We apply a weighted sieve with weights of Richert's type and proceed as in chapter 9 of Halberstam and Richert's book [4].

In many places we omit the calculations because they are similar to those in other books or papers, or because they are standard and straightforward. We note that one can obtain slightly stronger result (with smaller power in (1.5) and with variable $x$ having fewer prime factors) by means of more elaborate computational work.

## 2. Notations and some definitions

Throughout the paper we use standard number–theoretic notations. As usual, $\mu(n)$ denotes the Möbius function, $\varphi(n)$ is the Euler function, $\Omega(n)$ is the number of prime divisors of $n$, counted according to multiplicity,

$\tau(n)$ is the number of positive divisors of $n$. The greatest common divisor and, respectively, the least common multiple of the integers $m_1, m_2$ are denoted by $(m_1, m_2)$ and $[m_1, m_2]$. However, if $u$ and $v$ are real numbers then $(u, v)$ means the interval with endpoints $u$ and $v$. The meaning is always clear from the context. We use bold style letters to denote four–dimensional vectors. The letter $p$ is reserved for prime numbers. If $p > 2$ then $\left( \frac{\cdot}{p} \right)$ stands for the Legendre symbol. To denote summation over the positive integers $n \leq Z$ we write $\sum_{n \leq Z}$. Furthermore, $\sum_{x(q)}$, respectively, $\sum_{x(q)^*}$ means that $x$ runs over a complete, respectively, reduced system of residues modulo $q$. By $[\alpha]$ we denote the integer part of the real number $\alpha$, $e(\alpha) = e^{2\pi i \alpha}$ and $e_q(\alpha) = e(\alpha/q)$.

We assume that $\varepsilon > 0$ is an arbitrarily small positive number and $A$ is an arbitrarily large number; they can take different values in different formulas. Unless it is not specified explicitly, the constants in the $\mathcal{O}$ – terms and $\ll$ – symbols depend on $\varepsilon$ and $A$. For positive $U$ and $V$ we write $U \asymp V$ as an abbreviation of $U \ll V \ll U$.

Let $N$ be a sufficiently large real number. We define

$$(2.1) \qquad X = N^{1/2}, \qquad P = X^\delta \quad \text{for some constant} \quad \delta \in (0, 9/40),$$

$$(2.2) \qquad Q = NP^{-1}(\log N)^{-E}, \qquad M = X(\log N)^{-4E-4},$$

where $E > 1$ is a large constant, which we shall specify later.

To apply the sieve method we need information about the number of solutions of (1.4) in integers $x$ lying in arithmetical progressions and in primes $p_1, p_2, p_3$. For technical reasons we attach logarithmic weights to the primes and a smooth weight to the variable $x$. More precisely, we consider the function

$$(2.3) \qquad \omega_0(t) = \begin{cases} \exp\left( \frac{1}{(20t - 10)^2 - 1} \right) & \text{if} \quad t \in (9/20, 11/20), \\ 0 & \text{otherwise} \end{cases}$$

and let

$$(2.4) \qquad \omega(x) = \omega_0(x X^{-1}).$$

For any integer $n \in (N/2, N]$ and for any squarefree integer $k$, such that $(k, 6) = 1$, we define

$$(2.5) \qquad I(n, k) = \sum_{\substack{p_1^2 + p_2^2 + p_3^2 + x^2 = n \\ M < p_1, p_2, p_3 \leq X \\ x \equiv 0 \pmod{k}}} (\log p_1)(\log p_2)(\log p_3)\, \omega(x).$$

We expect that this quantity can be approximated, at least on average, by the expression

$$(2.6) \qquad \Sigma(n, k) = \frac{\pi}{4} X^2 \kappa(n)\, k^{-1}\, \mathfrak{S}(n, k),$$

which arises as a mean term when we apply formally the circle method.

The quantity $\kappa(n)$ from the right–hand side of (2.6) comes from the singular integral and is defined by

$$(2.7) \qquad \kappa(n) = \int_{-\infty}^{\infty} \omega_0(t) \left(nN^{-1} - t^2\right)^{1/2} dt.$$

Having in mind (2.3) we see that

$$(2.8) \qquad \kappa(n) \asymp 1 \qquad \text{for} \qquad n \in (N/2, \, N].$$

Furthermore, $\mathfrak{S}(n, k)$ comes from the singular series and is defined by

$$(2.9) \qquad \mathfrak{S}(n, k) = \sum_{q=1}^{\infty} f(q, n, k),$$

where

$$(2.10) \qquad f(q, n, k) = (q, k) \, q^{-1} \, \varphi(q)^{-3} \sum_{a(q)^*} S_k(q, a) \, S^*(q, a)^3 \, e_q(-an),$$

$$(2.11) \qquad S_k(q, a) = \sum_{\substack{x(q) \\ x \equiv 0 \pmod{(q,k)}}} e_q(ax^2), \qquad S^*(q, a) = \sum_{x(q)^*} e_q(ax^2).$$

We shall consider $\mathfrak{S}(n, k)$ in detail in the next section.

## 3. Some properties of the sum $\mathfrak{S}(n, k)$

It is not difficult to see that the function $f(q, n, k)$, defined by (2.10), is multiplicative with respect to $q$. We shall compute it for $q = p^l$.

From this point onwards we assume that the integers $n$ and $k$ satisfy

$$(3.1) \qquad n \equiv 4 \pmod{24}, \qquad (6, k) = 1, \qquad \mu(k) \neq 0.$$

Then we have

$$(3.2) \qquad f(2^l, n, k) = \begin{cases} 0 & \text{if } l = 1, \\ 1 & \text{if } l = 2, \\ 2 & \text{if } l = 3, \\ 0 & \text{if } l > 3. \end{cases}$$

Furthermore, if $p > 2$ is a prime, then

$$(3.3) \qquad f(p^l, n, k) = 0 \qquad \text{for} \qquad l > 1$$

and

$$(3.4) \qquad f(p,n,k) = \begin{cases} h_1(p,n) & \text{if } p \nmid kn\,, \\ h_2(p) & \text{if } p \nmid k\,,\ p \mid n\,, \\ h_3(p,n) & \text{if } p \mid k\,,\ p \nmid n\,, \\ h_4(p) & \text{if } p \mid k\,,\ p \mid n\,, \end{cases}$$

where the quantities $h_j$ are defined by

$$(3.5) \qquad h_1(p,n) = \frac{-1}{(p-1)^3} \left\{ p\left(1 + 3\left(\frac{-n}{p}\right)\right) + 3\left(\frac{-1}{p}\right) + \left(\frac{n}{p}\right) \right\},$$

$$(3.6) \qquad h_2(p) = \frac{1}{(p-1)^2} \left\{ p + 3\left(\frac{-1}{p}\right) \right\},$$

$$(3.7) \qquad h_3(p,n) = \frac{1}{(p-1)^3} \left\{ p^2\left(\frac{-n}{p}\right) + 3p\left(\left(\frac{-1}{p}\right) + \left(\frac{n}{p}\right)\right) + 1 \right\},$$

$$(3.8) \qquad h_4(p) = \frac{-1}{(p-1)^2} \left\{ 3p\left(\frac{-1}{p}\right) + 1 \right\}.$$

The proof of formulas (3.2) – (3.8) is standard and uses only the basic properties of the Gauss sums (see Hua [10], chapter 7, for example). We leave the verification to the reader.

