Khayyam J. Math. 1 (2015), no. 1, 82–106

Khayyam Journal of Mathematics

URL: www.emis.de/journals/KJM/

STAR SELECTION PRINCIPLES: A SURVEY

LJUBIŠA D.R. KOČINAC

Communicated by H.R. Ebrahimi Vishki

Abstract. We review selected results obtained in the last fifteen years on star selection principles in topology, an important subfield of the field of selection principles theory. The results which we discuss concern also uniform structures and, in particular, topological groups and their generalizations.

1. Introduction

There are many results in the literature which show that a number of topological properties can be characterized by using the method of stars. In particular it is the case with many covering properties of topological spaces. The method of stars has been used to study the problem of metrization of topological spaces, and for definitions of several important classical topological notions. More information on star covering properties can be found in [17], [45]. We use here such a method in investigation of selection principles for topological and uniform spaces.

Although Selection Principles Theory is a field of mathematics having a rich history going back to the papers by Borel, Menger, Hurewicz, Rothberger, Seirpiński in 1920–1930's, a systematic investigation in this area rapidly increased and attracted a big number of mathematicians in the last two-three decades after Scheeper's paper [54]. Nowadays, this theory has deep connections with many branches of mathematics such as Set theory and General topology, Game theory, Ramsey theory, Function spaces and hyperspaces, Cardinal invariants, Dimension theory, Uniform structures, Topological groups and relatives, Karamata theory. Researchers working in this area have organized four international mathematical forums called "Workshop on Coverings, Selections and Games in Topology".

Date: Received: 29 November 2014; Accepted: 30 December 2014.

2010 *Mathematics Subject Classification.* Primary 54D20; Secondary 54A35, 54B20, 54E15, 54H10, 91A44.

Key words and phrases. Star selection principles, ASSM, selectively (a), uniform selection principles.

There are several survey papers about selection principles theory (see, for example, [33, 34, 53] and the paper [73] for open problems).

Two basic ideas in this theory are simple and may be described by the following two schemes:

Scheme 1: To a topological property $\mathcal{P}$ associate selectively $\mathcal{P}$ as follows:

$\mathcal{P}$: for each A there is a B such that ...

selectively$\mathcal{P}$: For each sequence $\langle A_n : n \in \mathbb{N}\rangle$ there is a sequence $\langle B_n : n \in \mathbb{N}\rangle$ such that ...

Scheme 2: $\mathcal{A}$ and $\mathcal{B}$ are given collections, π is a procedure of selection. Apply π to $\mathcal{A}$ to arrive to $\mathcal{B}$.

For example, if $\mathcal{P}$ is compactness (for each open cover $\mathcal{U}$ of a space X there is a finite subcover $\mathcal{V}$), then selectively $\mathcal{P}$ is defined as follows: for each sequence $\langle \mathcal{U}_n; n \in \mathbb{N}\rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ of finite sets with $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ covers X. This property is called the Menger property (see below).

Many other selective versions of classical topological concepts have been defined in this way.

Three classical selection principles defined in general forms in [54] are:

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then the following selection hypothesis are defined:

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$: for each sequence $\langle A_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{A}$ there is a sequence $\langle B_n : n \in \mathbb{N}\rangle$ of finite sets such that for each n, $B_n \subset A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$: for each sequence $\langle A_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{A}$ there is a sequence $\langle b_n : n \in \mathbb{N}\rangle$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{U}_{fin}(\mathcal{A},\mathcal{B})$: for each sequence $\langle A_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{A}$ there is a sequence $\langle B_n : n \in \mathbb{N}\rangle$ such that for each n, B_n is a finite subset of A_n and $\{\bigcup B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

In this paper we use the following notation for collections of covers of a topological space X:

- $\mathcal{O}$ is the collection of all open covers of X;
- Ω is the collection of ω-covers of X. An open cover $\mathcal{U}$ of X is said to be an *ω-cover* if each finite subset of X is contained in a member of $\mathcal{U}$ and $X \notin \mathcal{U}$;
- Γ denotes the collection of γ-covers of X. An open cover $\mathcal{U}$ of X is said to be a *γ-cover* if each point of X does not belong to at most finitely many elements of $\mathcal{U}$.

Then:

M: $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ is the *Menger property* [47], [25];

R: $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ is the *Rothberger property* [50];

H: $\mathsf{U}_{fin}(\Gamma,\Gamma)$ is the *Hurewicz property* [25]

The paper is organized in the following way. Immediately after this introduction in Section 2 we give information about terminology and notation, and also about known topological constructions we use in this paper. In Section 3 we discuss in details star selection principles in topological spaces. The next two sections are devoted to neighbourhood and absolute star selection properties, two variations of the properties we considered in Section 3. In particular, in Subsection 5.2 we report results on selectively (a) spaces. In the second part of the paper we turn attention to appearance of star selection properties in special classes of topological structures: uniform and quasi-uniform spaces, and, especially, in topological and paratopological groups. Each section contains some open problems which can motivate new researches for work in this field.

