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HERMITE-HADAMARD TYPE INEQUALITIES FOR MAPPINGS WHOSE DERIVATIVES ARE $s-$CONVEX IN THE SECOND SENSE VIA FRACTIONAL INTEGRALS

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Abstract. In this paper we establish Hermite-Hadamard type inequalities for mappings whose derivatives are $s-$convex in the second sense and concave.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known that the Hermite-Hadamard inequality for convex function. Both inequalities hold in the reserved direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; see, for example see ([1]-[21]).

Definition 1.1. ([18]) A function $f : [0, \infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

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for all $x, y \in [0, \infty), \lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$. This class of $s-$convex functions is usually denoted by K_s^2.

In ([15]) Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense:

Theorem 1.2. *Suppose that $f : [0, \infty) \to [0, \infty)$ is an $s-$convex function in the second sense ,where $s \in (0, 1)$ and let $a, b \in [0, \infty), a < b$. If $f' \in L^1([a, b])$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1} \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2)

The following results are proved by M.I.Bhatti et al. (see [8]).

Theorem 1.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $|f''|$ is convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \qquad (1.3)\\ &\le \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{|f''(a)|+|f''(b)|}{2}\right]\\ &\le \frac{(b-a)^2}{\alpha+1}\beta(2, \alpha+1)\left[\frac{|f''(a)|+|f''(b)|}{2}\right] \end{aligned}$$

where β is Euler Beta function.

Theorem 1.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $p \in \mathbb{R}, p > 1$ such that $|f''|^{\frac{p}{p-1}}$ is convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \qquad (1.4)\\ &\le \frac{(b-a)^2}{\alpha+1}\beta^{\frac{1}{p}}(p+1, \alpha p+1)\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

where β is Euler Beta function.

Theorem 1.5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $q \ge 1$ such that $|f''|^q$ is convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \qquad (1.5)\\ &\le \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)}\left[\begin{array}{c}\left(\frac{2\alpha+4}{3\alpha+9}|f''(a)|^q + \frac{\alpha+5}{3\alpha+9}|f''(b)|^q\right)^{\frac{1}{q}}\\ +\left(\frac{\alpha+5}{3\alpha+9}|f''(a)|^q + \frac{2\alpha+4}{3\alpha+9}|f''(b)|^q\right)^{\frac{1}{q}}\end{array}\right]. \end{aligned}$$

Theorem 1.6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $p \in \mathbb{R}, p > 1$ with $q = \frac{p}{p-1}$ such that $|f''|^q$ is concave function on I. Suppose that $a, b \in I^{\circ}$ with $a < b$ and $f'' \in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) \right] \right| \\ \leq \; & \frac{(b-a)^2}{\alpha+1} \beta^{\frac{1}{p}} (p+1, \alpha p+1) \left| f'' \left(\frac{a+b}{2} \right) \right| \end{aligned} \tag{1.6}$$

where β is Euler Beta function.

We will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further this paper.

Definition 1.7. Let $f \in L[a,b]$. The Reimann-Liouville integrals $J_{a^+}^{\alpha} f(x)$ and $J_{b^-}^{\alpha} f(x)$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$ is the Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$ the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities, see [3]-[25].

In this paper, we establish fractional integral ineqalities of Hermite-Hadamard type for mappings whose derivatives are $s-$convex and concave.

2. Main results

In order to prove our main theorems we need the following lemma (see [8]).

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°,the interior of I. Assume that $a, b \in I^{\circ}$ with $a < b$ and $f'' \in L[a,b]$, then the following identity for fractional integral with $\alpha > 0$ holds:*

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) \right] \\ = \; & \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 t(1-t^{\alpha}) \left[f''(ta+(1-t)b) + f''((1-t)a+tb) \right] dt \end{aligned} \tag{2.1}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Theorem 2.2. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° and let $a,b \in I^\circ$ with $a < b$ and $f'' \in L[a,b]$. If $|f''|$ is $s-$convex in the second sense on I for some fixed $s \in (0,1]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \\ \leq\ & \frac{(b-a)^2}{2(\alpha+1)}\left[\frac{\alpha}{(s+2)(\alpha+s+2)} + \beta(2,s+1) - \beta(\alpha+2,s+1)\right] \\ & \times\left[|f''(a)| + |f''(b)|\right] \end{aligned} \tag{2.2}$$

where β is Euler Beta function.