From (3.2) – (3.8) we easily get

$$(3.9) \qquad f(q,n,k) \ll \tau^4(q)\, q^{-2}\, (q,kn)\,.$$

This estimate implies that the series (2.9) is absolutely convergent. We apply Euler's identity and we use (3.1) – (3.5) to obtain

$$\mathfrak{S}(n,k) = 8 \prod_{p>3} \left(1 + f(p,n,k)\right).$$

From this formula and (3.4), after some rearrangements, we get

$$(3.10) \qquad \mathfrak{S}(n,k) = 8\xi(n)\,\psi_n(k)\,,$$

where

$$(3.11) \qquad \xi(n) = \prod_{p>3} \left(1 + h_1(p,n)\right)\ \prod_{p\mid n} \frac{1 + h_2(p)}{1 + h_1(p,0)}\,,$$

$$(3.12) \qquad \psi_n(k) = \prod_{p\mid k} \psi_n(p)$$

and where

$$
(3.13) \qquad \psi_n(p) = \begin{cases} \dfrac{1 + h_3(p, n)}{1 + h_1(p, n)} & \text{if } p \nmid n, \\[3mm] \dfrac{1 + h_4(p)}{1 + h_2(p)} & \text{if } p \mid n. \end{cases}
$$

From (3.1), (3.5) – (3.8), (3.11) and (3.13) we obtain the estimates

$$(3.14) \qquad\qquad 1 \ll \xi(n) \ll \log\log n \,,$$

$$(3.15) \qquad 0 < \psi_n(p) < 5 \quad \text{if} \quad p \geq 7 \quad \text{or if} \quad p = 5 \quad \text{and} \quad 5 \nmid n \,,$$

$$(3.16) \qquad\qquad \psi_n(5) = 0 \quad \text{if} \quad 5 \mid n$$

and

$$(3.17) \qquad\qquad \psi_n(p) = 1 + \mathcal{O}\!\left(\frac{1}{p}\right),$$

where the constant in the $\mathcal{O}$ –term in the last formula is absolute. We leave the easy verification of formulas (3.14) – (3.17) to the reader.

Let us note that from (3.10), (3.12) and (3.16) it follows

$$(3.18) \qquad\qquad \mathfrak{S}(n, k) = 0 \quad \text{if} \quad 5 \mid (n, k) \,,$$

which we, of course, expect, having in mind the definition (2.5) of $I(n, k)$ and the conditions (3.1).

## 4. Proof of the Theorem

A central rôle in the proof of the Theorem plays the following Proposition, which asserts that the difference between the quantities $I(n, k)$ and $\Sigma(n, k)$, defined by (2.5) and (2.6), is small on average with respect to $n$ and $k$.

**Proposition.** *Suppose that the set $\mathcal{F}$ consists of integers $n \in (N/2, N]$, satisfying the congruence $n \equiv 4 \pmod{24}$, and denote by $F$ the cardinality of $\mathcal{F}$. Let $\gamma(k)$ be a real valued function, defined on the set of positive integers and such that*

$$(4.1) \qquad \gamma(k) = 0 \qquad if \qquad (6, k) > 1 \qquad or \qquad \mu(k) = 0$$

*and*

$$(4.2) \qquad\qquad |\gamma(k)| \leq \tau(k) \,.$$

*Suppose also that*

$$(4.3) \qquad D = X^\eta \qquad for \ some \ constant \qquad \eta \in (0, 1/8)$$

*and consider the sum*

$$(4.4) \qquad \mathcal{E} = \sum_{k \leq D} \gamma(k) \sum_{n \in \mathcal{F}} \big( I(n,k) - \Sigma(n,k) \big).$$

*Then we have*

$$(4.5) \qquad \mathcal{E} \ll F^{3/4} X^2 (\log N)^{262} + F X^2 (\log N)^{-E}.$$

*The constant in Vinogradov's symbol depends only on the constants $\delta$, $E$, and $\eta$, included, respectively, in (2.1), (2.2) and (4.3).*

We shall prove the Proposition in sections 5 – 8. In this section we shall use it to establish the Theorem.

Let $\mathcal{F}$ be the set of integers $n \in (N/2, N]$ satisfying $n \equiv 4 \pmod{24}$ and which cannot be represented in the form (1.4) with primes $p_1, p_2, p_3$ and with $x = \mathrm{P}_{11}$. Let $F$ be the cardinality of $\mathcal{F}$. We shall establish that

$$(4.6) \qquad F \ll (\log N)^{1052}.$$

Obviously, this implies the estimate (1.5).

To study the equation (1.4) with an almost–prime variable $x$ we apply a weighted sieve of Richert's type.

Let $\eta, \nu, \nu_1, \theta$ be constants such that

$$(4.7) \qquad 0 < \theta, \qquad 0 < \eta < 1/8, \qquad 0 < \nu < \nu_1, \qquad \nu + \nu_1 < \eta.$$

Denote

$$(4.8) \qquad z = X^\nu, \qquad z_1 = X^{\nu_1}, \qquad D = X^\eta$$

and

$$(4.9) \qquad \mathcal{P} = \prod_{3 < p < z} p.$$

Consider the sum

$$(4.10) \quad \Gamma = \sum_{n \in \mathcal{F}} \sum_{\substack{p_1^2 + p_2^2 + p_3^2 + x^2 = n \\ M < p_1, p_2, p_3 \leq X \\ (x, \mathcal{P}) = 1}} (\log p_1)(\log p_2)(\log p_3)\, \omega(x)\, \Delta(x; \theta, \nu, \nu_1),$$

where

$$(4.11) \qquad \Delta(x) = \Delta(x; \theta, \nu, \nu_1) = 1 - \theta \sum_{\substack{z \leq p < z_1 \\ p \mid x}} \left( 1 - \frac{\log p}{\log z_1} \right).$$

It is clear that

$$(4.12) \qquad \Gamma \leq \Gamma_1,$$

where $\Gamma_1$ is the contribution of the terms for which $\Delta(x) > 0$. We decompose $\Gamma_1$ as

(4.13) $$\Gamma_1 = \Gamma_2 + \Gamma_3\,,$$

where $\Gamma_2$ is the contribution of the terms with $x \equiv 0 \pmod{p^2}$ for some prime $p \in [z, z_1)$ and where $\Gamma_3$ comes from the other terms.

We note that the congruence condition $n \equiv 4 \pmod{24}$ and the size conditions on $p_i$ in the domain of summation in (4.10) imply that there are no terms with $(6, x) > 1$ counted in $\Gamma$ and, respectively, in $\Gamma_3$. Hence the condition $(x, \mathcal{P}) = 1$ from the domain of summation in (4.10) can be replaced by $(x, 6\mathcal{P}) = 1$. Furthermore, if $x$ is squarefree with respect to the prime numbers $p \in [z, z_1)$, if $(x, 6\mathcal{P}) = 1$ and $\Delta(x) > 0$, then

(4.14) $$\Omega(x) < \theta^{-1} + \nu_1^{-1}\,.$$

For explanation we refer the reader to Halberstam and Richert [4], chapter 9, p.256.