2. Definitions and terminology

Throughout the paper "space" means "topological space". By $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ we denote the set of natural numbers, integers, and real numbers, respectively. The symbol ω denotes the set of nonnegative integers and also the first infinite ordinal, while ω_1 is the first uncountable ordinal. The cardinality of continuum is denoted by $\mathfrak{c}$, and CH denotes the Continuum Hypothesis. Most of undefined notations and terminology are as in [18].

If X is a space, $\mathcal{K}$ a collection of subsets of X, A a subset of X, and $x \in X$, then $\mathrm{St}(A, \mathcal{K})$ is the union of all elements in $\mathcal{K}$ which meet A. We write $\mathrm{St}(x, \mathcal{K})$ instead of $\mathrm{St}(\{x\}, \mathcal{K})$.

We recall known topological constructions which will be used in next sections without special mention.

A. (**The Baire space** ${}^\omega\omega$) Let ${}^\omega\omega$ be the set of all functions $f : \omega \to \omega$ (in fact, the countable Tychonoff power of the discrete space $D(\omega)$). A natural pre-order $\prec^*$ on ${}^\omega\omega$ is defined by $f \prec^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many n. A subset F of ${}^\omega\omega$ is said to be *dominating* if for each $g \in {}^\omega\omega$ there is a function $f \in F$ such that $g \prec^* f$. A subset F of ${}^\omega\omega$ is called *bounded* if there is an $g \in {}^\omega\omega$ such that $f \prec^* g$ for each $f \in F$. The symbol $\mathfrak{b}$ (resp. $\mathfrak{d}$) denotes the least cardinality of an unbounded (resp. dominating) subset of ${}^\omega\omega$. Another uncountable small cardinal characterized (by Bartoszyński in 1987) in terms of subsets of ${}^\omega\omega$ is the cardinal $\mathsf{cov}(\mathcal{M})$, the *covering number of the ideal of meager subsets of* $\mathbb{R}$:

$$\mathsf{cov}(\mathcal{M}) = \min\{|F| : F \subset {}^\omega\omega \text{ such that } \forall g \in {}^\omega\omega\ \exists f \in F \text{ with } f(n) \neq g(n) \forall n \in \omega\}.$$

Recommended literature concerning uncountable small cardinals is [16] and [75].

B. (Ψ**-spaces**) A family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is called *almost disjoint* if the intersection of any two distinct sets in $\mathcal{A}$ is finite.

Let $\mathcal{A}$ be an almost disjoint family. The symbol $\Psi(\mathcal{A})$ denotes the space $\mathbb{N} \cup \mathcal{A}$ with the following topology: all points of $\mathbb{N}$ are isolated; a basic neighborhood of a point A in $\mathcal{A}$ is of the form $\{A\} \cup (\mathbb{N} \setminus F)$, where F is a finite subset of $\mathbb{N}$.

C. (**Pixley-Roy space**) For a space X, let $\mathsf{PR}(X)$ be the space of all nonempty finite subsets of X with the Pixley-Roy topology [15]: for $A \in \mathsf{PR}(X)$

and an open set $U \subset X$, let $[A, U] = \{B \in \mathsf{PR}(X) : A \subset B \subset U\}$; the family $\{[A, U] : A \in \mathsf{PR}(X), U \text{ open in } X\}$ is a base for the Pixley-Roy topology.

Obviously $\{\{x\} : x \in X\}$ is closed and discrete in $\mathsf{PR}(X)$. Therefore, $\mathsf{PR}(X)$ is Lindelöf if and only if X is countable. It is known that (1) for a T_1-space X, $\mathsf{PR}(X)$ is always zero-dimensional, Tychonoff and hereditarily metacompact, and (2) $\mathsf{PR}(X)$ is developable if and only if X is first-countable (see [15]).

D. (**Alexandroff duplicate**) Let (X, τ) be a topological space. The Alexandroff duplicate of X (see [18], [12]) is the set $\mathsf{AD}(X) := X \times \{0, 1\}$ equipped with the following topology. For each $U \in \tau$ let $\widehat{U} = U \times \{0, 1\}$. Define a base for a topology on $\mathsf{AD(X)}$ by $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$, where $\mathcal{B}_0$ is the family of all sets $\widehat{U} \setminus (F \times \{1\}) \subset \mathsf{AD}(X)$, with $U \in \tau$ and F a finite subset of X, and $\mathcal{B}_1 = \{\langle x, 1\rangle : x \in X\}$. For every $x \in X$ put $\tau_x = \{U \in \tau : x \in U\}$ and $\mathcal{B}_{\langle x,0\rangle} = \{\widehat{U} \setminus \{\langle x, 1\rangle\} : U \in \tau_x\}$, and $\mathcal{B}_{\langle x,1\rangle} = \{\{\langle x, 1\rangle\}\}$. Then, if X is a T_1-space, $\mathcal{B}_{\langle x,0\rangle}$ is a local base at each $\langle x, 0\rangle \in \mathsf{AD}(X)$, and $\mathcal{B}' = \bigcup_{x\in X}(\mathcal{B}_{\langle x,0\rangle} \cup B_{\langle x,1\rangle})$ is a base in $\mathsf{AD}(X)$ such that $\mathcal{B}' \subset \mathcal{B}$. If $\mathcal{U}$ is a family of open sets in X, then we say that the family $\mathcal{U}^* := \{\widehat{U} \setminus (F \times \{1\}) : U \in \mathcal{U}, F \text{ a finite subset of } X\}$ of open subsets of $\mathsf{AD}(X)$ is associated to $\mathcal{U}$ and vice versa.