Proof. From Lemma 2.1 since $|f''|$ is $s-$convex in the second sense on I, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \\ \leq\ & \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 |t(1-t^\alpha)|\left[|f''(ta+(1-t)b)| + |f''((1-t)a+tb)|\right]dt \\ \leq\ & \frac{(b-a)^2}{2(\alpha+1)}\left[\int_0^1 t(1-t^\alpha)\left[t^s|f''(a)| + (1-t)^s|f''(b)|\right]dt\right. \\ & \left. + \int_0^1 t(1-t^\alpha)\left[(1-t)^s|f''(a)| + t^s|f''(b)|\right]dt\right] \\ =\ & \frac{(b-a)^2}{2(\alpha+1)}\left[\int_0^1 t^{s+1}(1-t^\alpha)\,dt + \int_0^1 t(1-t^\alpha)(1-t)^s dt\right]\left[|f''(a)| + |f''(b)|\right] \\ =\ & \frac{(b-a)^2}{2(\alpha+1)}\left[\frac{\alpha}{(s+2)(\alpha+s+2)} + \beta(2,s+1) - \beta(\alpha+2,s+1)\right] \\ & \times\left[|f''(a)| + |f''(b)|\right] \end{aligned}$$

where we used the fact that

$$\int_0^1 t^{s+1}(1-t^\alpha)\,dt = \frac{\alpha}{(s+2)(\alpha+s+2)}$$

and

$$\int_0^1 t(1-t^\alpha)(1-t)^s dt = \beta(2,s+1) - \beta(\alpha+2,s+1)$$

which completes the proof. □

Remark 2.3. In Theorem 2.2 if we choose $s = 1$ then (2.2) reduces the inequality (1.3) of Theorem 1.3.

Theorem 2.4. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I°. Suppose that $a,b \in I^\circ$ with $a < b$ and $f'' \in L[a,b]$. If $|f''|^q$ is $s-$convex in the second sense on I for some fixed $s \in (0,1]$, $p,q > 1$, then the following inequality for fractional integrals with $\alpha \in (0,1]$ holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \tag{2.3}$$
$$\leq \frac{(b-a)^{2}}{\alpha+1}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\left[\frac{|f''(a)|^{q}+|f''(b)|^{q}}{s+1}\right]^{\frac{1}{q}}$$

where β is Euler Beta function and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1, using the well known Hölder inequality and $|f''|^{q}$ is $s-$convex in the second sense on I, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{(b-a)^{2}}{2(\alpha+1)}\int_{0}^{1}|t(1-t^{\alpha})|\left[|f''(ta+(1-t)b)|+|f''((1-t)a+tb)|\right]dt \\
\leq\ & \frac{(b-a)^{2}}{2(\alpha+1)}\left(\int_{0}^{1}t^{p}(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}} \\
&\times\left[\left(\int_{0}^{1}|f''(ta+(1-t)b)|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{0}^{1}|f''((1-t)a+tb)|^{q}dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)^{2}}{2(\alpha+1)}\left(\int_{0}^{1}t^{p}(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}} \\
&\times\left[\begin{array}{c}\left(\int_{0}^{1}(t^{s}|f''(a)|^{q}+(1-t)^{s}|f''(b)|^{q})dt\right)^{\frac{1}{q}} \\ +\left(\int_{0}^{1}((1-t)^{s}|f''(a)|^{q}+t^{s}|f''(b)|^{q})dt\right)^{\frac{1}{q}}\end{array}\right] \\
=\ & \frac{(b-a)^{2}}{2(\alpha+1)}\left(\int_{0}^{1}t^{p}(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(|f''(a)|^{q}\frac{1}{s+1}+|f''(b)|^{q}\frac{1}{s+1}\right)^{\frac{1}{q}} \\ +\left(|f''(a)|^{q}\frac{1}{s+1}+|f''(b)|^{q}\frac{1}{s+1}\right)^{\frac{1}{q}}\end{array}\right] \\
\leq\ & \frac{(b-a)^{2}}{\alpha+1}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\left[\frac{|f''(a)|^{q}+|f''(b)|^{q}}{s+1}\right]^{\frac{1}{q}}
\end{aligned}$$

where we used the fact that

$$\int_{0}^{1}t^{s}dt=\int_{0}^{1}(1-t)^{s}dt=\frac{1}{s+1}$$

and

$$\int_{0}^{1}t^{p}(1-t^{\alpha})^{p}dt\leq\int_{0}^{1}t^{p}(1-t)^{\alpha p}dt=\beta(p+1,\alpha p+1)$$

which completes the proof. □

Remark 2.5. In Theorem 2.4 if we choose $s=1$ then (2.3) reduces the inequality (1.4) of Theorem 1.4.