From this point onwards we assume also that

(4.15) $$\theta^{-1} + \nu_1^{-1} < 12\,.$$

Then using (4.14), (4.15) and the definition of the set $\mathcal{F}$ we conclude that the sum $\Gamma_3$ is empty, i.e.

(4.16) $$\Gamma_3 = 0\,.$$

Indeed, if this were not true, then for some $n \in \mathcal{F}$ the equation (1.4) would have a solution $p_1, p_2, p_3, x$ with $x$ satisfying (4.14). However, this is not possible due to (4.15) and the definition of $\mathcal{F}$.

It is easy to estimate $\Gamma_2$ from above. We have

$$\Gamma_2 \ll (\log N)^3 \sum_{n \in \mathcal{F}} \sum_{z \le p < z_1} \sum_{\substack{p_1^2 + p_2^2 + p_3^2 + x^2 = n \\ x \equiv 0 \pmod{p^2}}} 1$$

$$\ll (\log N)^3 \sum_{n \in \mathcal{F}} \sum_{z \le p < z_1} \sum_{l \le n} \left( \sum_{p_1^2 + p_2^2 = l} 1 \right) \sum_{\substack{p_3^2 + x^2 = n - l \\ x \equiv 0 \pmod{p^2}}} 1$$

$$\ll N^\varepsilon \sum_{n \in \mathcal{F}} \sum_{z \le p < z_1} \sum_{\substack{x, y \le X \\ x \equiv 0 \pmod{p^2}}} 1$$

$$\ll N^\varepsilon F \left( X^2 z^{-1} + z_1 X \right)$$

and, having in mind (4.7) and (4.8), we get

(4.17) $$\Gamma_2 \ll F X^{2-\varepsilon}\,.$$

From (4.12), (4.13), (4.16) and (4.17) it follows that

(4.18) $$\Gamma + \mathcal{O}(F X^{2-\varepsilon}) \le 0\,.$$

We shall now estimate $\Gamma$ from below. Using (4.10) and (4.11) we represent it in the form

$$(4.19) \qquad\qquad \Gamma = \Gamma_4 - \theta\Gamma_5\,,$$

where

$$(4.20) \qquad \Gamma_4 = \sum_{n\in\mathcal{F}} \sum_{\substack{p_1^2+p_2^2+p_3^2+x^2=n \\ M<p_1,p_2,p_3\leq X \\ (x,\mathcal{P})=1}} (\log p_1)\,(\log p_2)\,(\log p_3)\,\omega(x)$$

and where $\Gamma_5$ comes from the second term in the right–hand side of (4.11). Changing the order of summation we get

$$(4.21) \quad \Gamma_5 = \sum_{z\leq p<z_1} \left(1 - \frac{\log p}{\log z_1}\right) \sum_{n\in\mathcal{F}}$$
$$\times \sum_{\substack{p_1^2+p_2^2+p_3^2+x^2=n \\ M<p_1,p_2,p_3\leq X \\ (x,\mathcal{P})=1 \\ x\equiv 0 \pmod{p}}} (\log p_1)\,(\log p_2)\,(\log p_3)\,\omega(x)\,.$$

To find a non–trivial lower bound for $\Gamma$ we have to estimate $\Gamma_4$ from below and $\Gamma_5$ from above. We shall apply Rosser's sieve (see Iwaniec [12], [13]).

First we get rid of the condition $(x,\mathcal{P})=1$ from the domains of summation in (4.20) and (4.21) by introducing the weight

$$(4.22) \qquad\qquad \sum_{d|(x,\mathcal{P})} \mu(d) = \begin{cases} 1 & \text{if } (x,\mathcal{P})=1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Denote by $\lambda^-(d)$ the lower Rosser function of order $D$ and for each prime $p\in[z,z_1)$ denote by $\lambda_p^+(d)$ the upper Rosser function of order $D/p$. They satisfy

$$(4.23) \quad |\lambda^-(d)| \leq 1\,, \qquad \lambda^-(d) = 0 \quad \text{if} \quad \mu(d)=0 \quad \text{or} \quad d>D\,,$$
$$(4.24) \quad |\lambda_p^+(d)| \leq 1\,, \qquad \lambda_p^+(d) = 0 \quad \text{if} \quad \mu(d)=0 \quad \text{or} \quad d>D/p$$

and

$$(4.25) \qquad\qquad \sum_{d|(x,\mathcal{P})} \lambda^-(d) \leq \sum_{d|(x,\mathcal{P})} \mu(d) \leq \sum_{d|(x,\mathcal{P})} \lambda_p^+(d)\,.$$

Furthermore, let $f(s)$ and $F(s)$ be the functions of the linear sieve. We consider separately the cases $5 \mid n$ and $5 \nmid n$ and use (3.15) – (3.17), (4.7) – (4.9) to find that

$$(4.26) \qquad \sum_{\substack{d \mid \mathcal{P} \\ 5 \nmid (d,n)}} \frac{\lambda^-(d)}{d}\, \psi_n(d) \geq \mathfrak{N}(n)\Big\{ f\Big(\frac{\log D}{\log z}\Big) + \mathcal{O}\Big((\log N)^{-1/3}\Big)\Big\},$$

$$(4.27) \qquad \sum_{\substack{d \mid \mathcal{P} \\ 5 \nmid (d,n)}} \frac{\lambda_p^+(d)}{d}\, \psi_n(d) \leq \mathfrak{N}(n)\Big\{ F\Big(\frac{\log(D/p)}{\log z}\Big) + \mathcal{O}\Big((\log N)^{-1/3}\Big)\Big\},$$

where

$$(4.28) \qquad \mathfrak{N}(n) = \prod_{p \mid \mathcal{P}} \Big(1 - \frac{\psi_n(p)}{p}\Big) \asymp (\log N)^{-1}\,.$$

For the definition and properties of Rosser's functions and the functions $f(s)$, $F(s)$ as well as for explanation of (4.23) – (4.27) we refer the reader to Iwaniec [12], [13].

Consider the sum $\Gamma_4$. From (2.5), (4.20), (4.22), (4.23) and (4.25) we get

$$(4.29) \qquad\qquad\qquad \Gamma_4 \geq \Gamma_6\,,$$

where

$$\Gamma_6 = \sum_{n \in \mathcal{F}} \sum_{\substack{p_1^2 + p_2^2 + p_3^2 + x^2 = n \\ M < p_1, p_2, p_3 \leq X}} (\log p_1)(\log p_2)(\log p_3)\,\omega(x) \sum_{d \mid (x, \mathcal{P})} \lambda^-(d)$$

$$= \sum_{k \leq D} \gamma'(k) \sum_{n \in \mathcal{F}} I(n, k)$$

and where

$$(4.30) \qquad\qquad \gamma'(k) = \begin{cases} \lambda^-(k) & \text{if } k \mid \mathcal{P}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Consider now $\Gamma_5$. Using (4.21), (4.22) and (4.25) we find that

$$(4.31) \qquad\qquad\qquad \Gamma_5 \leq \Gamma_7\,,$$

where $\Gamma_7$ comes from the quantity from the right–hand side of (4.25). Changing the order of summation and applying (2.5), (4.9) and (4.24) we write $\Gamma_7$ in the form

$$\Gamma_7 = \sum_{k \leq D} \gamma''(k) \sum_{n \in \mathcal{F}} I(n, k)\,,$$

where

$$(4.32) \qquad \gamma''(k) = \sum_{z \le p < z_1} \sum_{\substack{d \mid \mathcal{P} \\ dp = k}} \left(1 - \frac{\log p}{\log z_1}\right) \lambda_p^+(d).$$