For many topological properties $\mathcal{P}$ the space $\mathsf{AD}(X)$ has $\mathcal{P}$ if X has $\mathcal{P}$ (see, for example, [12]). Such properties are, for instance, complete regularity, normality, compactness, Lindelöfness, (hereditary) paracompactness.

Recall also the definition of subspaces (called lines) of $\mathsf{AD}(X)$. Let A and B be disjoint subspaces of X. The subspace $Z = (A \times \{1\}) \cup (B \times \{0\})$ of $\mathsf{AD}(X)$ is called a *Michael-type line* (see [12, Definition 3.14]).

3. Star selection principles

In [19] it was proved that a Hausdorff space X is countably compact if and only if for every open cover $\mathcal{U}$ of X there exists a finite subset $F \subset X$ such that $\mathrm{St}(F, \mathcal{U}) = X$.

This result was a motivation for the following two definitions that appeared in [17].

A space X is *starcompact* if for every open cover $\mathcal{U}$ of X there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}(\cup\mathcal{V}, \mathcal{U}) = X$.

A space X is *strongly starcompact* if for every open cover $\mathcal{U}$ of X there exists a finite subset $F \subset X$ such that $\mathrm{St}(F, \mathcal{U}) = X$.

Applying now Schemes 1 and 2 we define selective versions of these notions, and modifying them we obtain the following star selection principles introduced by the author of this article in [30] (see also [31]).

Let $\mathcal{O}$ be the collection of all open covers of a space X, $\mathcal{B}$ a subcollection of $\mathcal{O}$, and $\mathcal{K}$ a family of subsets of X. Then:

1. The symbol $\mathsf{S}^*_{fin}(\mathcal{O}, \mathcal{B})$ denotes the selection hypothesis:

> For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{O}$ there is a sequence $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\{\mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$;

2. The symbol $\mathsf{S}_1^*(\mathcal{O},\mathcal{B})$ denotes the selection hypothesis:

> For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{O}$ there is a sequence $\langle U_n : n \in \mathbb{N}\rangle$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{\mathrm{St}(U_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$;

3. $\mathsf{SS}_{\mathcal{K}}^*(\mathcal{O},\mathcal{B})$ denotes the selection hypothesis:

> For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{O}$ there exists a sequence $\langle K_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{K}$ such that $\{\mathrm{St}(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

When $\mathcal{K}$ is the collection of all finite (resp. one-point, compact) subspaces of X we write $\mathsf{SS}_{fin}^*(\mathcal{O},\mathcal{B})$ (resp., $\mathsf{SS}_1^*(\mathcal{O},\mathcal{B})$, $\mathsf{SS}_{\mathrm{K}}^*(\mathcal{O},\mathcal{B})$) instead of $\mathsf{SS}_{\mathcal{K}}^*(\mathcal{O},\mathcal{B})$.

The following terminology we borrow from [30]. For a space X we have:

SM: the *star-Menger property* $= \mathsf{S}_{fin}^*(\mathcal{O},\mathcal{O})$;
SR: the *star-Rothberger property* $= \mathsf{S}_1^*(\mathcal{O},\mathcal{O})$;
SSM: the *strongly star-Menger property* $= \mathsf{SS}_{fin}^*(\mathcal{O},\mathcal{O})$;
SSR: the *strongly star-Rothberger property* $= \mathsf{SS}_1^*(\mathcal{O},\mathcal{O})$;
SS-K-M: the *star-K-Menger property* $= \mathsf{SS}_{\mathrm{K}}^*(\mathcal{O},\mathcal{O})$.

In [7], two star versions of the Hurewicz property were studied:

SH: the *star-Hurewicz property* $= \mathsf{S}_{fin}^*(\mathcal{O},\Gamma)$;
SSH: the *strongly star-Hurewicz property* $= \mathsf{SS}_{fin}^*(\mathcal{O},\Gamma)$.