Theorem 2.6. *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a twice differentiable function on I°. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$. If $|f''|^q$ is $s-$convex in the second sense on I for some fixed $s \in (0, 1]$ and $q \geq 1$ then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right| \qquad (2.4)\\
\leq\ & \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)}\\
&\times\left[\left(|f''(a)|^q \tfrac{2\alpha+4}{(s+2)(\alpha+s+2)}+|f''(b)|^q \tfrac{[\beta(2,s+1)-\beta(\alpha+2,s+1)](2\alpha+4)}{\alpha}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(|f''(a)|^q \tfrac{[\beta(2,s+1)-\beta(\alpha+2,s+1)](2\alpha+4)}{\alpha}+|f''(b)|^q \tfrac{2\alpha+4}{(s+2)(\alpha+s+2)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. From Lemma 2.1, using power mean inequality and $|f''|^q$ is $s-$convex in the second sense on I we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\\
\leq\ & \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 |t(1-t^\alpha)|\left[|f''(ta+(1-t)b)|+|f''((1-t)a+tb)|\right]dt\\
\leq\ & \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t(1-t^\alpha)\,dt\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 t(1-t^\alpha)\,|f''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 t(1-t^\alpha)\,|f''((1-t)a+tb)|^q\,dt\right)^{\frac{1}{q}}\right]\\
\leq\ & \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t(1-t^\alpha)\,dt\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_0^1 \left[t^{s+1}(1-t^\alpha)|f''(a)|^q+t(1-t^\alpha)(1-t)^s|f''(b)|^q\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 t(1-t^\alpha)(1-t)^s|f''(a)|^q+t^{s+1}(1-t^\alpha)|f''(b)|^q\,dt\right)^{\frac{1}{q}}\right]\\
=\ & \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{\alpha}{2(\alpha+2)}\right)^{1-\frac{1}{q}}\\
&\times\left[\left(|f''(a)|^q \tfrac{\alpha}{(s+2)(\alpha+s+2)}+|f''(b)|^q\left[\beta(2,s+1)-\beta(\alpha+2,s+1)\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\left(|f''(a)|^q\left[\beta(2,s+1)-\beta(\alpha+2,s+1)\right]+|f''(b)|^q \tfrac{\alpha}{(s+2)(\alpha+s+2)}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$= \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)} \times \left[\left(|f''(a)|^q \tfrac{(2\alpha+4)}{(s+2)(\alpha+s+2)} + |f''(b)|^q \tfrac{[\beta(2,s+1)-\beta(\alpha+2,s+1)](2\alpha+4)}{\alpha} \right)^{\frac{1}{q}} + \left(|f''(a)|^q \tfrac{[\beta(2,s+1)-\beta(\alpha+2,s+1)](2\alpha+4)}{\alpha} + |f''(b)|^q \tfrac{2\alpha+4}{(s+2)(\alpha+s+2)} \right)^{\frac{1}{q}} \right]$$

where we used the fact that

$$\int_0^1 t^{s+1}(1-t^\alpha)\,dt = \frac{\alpha}{(s+2)(\alpha+s+2)}$$

and

$$\int_0^1 t(1-t^\alpha)(1-t)^s\,dt = \beta(2,s+1) - \beta(\alpha+2,s+1)$$

which completes the proof. □

Remark 2.7. In Theorem 2.6 if we choose $s=1$ then (2.4) reduces the inequality (1.5) of Theorem 1.5.

The following result holds for $s-$concavity.

Theorem 2.8. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I°. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a,b]$. If $|f''|^q$ is $s-$concave in the second sense on I for some fixed $s \in (0,1]$ and $p, q > 1$, then the following inequality for fractional integrals with $\alpha \in (0,1]$ holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right] \right| \leq \frac{(b-a)^2}{\alpha+1}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\,2^{\frac{s-1}{q}} \left| f''\left(\frac{a+b}{2}\right) \right| \tag{2.5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and β is Euler Beta function.

Proof. From Lemma 2.1 and using the Hölder inequality we have

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right] \right| \qquad (2.6)\\ &\leq \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 |t(1-t^\alpha)| \left[|f''(ta+(1-t)b)| + |f''((1-t)a+tb)| \right] dt \\ &\leq \frac{(b-a)^2}{2(\alpha+1)} \left(\int_0^1 t^p (1-t^\alpha)^p\,dt \right)^{\frac{1}{p}} \\ &\quad \times \left[\left(\int_0^1 |f''(ta+(1-t)b)|^q\,dt \right)^{\frac{1}{q}} + \left(\int_0^1 |f''((1-t)a+tb)|^q\,dt \right)^{\frac{1}{q}} \right] \end{aligned}$$

Since $|f''|^q$ is $s-$concave using inequality (1.2) we get (see [2])

$$\int_0^1 |f''(ta+(1-t)b)|^q \, dt \leq 2^{s-1} \left| f''\left(\frac{a+b}{2}\right)\right|^q \tag{2.7}$$

and

$$\int_0^1 |f''((1-t)a+tb)|^q \, dt \leq 2^{s-1} \left| f''\left(\frac{b+a}{2}\right)\right|^q \tag{2.8}$$

Using (2.7) and (2.8) in (2.6), we have

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \\ \leq\ & \frac{(b-a)^2}{\alpha+1}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\, 2^{\frac{s-1}{q}} \left| f''\left(\frac{a+b}{2}\right)\right| \end{aligned}$$

which completes the proof. □

Remark 2.9. In Theorem 2.8 if we choose $s=1$ then (2.5) reduces inequality (1.6) of Theorem 1.6.