Using (4.8), (4.9), (4.23), (4.24), (4.30) and (4.32) we see that both functions $\gamma'(k)$ and $\gamma''(k)$ satisfy the conditions (4.1) and (4.2). We consider the sums

$$(4.33) \qquad \Gamma_8 = \sum_{k \le D} \gamma'(k) \sum_{n \in \mathcal{F}} \Sigma(n, k), \quad \Gamma_9 = \sum_{k \le D} \gamma''(k) \sum_{n \in \mathcal{F}} \Sigma(n, k)$$

and using the Proposition we conclude that

$$(4.34) \qquad |\Gamma_6 - \Gamma_8|, \ |\Gamma_7 - \Gamma_9| \ll F^{3/4} X^2 (\log N)^{262} + F X^2 (\log N)^{-E}.$$

According to (4.19), (4.29), (4.31) and (4.34) we have

$$(4.35) \qquad \Gamma \ge \Gamma_8 - \theta \Gamma_9 + \mathcal{O}\big(F^{3/4} X^2 (\log N)^{262}\big) + \mathcal{O}\big(F X^2 (\log N)^{-E}\big).$$

Consider more precisely the sums $\Gamma_8$ and $\Gamma_9$. From (2.6), (3.10), (3.18), (4.30) and (4.33) it follows that

$$(4.36) \qquad \Gamma_8 = 2\pi X^2 \sum_{n \in \mathcal{F}} \kappa(n)\, \xi(n) \sum_{\substack{k \mid \mathcal{P} \\ 5 \nmid (k, n)}} \frac{\lambda^-(k)}{k}\, \psi_n(k)$$

and, respectively, we apply (2.6), (3.10), (3.18) and (4.33) to get

$$(4.37) \qquad \Gamma_9 = 2\pi X^2 \sum_{n \in \mathcal{F}} \kappa(n)\, \xi(n) \sum_{\substack{k \le D \\ 5 \nmid (k, n)}} \frac{\gamma''(k)}{k}\, \psi_n(k).$$

Furthermore, according to (3.12), (4.24) and (4.32) we have

$$(4.38) \qquad \sum_{\substack{k \le D \\ 5 \nmid (k, n)}} \frac{\gamma''(k)}{k}\, \psi_n(k) = \sum_{z \le p < z_1} \frac{\psi_n(p)}{p} \left(1 - \frac{\log p}{\log z_1}\right) \sum_{\substack{d \mid \mathcal{P} \\ 5 \nmid (d, n)}} \frac{\lambda_p^+(d)}{d}\, \psi_n(d).$$

From (2.8), (3.14) – (3.17), (4.26) – (4.28) and (4.36) – (4.38) we obtain

$$(4.39) \quad \Gamma_8 - \theta \Gamma_9 \ge 2\pi X^2 \sum_{n \in \mathcal{F}} \kappa(n)\, \xi(n)$$

$$\times \mathfrak{N}(n) \left\{ \Phi(\eta, \nu, \nu_1, \theta) + \mathcal{O}\big((\log N)^{-1/3}\big) \right\},$$

where

$$\Phi(\eta, \nu, \nu_1, \theta) = f\left(\frac{\log D}{\log z}\right) - \theta \sum_{z \le p < z_1} \frac{1}{p} \left(1 - \frac{\log p}{\log z_1}\right) F\left(\frac{\log(D/p)}{\log z}\right).$$

Now we apply the arguments of Halberstam and Richert [4], chapter 9, p. 246 and we find

$$(4.40) \qquad \Phi(\eta, \nu, \nu_1, \theta) = \Phi_0(\eta, \nu, \nu_1, \theta) + \mathcal{O}\big((\log N)^{-1}\big),$$

where

$$\Phi_0(\eta, \nu, \nu_1, \theta) = f(\eta\nu^{-1}) - \theta \int_{\nu_1^{-1}}^{\nu^{-1}} F\left(\nu^{-1}\left(\eta - \frac{1}{t}\right)\right) \left(1 - \frac{1}{\nu_1 t}\right) \frac{dt}{t}.$$

We specify the constants included by

$$(4.41) \quad \theta = 1.5, \qquad \eta = 0.12499, \qquad \nu = 0.03125, \qquad \nu_1 = 0.09373.$$

It is easy to see that they satisfy the conditions (4.7) and (4.15). Furthermore, using Lemma 9.1 of Halberstam and Richert [4], we can verify that if $\eta$, $\nu$, $\nu_1$ and $\theta$ are specified by (4.41), then

$$(4.42) \qquad\qquad \Phi_0(\eta, \nu, \nu_1, \theta) > 0.$$

From (2.8), (3.14), (4.28), (4.39), (4.40) and (4.42) we get

$$(4.43) \qquad\qquad \Gamma_8 - \theta\Gamma_9 \geq c_0\, F\, X^2\, (\log N)^{-1},$$

where $c_0 > 0$ is a constant. Inequalities (4.18), (4.35) and (4.43) imply

$$(4.44) \qquad F\, X^2\, (\log N)^{-1} \ll F^{3/4}\, X^2\, (\log N)^{262} + F\, X^2\, (\log N)^{-E}.$$

We are now in a position to apply the main idea of Wooley [22]. Since $E > 1$ we can omit the second term from the right–hand side of (4.44). We get

$$F\, X^2\, (\log N)^{-1} \ll F^{3/4}\, X^2\, (\log N)^{262},$$

which implies the estimate (4.6) and proves the Theorem.

## 5. The proof of the Proposition — beginning

We represent the sum $I(n, k)$, defined by (2.5), in the form

$$(5.1) \qquad\qquad I(n, k) = \int_0^1 S^3(\alpha)\, f_k(\alpha)\, e(-n\alpha)\, d\alpha,$$

where

$$(5.2) \quad S(\alpha) = \sum_{M < p \leq X} (\log p)\, e(\alpha p^2), \quad f_k(\alpha) = \sum_{\substack{x \in \mathbb{Z} \\ x \equiv 0 \pmod{k}}} \omega(x)\, e(\alpha x^2).$$

The integration in (5.1) can be taken over any interval of length one and, in particular, over

$$J_0 = \left( (1 + [X])^{-1},\ 1 + (1 + [X])^{-1} \right].$$

We represent $J_0$ as an union of disjoint Farey intervals

$$J_0 = \bigcup_{q \le X} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{B}(q,a) \,,$$

where

(5.3) $$\mathcal{B}(q,a) = \Big( \frac{a}{q} - \frac{1}{q(q+q')} \,, \, \frac{a}{q} + \frac{1}{q(q+q'')} \Big]$$

and where $q'$ and $q''$ are specified by the conditions

(5.4) $X < q+q', q+q'' \le q+X \,; \quad aq' \equiv 1 \pmod{q}, \quad aq'' \equiv -1 \pmod{q}$

(for more details we refer the reader, for example, to Hardy and Wright [5], chapter 7).