It is clear that each of properties SM, SH, SR can be viewed as a selective version of starcompactnes, while the properties SSM, SSH, SSR, SS-K-M can be viewed as selective versions of strong starcompactness. Starcompctness implies SH, hence also SM, and strong starcompactness implies SSH and thus SSM. In [30, Example 2.3] we have shown that the Tychonoff Plank $[0,\omega_1]\times[0,\omega]\setminus\{\langle\omega_1,\omega\rangle\}$ is SSM but not strongly starcompact. On the other hand, in [61, Example 2.1] it is proved that the Tychonoff Plank is SSH but not starcompact (thus not strongly starcompact). It is worth to mention that for each ordinal α, the space $[0,\alpha)$ with the order topology is SSR.

Of course, Menger spaces are SSM, and every SSM space is SM. Similarly for the Hurewicz and Rothberger properties.

The simplest example which shows that the converse need not be true is the ordinal space $[0,\omega_1)$ which is SSH (hence SSM, SH, SM) but not M (thus not H) (see [30] and [7]).

By results in [30] and [7] we have that every metacompact (every open cover $\mathcal{U}$ has a point-finite open refinement $\mathcal{V}$) strongly star-Menger space is Menger, and that for paracompact Hausdorff spaces the three Menger-type properties, SM, SSM and M are equivalent [30]. The same situation is with the classes SSR, SR and R [30] and SSH, SH and H [7].

Let us mention the following

Example 3.1. ([61, Example 2.2]) There is a Tychonoff SH space which is not SSH.

Such a space is $\alpha D(\mathfrak{c}) \times [0, \mathfrak{c}^+] \setminus \{\langle \infty, \mathfrak{c}^+ \rangle\}$ of the product $\alpha D(\mathfrak{c}) \times [0, \mathfrak{c}^+]$, where $\alpha D(\mathfrak{c}) = D(\mathfrak{c}) \cup \{\infty\}$ is the one-point compactification of the discrete space $D(\mathfrak{c})$ of cardinality $\mathfrak{c}$.

Following the general definition of $\mathsf{SS}^*_{\mathcal{K}}(O, O)$ (the beginning of this section) and taking $\mathcal{K}$ to be the collection of countably compact spaces Song defined star-C-Menger spaces in [68] (he also studied star-K-Menger spaces in [65]). He proved:

Example 3.2. ([68, Example 2.2]) There exists a Tychonoff star-C-Menger space which is not star-K-Menger.

Now we are going to see how above mentioned star selection properties are related to Ψ-spaces and Pixley-Roy spaces. In fact, in Ψ-spaces $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$ star selection properties strongly depend on the cardinality of the almost disjoint family $\mathcal{A}$ and are related to small infinite cardinals. The first results of this kind appeared in the preprint/draft [46] sent me by the author in July 1998 (see [30, Example 2.2] and [7]), and then included in the paper [9]. By combining the results from [46] and [9] we can formulate the following

Theorem 3.3. *The following hold for a Ψ-space $\Psi(\mathcal{A})$:*

(1) *$\Psi(\mathcal{A})$ is* SSM *if and only if $|\mathcal{A}| < \mathfrak{d}$. If $|\mathcal{A}| = \mathfrak{c}$, then $\Psi(\mathcal{A})$ is not* SM*, and if $\mathcal{A}| < \aleph_\omega$, then $\Psi(\mathcal{A})$ is* SM *if and only if it is* SSM*;*
(2) *$\Psi(\mathcal{A})$ is* SSH *if and only if $|\mathcal{A}| < \mathfrak{b}$;*
(3) *If $|\mathcal{A}| < \mathsf{cov}(\mathcal{M})$, then $\Psi(\mathcal{A})$ is* SSR*. There is an almost disjoint family $\mathcal{A}$ of cardinality $\mathsf{cov}(\mathcal{M})$ such that $\Psi(\mathcal{A})$ is not* SSR*.*

In [52], Sakai investigated star-Mengerness in the Pixley-Roy space. He established the following:

Theorem 3.4. (1) *If $\mathsf{PR}(X)$ is star-Menger, then $|X| < \mathfrak{c}$ holds. Hence, under* CH*, $\mathsf{PR}(X)$ is star-Menger if and only if X is countable;*
(2) *If $\mathsf{PR}(X)$ is star-Menger, then every finite power of X is Menger.*
(3) *If X is a cosmic space of cardinality less than $\mathfrak{d}$, then every finite power of $\mathsf{PR}(X)$ is star-Menger;*
(4) *Let X be a semi-stratifiable space* [13]*. If $\mathsf{PR}(X)$ is star-Menger, then $\mathsf{PR}(X)^\kappa$ is weakly Menger for any cardinal κ;*
(5) *If X is first-countable and $\mathsf{PR}(X)$ is star-Menger, then $\mathsf{PR}(X)$ is weakly Menger.*

A space X is said to be *weakly Menger* [14] if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \mathbb{N} \rangle$ of finite sets such that for each n, $\mathcal{V}_n \subset \mathcal{U}_n$ and $\overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n} = X$.