References

1. M. Alomari, M. Darus, *On the Hadamard's inequality for log-convex functions on the coordinates*, J. Inequal. Appl. **2009** (2009), 13. Article ID 283147.
2. M. Alomari, M. Darus, S.S. Dragomir, P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*, Appl. Math. Lett. **23** (2010), 1071–1076.
3. G. Anastassiou, M.R. Hooshmandasl, A. Ghasemi, F. Moftakharzadeh, *Montogomery identities for fractional integrals and related fractional inequalities*, J. Ineq. Pure Appl. Math. **10** (2009), no. 4, Art 97.
4. A.G. Azpeitia, *Convex functions and the Hadamard inequality*, Revista Colombiana Mat. **28** (1994), 7–12.
5. M.K. Bakula, M.E. Özdemir, J. Pečarić, *Hadamard tpye inequalities for m-convex and (α,m)-convex functions*, J. Ineq. Pure Appl. Math. **9** (2008), no. 4, Art. 96.
6. M.K. Bakula, J. Pečarić, *Note on some Hadamard-type inequalities*, J. Ineq. Pure Appl. Math. **5** (2004), no. 3, Article 74.
7. S. Belarbi, Z. Dahmani, *On some new fractional integral inequalities*, J. Ineq. Pure Appl. Math. **10** (2009), no. 3, Art. 86.
8. M.I. Bhatti, M. Iqbal and S.S. Dragomir, *Some new fractional integral Hermite-Hadamard type inequalities*, RGMIA Res. Rep. Coll., **16** (2013), Article 2.
9. Z. Dahmani, *New inequalities in fractional integrals*, Int. J. Nonlinear Sci. **9** (2010), no. 4, 493–497.
10. Z. Dahmani, *On Minkowski and Hermite–Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. **1** (2010), no. 1, 51–58.
11. Z. Dahmani, L. Tabharit, S. Taf, *Some fractional integral inequalities*, Nonlinear. Sci. Lett. A **1** (2010), no. 2, 155–160.
12. Z. Dahmani, L. Tabharit, S. Taf,*New generalizations of Gruss inequality using Riemann–Liouville fractional integrals*, Bull. Math. Anal. Appl. **2** (2010), no. 3, 93–99.
13. S.S. Dragomir, *On some new inequalities of Hermite–Hadamard type for m-convex functions*, Tamkang J. Math. **3** (2002), no. 1.
14. S.S. Dragomir, R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett. **11** (1998), no. 5, 91–95.

15. S.S.Dragomir, S. Fitzpatrik, *The Hadamard's inequality for s-convex functions in the second sense*, Demonstratio Math **32** (1999), no. 4, 687–696.
16. S.S. Dragomir, C.E.M. Pearce, *Selected topics on Hermite–Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000.
17. P.M. Gill, C.E.M. Pearce, J. Pečarić, *Hadamard's inequality for r-convex functions*, J. Math. Anal. Appl. **215** (1997), no. 2, 461–470.
18. H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math. **48** (1994), 100–111.
19. U.S. Kırmacı, M.K. Bakula, M.E. Özdemir, J. Pečarić, *Hadamard-tpye inequalities for s-convex functions*, Appl. Math. Comput. **193** (2007), 26–35.
20. M.E. Özdemir, M. Avci, E. Set, *On some inequalities of Hermite–Hadamard type via m-convexity*, Appl. Math. Lett. **23** (2010), no. 9, 1065–1070.
21. J.E. Pečarić, F. Proschan, Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.
22. M.Z. Sarikaya, H. Ogunmez, *On new inequalities via Riemann–Liouville fractional integration*, Abstract and Applied Analysis, Volume 2012, Article ID 428983, 10 pages, doi:10.1155/2012/428983.
23. M.Z. Sarikaya and H. Yaldiz, *On weighted Montogomery identities for Riemann-Liouville fractional integrals*, Konuralp Journal of Mathematics, **1** (2013), no. 1, 48–53.
24. M.Z. Sarikaya, E.Set, H.Yaldiz and N.Basak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, **57** (2013), 2403–2407.
25. E. Set, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals*, Comp. Math. Appl. **63** (2012), no. 7, 1147–1154.
26. E. Set, M.E. Özdemir, S.S. Dragomir, *On the Hermite–Hadamard inequality and other integral inequalities involving two functions*, J. Inequal. Appl. (2010), 9. Article ID 148102.
27. E. Set, M.E. Özdemir, S.S. Dragomir, *On Hadamard-type inequalities involving several kinds of convexity*, J. Inequal. Appl. (2010), 12. Article ID 286845.

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