Consider the set

(5.5) $$\mathcal{M} = \bigcup_{q \le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{N}(q,a) \,,$$

where

(5.6) $$\mathfrak{N}(q,a) = \Big( \frac{a}{q} - \frac{1}{qQ} \,, \, \frac{a}{q} + \frac{1}{qQ} \Big] \,.$$

It is clear that if $1 \le a \le q \le P$, $(a,q) = 1$, then $\mathfrak{N}(q,a) \subset \mathcal{B}(q,a)$, hence we can represent $J_0$ in the form

(5.7) $$J_0 = \mathcal{M} \cup m \,,$$

where

(5.8) $$m = m_1 \cup m_2 \,,$$

(5.9) $$m_1 = \bigcup_{q \le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \Big( \mathcal{B}(q,a) \setminus \mathfrak{N}(q,a) \Big) \,, \quad m_2 = \bigcup_{P < q \le X} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{B}(q,a) \,.$$

Hence we have

(5.10) $$I(n,k) = I_1 + I_2 \,,$$

where

(5.11) $$I_1 = \int_{\mathcal{M}} S^3(\alpha) \, f_k(\alpha) \, e(-n\alpha) \, d\alpha \,, \quad I_2 = \int_m S^3(\alpha) \, f_k(\alpha) \, e(-n\alpha) \, d\alpha \,.$$

From (4.4), (5.10) and (5.11) we see that the sum $\mathcal{E}$ can be represented as

(5.12) $$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \,,$$

where

(5.13) $\qquad \mathcal{E}_1 = \sum_{k \leq D} \gamma(k) \sum_{n \in \mathcal{F}} \left( I_1 - \Sigma(n, k) \right), \qquad \mathcal{E}_2 = \sum_{k \leq D} \gamma(k) \sum_{n \in \mathcal{F}} I_2 .$

To prove the Proposition we have to estimate the sums $\mathcal{E}_1$ and $\mathcal{E}_2$. We shall consider $\mathcal{E}_1$ in section 6 and $\mathcal{E}_2$ – in sections 7 and 8.

## 6. The estimation of the sum $\mathcal{E}_1$

We apply Lemma 3.1 of Wooley [22], which is based on the earlier result of Bauer, Liu and Zhan [1]. It states that if the set $\mathcal{M}$ is defined by (5.5), then for any integer $h$ such that $1 \leq h \leq N$, we have

(6.1) $\quad \displaystyle\int_{\mathcal{M}} S^3(\alpha) \, e(-h\alpha) \, d\alpha = \frac{\pi}{4} \, \mathfrak{S}^*(h, P) \, h^{1/2}$

$$+ \mathcal{O}\big( (\, |\mathfrak{S}^*(h, P)| + 1 \,) \, X \, (\log N)^{-2E} \big),$$

where

(6.2) $\qquad \mathfrak{S}^*(h, P) = \displaystyle\sum_{q \leq P} \varphi(q)^{-3} \sum_{a(q)^*} S^*(q, a)^3 \, e_q(-ah) .$

Using (5.2) and (5.11) we write $I_1$ in the form

$$I_1 = \sum_{\substack{x \in \mathbb{Z} \\ x \equiv 0 \pmod{k}}} \omega(x) \int_{\mathcal{M}} S^3(\alpha) \, e(-\alpha(n - x^2)) \, d\alpha .$$

We now apply (2.3), (2.4) and (6.1) to get

(6.3) $\qquad\qquad\qquad I_1 = I_1^{(1)} + \mathcal{O}\big( I_1^{(2)} \big),$

where

(6.4) $\qquad I_1^{(1)} = \dfrac{\pi}{4} \displaystyle\sum_{\substack{x \in \mathbb{Z} \\ x \equiv 0 \pmod{k}}} \omega(x) \, \mathfrak{S}^*(n - x^2, P) \, (n - x^2)^{1/2} ,$

(6.5) $\qquad I_1^{(2)} = X \, (\log N)^{-2E} \displaystyle\sum_{\substack{x \leq \frac{2}{3} X \\ x \equiv 0 \pmod{k}}} \big( |\mathfrak{S}^*(n - x^2, P)| + 1 \big) .$

We note that, due to the choice of our weight $\omega(x)$, it is not necessary to consider analogs of formula (6.1) for non–positive integers $h$, as it was done in [22].

Using (4.2), (5.13) and (6.3) we find

(6.6) $\qquad\qquad\qquad \mathcal{E}_1 = \mathcal{E}_1^{(1)} + \mathcal{O}\big( \mathcal{E}_1^{(2)} \big),$

where

$$(6.7) \quad \mathcal{E}_1^{(1)} = \sum_{k \leq D} \gamma(k) \sum_{n \in \mathcal{F}} \left( I_1^{(1)} - \Sigma(n, k) \right), \qquad \mathcal{E}_1^{(2)} = \sum_{k \leq D} \tau(k) \sum_{n \in \mathcal{F}} I_1^{(2)}.$$

Consider first $\mathcal{E}_1^{(2)}$. It was established in [22] that

$$(6.8) \qquad \mathfrak{S}^*(h, P) \ll \tau(h) \log P,$$

so, using (2.1), (6.5), (6.7) and (6.8) we find

$$(6.9) \qquad \mathcal{E}_1^{(2)} \ll X (\log N)^{-2E+1} \sum_{n \in \mathcal{F}} \mathcal{H}_n,$$

where

$$\mathcal{H}_n = \sum_{k \leq D} \tau(k) \sum_{\substack{x \leq \frac{2}{3} X \\ x \equiv 0 \pmod{k}}} \tau(n - x^2).$$

It is clear that

$$\mathcal{H}_n = \sum_{x \leq \frac{2}{3} X} \left( \sum_{\substack{k \leq D \\ k \mid x}} \tau(k) \right) \tau(n - x^2) \leq \sum_{x \leq \frac{2}{3} X} \tau^2(x) \, \tau(n - x^2)$$

$$\leq \sum_{x \leq \frac{2}{3} X} \tau^4(x) + \sum_{x \leq \frac{2}{3} X} \tau^2(n - x^2).$$

Applying Theorem 3 of Hua [11] we find

$$(6.10) \qquad \mathcal{H}_n \ll X (\log N)^B$$

for some absolute constant $B > 0$. From (6.9) and (6.10) we conclude that if $E \geq B + 1$, which we shall assume, then

$$(6.11) \qquad \mathcal{E}_1^{(2)} \ll F X^2 (\log N)^{-E}.$$

Consider now the sum $\mathcal{E}_1^{(1)}$. First we apply (6.2) and (6.4) to write the expression $I_1^{(1)}$ in the form

$$(6.12) \qquad I_1^{(1)} = \frac{\pi}{4} \sum_{q \leq P} \varphi(q)^{-3} \sum_{a(q)^*} S^*(q, a)^3 \, W(n, k, q, a),$$

where

$$W = W(n, k, q, a) = \sum_{\substack{x \in \mathbb{Z} \\ x \equiv 0 \pmod{k}}} \omega(x) \, e_q(-a(n - x^2)) \, (n - x^2)^{1/2}.$$