Since the very beginning of the theory of star selection principles one the following question was one of the most interesting: how large the extent of SM or SSM spaces can be. Recall that the *extent* $e(X)$ of a space X is the supremum of cardinalities of closed discrete subspaces of X. Recently, some results in this connection have been obtained by Y.-K. Song [62] and M. Sakai [52], and also by B. Tsaban [74].

Song [62, Example 2.4] observed that the extent of a T_1 strongly star-Menger space can be arbitrarily large, and asked whether there is a Tychonoff strongly star-Menger space X such that $e(X) \geq \mathfrak{c}$. Answering this question, Sakai proved in [52, Corollaries 2.2, 2.6]:

Theorem 3.5. (1) *The extent of a regular strongly star-Menger space cannot exceed* $\mathfrak{c}$*;*

(2) *If* X *is a star-Menger space with* $w(X) = \mathfrak{c}$*, then every closed and discrete subspace of* X *has cardinality less than* $\mathfrak{c}$*;*

(3) *Let* X *be a normal star-Menger space. Then* $e(X) \leq \mathfrak{d}$*;*

(4) *The assertion every developable strongly star-Menger space is separable and metrizable is equivalent to* $\omega_1 = \mathfrak{d}$*;*

(5) *The statements* $\omega_1 = \mathfrak{d}$ *is equivalent to the statement that for every strongly star-Menger space* X*,* $e(X) \leq \omega$ *holds.*

The following problem was posed by Sakai.

Problem 3.6. ([52, Question 3.3]) Can the extent of a metacompact (or, subparacompact) star-Menger space be arbitrarily large

Another interesting question regarding star selection principles is their relations with the Alexandroff double. Some of results in this direction are listed below.

1. ([62, Corollary 2.9]) If X is an SSM T_1-space, then $\mathsf{AD}(X)$ is SSM if and only if $e(X) < \omega_1$.
2. It was observed in [61] that the Alexandroff double of the SH space in Example 3.1 is not SH.
3. ([60, Theorem 2.4]) If X is a T_1-space and $\mathsf{AD}(X)$ is an SSH space, then $e(X) < \omega_1$.

The last result suggests the following problem.

Problem 3.7. Is the Alexandorff duplicate $\mathsf{AD}(X)$ of an SSH space X with $e(X) < \omega_1$ also SSH

3.1. **Operations.** Most of star selection properties are not hereditary. Even more, they are not preserved by nice subspaces such as (regular) closed. It was proved for SM and SSM spaces in [66], for SH and SSH spaces in [67], and for SR and SSR spaces in [64].

Let us formulate once again a still open question from [30].

Problem 3.8. Characterize hereditarily SM (SSM, SR, SSR, SH, SSH) spaces.

There are some partial answers to this question. For example, SSM and SSH spaces are preserved by open F_σ-sets (see [66] and [67], respectively), while SSR property is preserved by clopen subspaces [64].

It is known and easy to prove that continuous mappings preserve SSM, SH, and SSH) spaces (see [62], [61], [60], respectively).

Open and closed finite-to-one mappings preserve SSM and SSH spaces ([62] and [60]) in the preimage direction, while open, perfect mappings preserve SH

spaces in the preimage direction [61]. On the other hand, it was proved in [60] that assuming $\mathfrak{b} = \mathfrak{c}$ and $\neg\mathsf{CH}$, there exists a closed 2-to-1 continuous mapping $f : X \to Y$ such that Y is SSH, but X is not.

The product of two SM (resp. SH) spaces need not be in the same class. For SSH spaces, for instance, it was shown in [60]. But if one factor is compact, then the product is in the same class [30], [7]. Similarly, the product a star-C-Menger space and a compact space is also star-C-Menger [68]. However, under $\mathfrak{b} = \mathfrak{c}$ and $\neg CH$, there exist an SSH space X and a compact space Y such that $X \times Y$ is not SSH [60].

Let us observe that a Lindelöf space is not a preserving factor for the classes SSM and SSH [30, 7].

The following question is an open problem.

Problem 3.9. ([60]) Do there exist a ZFC example of an SSH space X and a compact space Y such that $X \times Y$ is not SSH

In [30] we posed the following still open problem.

Problem 3.10. Characterize spaces X which are SM (SSM, SR, SSR) in all finite powers.

A partial solution of this problem was given in [7].

Theorem 3.11. *The following statements hold:*

(1) *If each finite power of a space X is SM, then X satisfies $\mathsf{S}^*_{fin}(\mathcal{O}, \Omega)$;*
(2) *If all finite powers of a space X are SSM, then X satisfies $\mathsf{SS}^*_{fin}(\mathcal{O}, \Omega)$.*

In the same paper we have the following two assertions. (We remind the reader that the symbol $\mathcal{O}^{wgp}$ denotes the collection of weakly groupable covers of a space. A countable open cover $\mathcal{U}$ of a space X is said to be *weakly groupable* if there is a partition $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ of $\mathcal{U}_n$ into finite, pairwise disjoint subcollections, so that for each finite subset F of X there is $n \in \mathbb{N}$ with $F \subset \bigcup \mathcal{U}_n$.)

Theorem 3.12. *For a space X the following are equivalent:*

(1) *X satisfies $\mathsf{S}^*_{fin}(\mathcal{O}, \Omega)$;*
(2) *X satisfies $\mathsf{S}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$.*

Theorem 3.13. *For a space X the following are equivalent:*

(1) *X satisfies $\mathsf{SS}^*_{fin}(\mathcal{O}, \Omega)$;*
(2) *X satisfies $\mathsf{SS}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$.*

So, we have actually the following problem.

Problem 3.14. Does $X \in \mathsf{S}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$ imply that all finite powers of X are star-Menger Is it true that $\mathsf{S}^*_{fin}(\mathcal{O}, \Omega) = \mathsf{S}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$ Does $X \in \mathsf{SS}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$ imply that each finite power of X is SSM

The following result was proved in [7]. First we recall that $\mathcal{O}^{gp}$ denotes the collection of groupable covers of a space. A countable open cover $\mathcal{U}$ of a space X is said to be *groupable* if there is a partition $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ of $\mathcal{U}_n$ into finite, pairwise disjoint subcollections, so that each $x \in X$ belongs to all but finitely many $\bigcup \mathcal{U}_n$.

Theorem 3.15. *For a space X the following are equivalent:*

(1) *X has the strongly star-Hurewicz property;*
(2) *X satisfies the selection principle $\mathsf{SS}^*_{fin}(\mathcal{O}, \mathcal{O}^{gp})$.*

This result naturally suggests the following

Problem 3.16. Is it true that $\mathsf{S}^*_{fin}(\mathcal{O}, \Gamma) = \mathsf{S}^*_{fin}(\mathcal{O}, \mathcal{O}^{gp})$

Let us end this section by some comments.

1. In this paper we did not consider connections between star selection properties and games naturally associated to them.

[For example, the *strongly star-Hurewicz game* illustrates this situation; it is defined as follows. Let X be a space. Two players, ONE and TWO, play a round per each natural number n. In the n–th round ONE chooses an open cover $\mathcal{U}_n$ of X and TWO responds by choosing a finite set $A_n \subset X$. A play $\mathcal{U}_1, A_1; \cdots; \mathcal{U}_n, A_n; \cdots$ is won by TWO if $\{\mathrm{St}(A_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ-cover of X; otherwise, ONE wins.

Evidently, if ONE has no winning strategy in the strongly star-Hurewicz game, then X is an SSH space. But the converse need not be true.]

It would be interesting to study these connections for all classes we discussed in this section.

2. We also did not discuss relative versions of star selection principles (initiated by the author) that can be found in the literature (see, for instance, [7], [10]).

3. Recently, I introduced *selection principles in relator spaces* as generalizations of uniform selection principles. My PhD student Kocev studied these selection properties in [27], [28], [29]. We did not include these results in this survey although there are many interesting results and open questions in this connection.

4. Selection properties of fuzzy metric spaces [35] are a kind of star selection properties.

4. Neighbourhood star selection principles

In this section we investigate star selection principles which are very close to the already considered star selection properties, but defined by neighbourhoods and stars. Selection properties defined in this way are weaker than the Menger, Rothberger and Hurewicz properties and are between strong star versions and star versions of the corresponding properties. The definitions of these selection principles were given in [31, Def. 0.2], and studied in details in [8]. Our exposition here mainly follows the last mentioned paper.

Definition 4.1. Let $\mathcal{O}$ and $\mathcal{B}$ be as in the previous section. A space X satisfies:

$\mathsf{NSM}(\mathcal{O}, \mathcal{B})$ if for every sequence $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$ of elements of $\mathcal{A}$ one can choose finite $A_n \subset X$, $n \in \mathbb{N}$, so that for every open $O_n \supset A_n$, $n \in \mathbb{N}$, $\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$;

$\mathsf{NSR}(\mathcal{O}, \mathcal{B})$ if for every sequence $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$ of elements of $\mathcal{A}$ one can choose $x_n \in X$, $n \in \mathbb{N}$, so that for every open $O_n \ni x_n$, $n \in \mathbb{N}$, $\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$;

$\mathsf{NSH}(\mathcal{O},\mathcal{B})$ if if for every sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{A}$ one can choose finite $A_n \subset X$, $n \in \mathbb{N}$, so that for every open $O_n \supset A_n$, $n \in \mathbb{N}$, and for every $x \in X$, $x \in \mathrm{St}(O_n, \mathcal{U}_n)$ for all but finitely many n.