Furthermore, we have

$$(6.13) \qquad W = \sum_{\substack{m(q) \\ m \equiv 0 \pmod{(k,q)}}} e_q(a(m^2 - n)) \, W^*(n, k, q, a),$$

where

$$(6.14) \qquad W^* = W^*(n, k, q, a) = \sum_{\substack{x \in \mathbb{Z} \\ x \equiv 0 \pmod{k} \\ x \equiv m \pmod{q}}} \omega(x)\,(n - x^2)^{1/2}\,.$$

Obviously, if $(k, q) \nmid m$, then $W^* = 0$. If $(k, q) \mid m$, then there exists unique $h = h_{m,q,k} \pmod{[k, q]}$, such that the system of congruences in the domain of summation of (6.14) is equivalent to $x \equiv h \pmod{[k, q]}$. We apply Poisson's summation formula and use (2.3) and (2.4). After some calculations we get

$$W^* = \frac{X^2}{[k, q]} \sum_{l \in \mathbb{Z}} e\Big(\frac{lh}{[k, q]}\Big) \int_{-\infty}^{\infty} \omega_0(t)\,(nN^{-1} - t^2)^{1/2}\, e\Big(-\frac{lX}{[k, q]}\,t\Big)\, dt\,.$$

If $|l| \geq 1$ then we integrate the last integral by parts $m$ times. Having in mind the conditions $k \leq D$, $q \leq P$ and using (2.1) and (4.3) we find that the integral is

$$\ll \Big(\frac{|l|X}{[k, q]}\Big)^{-m} \ll |l|^{-m}\, X^{-m/2}\,,$$

where the constant in Vinogradov's symbol depends only on $m$. From this observation we conclude that the contribution to $W^*$, coming from the terms with $|l| \geq 1$, is negligible. More precisely, we have

$$(6.15) \qquad W^* = \frac{X^2}{[k, q]}\, \kappa(n) + \mathcal{O}(X^{-A})\,,$$

where $\kappa(n)$ is defined by (2.7) and where the constant $A > 0$ is arbitrarily large.

From (2.10), (2.11), (6.12), (6.13) and (6.15) we get

$$I_1^{(1)} = \frac{\pi}{4}\, X^2\, \kappa(n)\, k^{-1} \sum_{q \leq P} f(q, n, k) + \mathcal{O}(X^{-A})\,.$$

It remains to take into account also (2.6), (2.8), (2.9) and (3.9) and we find

$$I_1^{(1)} = \Sigma(n, k) + \mathcal{O}\big(X^{2+\varepsilon}\, k^{-1}\, P^{-1}\big)\,.$$

This formula and (2.1), (4.2), (6.7) imply

$$(6.16) \qquad \mathcal{E}_1^{(1)} \ll F\, X^{2+\varepsilon}\, P^{-1} \ll F\, X^{2-\varepsilon}\,.$$

We note that any, arbitrarily small positive value of the constant $\delta$ from the definition of $P$, suffices for the proof of the last estimate. This happens wherever $P$ occurs, so any progress in obtaining asymptotic formulas of type (6.1) for larger sets of major arcs is not relevant to our problem.

Finally, from (6.6), (6.11) and (6.16) we obtain the estimate

$$(6.17) \qquad \mathcal{E}_1 \ll F\, X^2\, (\log N)^{-E}\,.$$

## 7. The estimation of the sum $\mathcal{E}_2$

Using (5.11) and (5.13) we represent the sum $\mathcal{E}_2$ in the form

$$\mathcal{E}_2 = \int_m S^3(\alpha)\, H(\alpha)\, K(-\alpha)\, d\alpha\,,$$

where

(7.1)        $$H(\alpha) = \sum_{k \leq D} \gamma(k)\, f_k(\alpha)\,, \qquad K(\alpha) = \sum_{n \in \mathcal{F}} e(\alpha n)\,.$$

Applying Hölder's inequality we get

(7.2)        $$\mathcal{E}_2 \ll \int_m |\, S^3(\alpha)\, H(\alpha)\, K(-\alpha)\,|\, d\alpha$$

$$\ll F^{1/2} \int_m |S(\alpha)|^3\, |H(\alpha)|\, |K(\alpha)|^{1/2}\, d\alpha$$

$$\ll F^{1/2}\, T_1^{3/4}\, T_2^{1/4}\,,$$

where

(7.3)        $$T_1 = \int_m |S(\alpha)|^4\, d\alpha\,, \qquad T_2 = \int_m |H(\alpha)|^4\, |K(\alpha)|^2\, d\alpha\,.$$

For $T_1$ we apply the well known estimate

(7.4)        $$T_1 \ll \int_0^1 |S(\alpha)|^4\, d\alpha \ll X^2\, (\log N)^7\,.$$

Consider $T_2$. From (7.1) and (7.3) we get

(7.5)        $$T_2 = \int_m |H(\alpha)|^4 \sum_{n_1, n_2 \in \mathcal{F}} e(\alpha(n_1 - n_2))\, d\alpha$$

$$= F \int_m |H(\alpha)|^4\, d\alpha + \sum_{\substack{n_1, n_2 \in \mathcal{F} \\ n_1 \neq n_2}} \int_m |H(\alpha)|^4\, e(\alpha(n_1 - n_2))\, d\alpha$$

$$\ll F \int_0^1 |H(\alpha)|^4\, d\alpha + F^2 \max_{1 \leq l \leq N} |\Phi(l)|\,,$$

where

(7.6)        $$\Phi(l) = \int_m |H(\alpha)|^4\, e(\alpha l)\, d\alpha\,.$$

To estimate the first term from the last line of (7.5) we apply the inequality

(7.7)        $$\int_0^1 |H(\alpha)|^4\, d\alpha \ll X^2\, (\log N)^{1027}\,.$$

Its proof is very similar to the proof of formula (4.3) from author's paper [20], so we omit it.

In the next section we shall estimate $\Phi(l)$ and we shall prove that

$$(7.8) \qquad \max_{1 \le l \le N} |\Phi(l)| \ll D^4 \, X^{3/2+\varepsilon} + X^{2+\varepsilon} \, P^{-1/2} \, .$$

From (2.1), (4.3), (7.5), (7.7) and (7.8) we obtain

$$(7.9) \qquad T_2 \ll F \, X^2 \, (\log N)^{1027} + F^2 \, X^{2-\varepsilon} \, .$$

Applying (7.2), (7.4) and (7.9) we get

$$(7.10) \qquad \mathcal{E}_2 \ll F^{3/4} \, X^2 \, (\log N)^{262} + F \, X^{2-\varepsilon} \, .$$

It remains to combine (5.12), (6.17) and (7.10) and we find that the estimate (4.5) holds, which proves the Proposition.