In particular we have the following definitions:

Definition 4.2. A space X is:

- NSM: (*neighbourhood star-Menger*) if the selection hypothesis $\mathsf{NSM}(\mathcal{O},\mathcal{O})$ is true for X;
- NSR: (*neighbourhood star-Rothberger*) if the property $\mathsf{NSR}(\mathcal{O},\mathcal{O})$ is true for X;
- NSH: (*neighbourhood star-Hurewicz*) if the selection hypothesis $\mathsf{NSH}(\mathcal{O},\Gamma)$ is true for X.

Note. NSR and NSM spaces (as well as neighbourhood star-K-spaces) were defined in [31] under different names (nearly strongly star-Rothberger and nearly strongly star-Menger spaces).

Remark 4.3. Since in the class of paracompact Hausdorff we have that $\mathsf{R} \Leftrightarrow \mathsf{SR}$, $\mathsf{M} \Leftrightarrow \mathsf{SM}$ (see [30]) and $\mathsf{H} \Leftrightarrow \mathsf{SH}$ (see [7]), we have that in the class of paracompact Hausdorff spaces all Rothberger-type properties, all Menger-type properties and all Hurewicz-type properties considered are equivalent respectively (see Diagram 1).

The implications $\mathsf{NSM} \Rightarrow \mathsf{SM}$, $\mathsf{NSH} \Rightarrow \mathsf{SH}$ and $\mathsf{NSR} \Rightarrow \mathsf{SR}$ can not be reversed as the following example shows.

Example 4.4. ([8, Example 3.7]) A Tychonoff space which is SR and SH (and thus SM), but is neither of NSR, NSH, NSM.

Such a space X is constructed in the following way. Let κ be an uncountable cardinal and $\alpha(D(\kappa)) = D(\kappa) \cup \{\infty\}$ the one point compactification of the discrete space $D(\kappa)$. Set $X_0 = \alpha D(\kappa) \times [0, \kappa^+)$, $X_1 = D(\kappa) \times \{\kappa^+\}$, $X = X_0 \cup X_1$. Endow X with the topology inherited from the product $\alpha D(\kappa) \times [0, \kappa^+]$.

We show now that consistently, NSM, NSH and NSR do not imply SSM, SSH and SSR, respectively.

Example 4.5. ([8, Examples 3.1–3.3]) Let S be a subset of $\mathbb{R}$ such that $|S| = \omega_1$ and for every nonempty open $U \subset \mathbb{R}$, $|S \cap U| = \omega_1$. Set $X_S = S \times [0, \omega]$ topologized in the following way: (i) a basic neighbourhood of a point $\langle x, n\rangle \in X_S$ has the form $((U \cap S) \setminus A) \times \{n\}$, where U is a neighbourhood of x in the usual topology of $\mathbb{R}$ and A is a countable set not containing x; (ii) a point $\langle x, \omega\rangle$, $x \in S$, has basic neighbourhoods of the form $((U \cap S) \setminus A) \times (n, \omega) \cup \{\langle x, \omega\rangle\}$, where U is a neighbourhood of x in the usual topology of $\mathbb{R}$, A is a countable subset of S, and $n \in \omega$. Then X_S is a Urysohn space and:

(1) Under $\omega_1 < \mathfrak{d}$ the space X_S is an NSM space which is not SSM.
(2) Under $\omega_1 < \mathfrak{b}$, X_S is an NSH space which is not SSH.
(3) Under $\omega_1 < \mathsf{cov}(\mathcal{M})$, X_S is an NSR space which is not SSR.

The following problem is still open.

Problem 4.6. ([8, Problem 3.6]) Do there exist ZFC examples of spaces as in Example 4.5

5. Absolute versions of selection principles

In [43] Matveev introduced the class of absolutely countable compact spaces: A space X is *absolutely countable compact* (shortly acc) if for each open cover $\mathcal{U}$ of X and each dense subset D of X there is a finite $A \subset D$ such that $\mathrm{St}(A, \mathcal{U}) = X$.

In his subsequent paper [44], Matveev applied a similar idea to introduce the following property: a space X is said to be an (a)*-space* if for each open cover $\mathcal{U}$ of X and each dense subset D of X there is a closed discrete (in X) set $A \subset D$ such that $\mathrm{St}(A, \mathcal{U}) = X$. He also defined the class of (wa)-spaces replacing in the previous definition "closed discrete" by "discrete". These spaces were studied in a number of papers [21], [26], [51], [56], [69].

In 2010, we employed Matveev's idea to define selective versions of several star selection principles in the following general form (see [11, p. 1361]).