## 8. The estimation of $\Phi(l)$

For any integers $a, q$, satisfying $1 \le a \le q \le X$, $(a, q) = 1$, we consider the set $\mathfrak{M}(q, a)$, defined by

$$(8.1) \quad \mathfrak{M}(q, a) = \begin{cases} \left( - \frac{1}{q(q+q')}, \, \frac{1}{q(q+q'')} \right] \setminus \left( - \frac{1}{qQ}, \, \frac{1}{qQ} \right] & \text{if } q \le P, \\ \\ \left( - \frac{1}{q(q+q')}, \, \frac{1}{q(q+q'')} \right] & \text{if } P < q \le X, \end{cases}$$

where the integers $q'$ and $q''$ are specified by (5.4). Using (5.3), (5.6), (5.8), (5.9), (7.6) and (8.1) we find that

$$(8.2) \qquad \Phi(l) = \sum_{q \le X} \sum_{a(q)^*} \int_{\mathfrak{M}(q,a)} \left| H\left( \frac{a}{q} + \beta \right) \right|^4 e\left( l\left( \frac{a}{q} + \beta \right) \right) d\beta \, .$$

If $q \le X$, $|\beta| \le (qX)^{-1}$ and $k \le X$, then we have

$$(8.3) \quad f_k\left( \frac{a}{q} + \beta \right) = \frac{X}{qk} \sum_{|m| \le kX^\varepsilon} S(q, ak^2, m) \, I\left( \beta N, -\frac{Xn}{qk} \right) + \mathcal{O}(X^{-A}) \, ,$$

where

$$(8.4) \qquad S(q, h, m) = \sum_{x(q)} e_q(hx^2 + mx) \, ,$$

$$(8.5) \qquad I(\beta, u) = \int_{-\infty}^{\infty} \omega_0(x) \, e(\beta x^2 + ux) \, dx \, .$$

Formula (8.3) is a special case of Lemma 12 from the paper [8] by Heath–Brown and the author.

From (7.1) and (8.3) it follows that the integrand in (8.2) can be represented in the form

$$(8.6) \qquad \frac{X^4}{q^4} \sum_{k_1,\dots,k_4 \le D} \frac{\gamma(k_1)\dots\gamma(k_4)}{k_1\dots k_4} \sum_{\mathbf{n}\in\mathcal{N}_{\mathbf{k}}} e_q(al)\, S_{\mathbf{k}}(q,a,\mathbf{n})$$
$$\times\, e(\beta l)\, I_{\mathbf{k}}(\beta N, X q^{-1}\mathbf{n}) + \mathcal{O}(X^{-A}),$$

where

$$(8.7) \qquad \mathcal{N}_{\mathbf{k}} = \{\, \mathbf{n}\in\mathbb{Z}^4 \;:\; |n_i| \le k_i X^\varepsilon,\;\; i=1,2,3,4 \,\},$$

$$(8.8) \qquad S_{\mathbf{k}}(q,a,\mathbf{n}) = S(q, ak_1^2, n_1)\, S(q, ak_2^2, n_2)$$
$$\times\, S(q, -ak_3^2, -n_3)\, S(q, -ak_4^2, -n_4),$$

$$(8.9) \qquad I_{\mathbf{k}}(\gamma,\mathbf{u}) = I(\gamma, -u_1 k_1^{-1})\, I(\gamma, -u_2 k_2^{-1})$$
$$\times\, I(-\gamma, u_3 k_3^{-1})\, I(-\gamma, u_4 k_4^{-1})$$

and where $\mathbf{n}$, $\mathbf{k}$ and $\mathbf{u}$ are four dimensional vectors with components, respectively, $n_i$, $k_i$ and $u_i$, $i=1,2,3,4$.

We substitute the expression (8.6) for the integrand in (8.2). Then, in order to apply Kloosterman's method, we change the order of integration and summation over $a$. We find that

$$(8.10) \qquad \Phi(l) = X^4 \sum_{q\le X} q^{-4} \sum_{k_1,\dots,k_4 \le D} \frac{\gamma(k_1)\dots\gamma(k_4)}{k_1\dots k_4}$$
$$\times \sum_{\mathbf{n}\in\mathcal{N}_{\mathbf{k}}} \int_{\beta\in\Delta_q} \sum_{\substack{a(q)^* \\ \mathfrak{M}(q,a)\ni\beta}} e_q(al)\, S_{\mathbf{k}}(q,a,\mathbf{n})$$
$$\times\, e(\beta l)\, I_{\mathbf{k}}(\beta N, X q^{-1}\mathbf{n})\, d\beta \;+\; \mathcal{O}(X^{-A}),$$

where

$$(8.11) \qquad \Delta_q = \begin{cases} \left(-\frac{1}{qX},\,\frac{1}{qX}\right] \setminus \left(-\frac{1}{qQ},\,\frac{1}{qQ}\right] & \text{if } q \le P, \\[2mm] \left(-\frac{1}{qX},\,\frac{1}{qX}\right] & \text{if } P < q \le X. \end{cases}$$

To proceed further we have to express the condition $\mathfrak{M}(q,a) \ni \beta$ in more convenient form. We do this in a standard way by introducing a function $\sigma(v,q,\beta)$, defined for integers $q,v$ such that $1 \le q \le X$, $-q/2 < v \le q/2$ and real numbers $\beta \in [-(qX)^{-1}, (qX)^{-1}]$. For fixed $v$ and $q$ this function is integrable with respect to $\beta$ and satisfy

$$(8.12) \qquad |\sigma(v,q,\beta)| \le (1+|v|)^{-1} \qquad \text{for all} \qquad v,q,\beta.$$

Furthermore, if $(a, q) = 1$, if $\bar{a} \pmod{q}$ is defined by $a\bar{a} \equiv 1 \pmod{q}$ and if $q'$ and $q''$ are specified by (5.4), then

$$(8.13) \qquad \sum_{-\frac{q}{2} < v \le \frac{q}{2}} \sigma(v, q, \beta)\, e_q(\bar{a} v) = \begin{cases} 1 & \text{if } \beta \in \left( -\frac{1}{q(q+q')}, \frac{1}{q(q+q'')} \right], \\ 0 & \text{otherwise} . \end{cases}$$

A construction of a function with these properties is available, for example, in Heath–Brown [7], section 3.

Using (8.13) we can express the condition $\mathfrak{M}(q, a) \ni \beta$ from the domain of summation in (8.10).

If $P < q \le X$, then, according to (8.1), (8.11) and (8.13), we find that

$$(8.14) \qquad \int_{\beta \in \Delta_q} \sum_{\substack{a(q)^* \\ \mathfrak{M}(q,a) \ni \beta}} \cdots = \int_{\beta \in \Delta_q} \sum_{a(q)^*} \sum_{-\frac{q}{2} < v \le \frac{q}{2}} \sigma(v, q, \beta)\, e_q(\bar{a}\, v) \ldots .$$

If $q \le P$ then we integrate over $\beta \in \Delta_q = \left( -\frac{1}{qX}, \frac{1}{qX} \right] \setminus \left( -\frac{1}{qQ}, \frac{1}{qQ} \right]$, hence, having in mind (8.1), we see that the condition $\mathfrak{M}(q, a) \ni \beta$ in (8.10) is equivalent to $\left( -\frac{1}{q(q+q')}, \frac{1}{q(q+q'')} \right] \ni \beta$. Therefore we can use again (8.13) to obtain (8.14).

We conclude that $\Phi(l)$ can be written in the form

$$(8.15) \qquad \Phi(l) = X^4 \sum_{q \le X} q^{-4} \sum_{k_1,\ldots,k_4 \le D} \frac{\gamma(k_1)\ldots\gamma(k_4)}{k_1 \ldots k_4}$$

$$\times \sum_{\mathbf{n} \in \mathcal{N}_{\mathbf{k}}} \int_{\beta \in \Delta_q} \sum_{-\frac{q}{2} < v \le \frac{q}{2}} \sigma(v, q, \beta)\, e(\beta l)$$

$$\times I_{\mathbf{k}}(\beta N, X q^{-1}\mathbf{n})\, \mathcal{W}(q, \mathbf{k}, \mathbf{n}, l, v)\, d\beta + \mathcal{O}(X^{-A}) ,$$

where

$$(8.16) \qquad \mathcal{W} = \mathcal{W}(q, \mathbf{k}, \mathbf{n}, l, v) = \sum_{a(q)^*} e_q(\bar{a} v + al)\, S_{\mathbf{k}}(q, a, \mathbf{n}) .$$

Using (4.1), (4.2), (8.12) and (8.15) we find that

$$(8.17) \qquad \Phi(l) \ll X^4 \sum_{q \le X} q^{-4} \sideset{}{'}\sum_{k_1,\ldots,k_4 \le D} \frac{\tau(k_1)\ldots\tau(k_4)}{k_1 \ldots k_4} \sum_{\mathbf{n} \in \mathcal{N}_{\mathbf{k}}}$$

$$\times \sum_{|v| \le X} (1 + |v|)^{-1}\, T(q, \mathbf{k}, \mathbf{n})\, |\mathcal{W}(q, \mathbf{k}, \mathbf{n}, l, v)| ,$$

where $\sum'$ means that the summation is taken over squarefree odd $k_i$ and

$$(8.18) \qquad T = T(q, \mathbf{k}, \mathbf{n}) = \int_{\beta \in \Delta_q} |\, I_{\mathbf{k}}(\beta N, X q^{-1}\mathbf{n})\,|\, d\beta .$$

Consider the sum $\mathcal{W}$. From (8.4), (8.8) and (8.16) we see that it can be written as

$$(8.19) \quad \mathcal{W} = \sum_{x_1,\ldots,x_4(q)} \sum_{a(q)^*} e_q \Big( a\big( k_1^2 x_1^2 + k_2^2 x_2^2 - k_3^2 x_3^2 - k_4^2 x_4^2 \big)$$

$$+ \bar{a}v + al + n_1 x_1 + n_2 x_2 - n_3 x_3 - n_4 x_4 \Big).$$

In this form the sum $\mathcal{W}$ is very similar to a sum, considered by Brüdern and Fouvry [2]. Applying the method, developed for the proof of Lemma 1 from [2], we find that if $k_i$ are squarefree odd integers, then

$$(8.20) \qquad \mathcal{W} \ll q^{5/2+\varepsilon} \left\{ (q,l)\,(q,k_1^2) \, \ldots \, (q,k_4^2) \right\}^{1/2}.$$

Consider the quantity $T$, defined by (8.18). Obviously

$$(8.21) \qquad T(q,\mathbf{k},\mathbf{n}) \ll X^{-2} \int_{-\infty}^{\infty} |I_{\mathbf{k}}(\gamma, Xq^{-1}\mathbf{n})|\, d\gamma.$$

Applying (8.5), (8.7), (8.9), (8.21) and Lemma 10(ii) of [8] we get

$$(8.22) \quad T(q,\mathbf{k},\mathbf{n}) \ll X^{-3+\varepsilon}\, q \left( \sum_{i=1}^{4} \frac{|n_i|}{k_i} \right)^{-1} \quad \text{for} \quad \mathbf{n} \neq \mathbf{0}, \quad \mathbf{n} \in \mathcal{N}_{\mathbf{k}}.$$

If $\mathbf{n} = \mathbf{0}$ and $P < q \leq X$, then we use (8.5), (8.9), (8.21) and apply the well known estimate

$$(8.23) \qquad I(\beta, u) \ll \min\left( 1, |\beta|^{-1/2} \right).$$

We get

$$(8.24) \qquad T(q,\mathbf{k},\mathbf{0}) \ll X^{-2} \qquad \text{for} \qquad P < q \leq X.$$

If $\mathbf{n} = \mathbf{0}$ and $q \leq P$, then we use (2.2), (8.11), (8.18) and (8.23) to obtain

$$(8.25) \quad T(q,\mathbf{k},\mathbf{0}) \ll X^{-2} \int_{X^2/(qQ)}^{\infty} \frac{d\beta}{\beta^2} \ll q\, X^{-2}\, P^{-1} \qquad \text{for} \qquad q \leq P.$$

From (8.17) we find that

$$(8.26) \qquad \Phi(l) \ll \Phi_1 + \Phi_2 + \Phi_3,$$

where $\Phi_1$ is the contribution of the terms from the right–hand side of (8.17) such that $\mathbf{n} \neq \mathbf{0}$, $\Phi_2$ comes from the terms with $\mathbf{n} = \mathbf{0}$ and $P < q \leq X$ and, finally, $\Phi_3$ comes from the terms for which $\mathbf{n} = \mathbf{0}$ and $q \leq P$.

To estimate $\Phi_1$ we use (8.20) and (8.22) and we get

$$\Phi_1 \ll X^{1+\varepsilon} \sum_{q \leq X} q^{-1/2} (q,l)^{1/2} \sum_{k_1,\ldots,k_4 \leq D} \frac{\tau(k_1)\ldots\tau(k_4)}{k_1 \ldots k_4}$$
$$\times \left( (q,k_1^2)\ldots(q,k_4^2) \right)^{1/2} \sum_{\substack{\mathbf{n} \in \mathcal{N}_{\mathbf{k}} \\ \mathbf{n} \neq \mathbf{0}}} \left( \sum_{i=1}^{4} \frac{|n_i|}{k_i} \right)^{-1}.$$

After some standard calculations, which are very similar to those in section 5 of [8], we obtain

$$(8.27) \qquad \Phi_1 \ll D^4 X^{3/2+\varepsilon}.$$

We leave the verification of this estimate to the reader.

For $\Phi_2$ we apply (8.20) and (8.24) to get

$$(8.28) \qquad \Phi_2 \ll X^{2+\varepsilon} \sum_{P < q \leq X} q^{-3/2} (q,l)^{1/2} \left( \sum_{k \leq D} \frac{\tau(k)(q,k)}{k} \right)^4$$
$$\ll X^{2+\varepsilon} P^{-1/2}.$$

To estimate $\Phi_3$ we apply, respectively, (8.20) and (8.25) and we find

$$(8.29) \qquad \Phi_3 \ll X^{2+\varepsilon} P^{-1} \sum_{q \leq P} q^{-1/2} (q,l)^{1/2} \left( \sum_{k \leq D} \frac{\tau(k)(q,k)}{k} \right)^4$$
$$\ll X^{2+\varepsilon} P^{-1/2}.$$

The estimate (7.8) is a consequence of (8.26) – (8.29).

This proves the Proposition and now the proof of the Theorem is complete.

**Added in proof:** Two interesting results, concerning the quantity $E_1(Y)$, defined in section 1, appeared since the present paper was submitted for publication. J.Liu, Wooley and Yu [17] established the estimate $E_1(Y) \ll Y^{3/8+\varepsilon}$ and very recently Harman and Kumchev [6] proved that $E_1(Y) \ll Y^{5/14+\varepsilon}$.

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