Definition 5.1. ([11]) Let $\mathcal{O}$ and $\mathcal{B}$ be collections of open covers of a space X as mentioned above, and let $\mathcal{K}$ be a collection of subsets of X. Then X is said to be a *selectively* $(\mathcal{O}, \mathcal{B})$*-*$(a)_{\mathcal{K}}$*-space*, denoted by $X \in \mathsf{Sel}(\mathcal{O}, \mathcal{B})\text{-}(a)_{\mathcal{K}}$, if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{O}$ and each dense subset D of X there is a sequence $\langle K_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{K}$ such that each A_n is a subset of D and $\{\mathrm{St}(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

In this definition we have the following classes of spaces:

(1) selectively $(\mathcal{O}, \mathcal{O})\text{-}(a)_{\text{finite}}$-spaces are called *absolutely strongly star-Menger spaces* (shortly ASSM spaces), which form a subclass of SSM-spaces;

(2) selectively $(\mathcal{O}, \Gamma)\text{-}(a)_{\text{finite}}$ spaces are *absolutely strogly star-Hurewicz spaces* (shortly $=$ ASSH spaces), which form a subclass of SSH spaces;,

(3) $\mathsf{Sel}(\mathcal{O}, \mathcal{O})\text{-}(a)_{\text{singleton}}$ is the class of *absolutely strongly Rothberger spaces* (ASSR spaces for short), a subclass of the class SSR;

(4) For a space X satisfying $\mathsf{Sel}(\mathcal{O}, \mathcal{O})\text{-}(a)_{\text{closed discrete}}$ we say that X is a *selectively* (a)*-space*, and this is a direct generalization of the notion of (a)-spaces. This class of spaces will be discussed in a separate subsection of this section.

The following diagram shows relationships among the classes of spaces that we have defined so far. Let us mention that arrows in this diagram are not reversible; for some of them it was already demonstrated by examples in the previous sections, and for some other it will be done in what follows.

$$\begin{tikzcd}
 & \mathsf{R} \arrow[r] \arrow[d] & \mathsf{M} \arrow[d] & \mathsf{H} \arrow[l] \arrow[d] & \\
\mathsf{ASSR} \arrow[r] \arrow[d] & \mathsf{SSR} \arrow[r] \arrow[dd] & \mathsf{SSM} \arrow[dd] & \mathsf{SSH} \arrow[l] \arrow[dd] & \mathsf{ASSH} \arrow[l] \\
\mathsf{ASSM} \arrow[urr] & & & & \\
 & \mathsf{NSR} \arrow[r] \arrow[d] & \mathsf{NSM} \arrow[d] & \mathsf{NSH} \arrow[l] \arrow[d] & \\
 & \mathsf{SR} \arrow[r] & \mathsf{SM} & \mathsf{SH} \arrow[l] &
\end{tikzcd}$$

Diagram 1: Star selection properties

5.1. ASSM, ASSH, ASSR **spaces.** In this subsection we review very few basic results and examples concerning ASSM, ASSH, ASSR spaces. We begin with some examples.

Example 5.2. (1) The Tychonoff plank is a Tychonoff ASSM which is not acc.

(2) ([58, Examples 2.1]) There exists a Tychonoff ASSH space X which is not acc.

Let $X = [0, \omega] \times [0, \omega] \setminus \{(\omega, \omega\}$ as a subspace of the product $[0, \omega] \times [0, \omega]$.

(3) ([58, Example 2.2]) There exists a Tychonoff SSH space X which is not ASSH.

Such a space is $X = [0, \omega_1) \times [0, \omega_1]$.

Here are some properties of absolute star selection properties.

First, similarly to other star selection properties, these properties are not hereditary.

In [59] and [58], it is proved that in the class of Thychonoff spaces ASSM and ASSH properties are not preserved by regular-closed G_δ-subspaces.

Song noticed also that ASSM and ASSH properties are not invariants of continuous mappings. But he proved that these two properties, similarly to the acc property [43], are preserved by continuous varpseudoopen mappings. Recall that a continuous mapping $f : X \to Y$ is *varpseudoopen* provided $\mathrm{int}_Y(f(U)) \neq \emptyset$ for every nonempty open set U of X.

Theorem 2.15 in [58] states that if the the product of two spaces is ASSH, then both spaces are ASSH. On the other hand, in difference of some other star selection properties, the product of an ASSM or ASSH space X and a compact space Y need not be ASSM or ASSH as it was observed in [59] and [58]. For both cases the product $[0,\omega_1) \times [0,\omega_1]$ can serve as an example.

Matveev showed that the product of a Hausdorff acc space and a first countable compact space is acc (see [43, Theorem 2.3]. So, it is naturally to ask:

Problem 5.3. (Song) Is the product of an ASSH space and a first countable compact space also ASSH

Let us finish with the following fact [59, Theorem 3.8]: If X is an ASSM space with $e(X) < \omega_1$, then $\mathsf{AD}(X)$ is ASSM.

5.2. **Selectively** (a) **and related spaces.** The importance of property (a) was established in the literature: there are strong connections of this property with countable compactness, normality and metrizability (see the already mentioned papers [21, 26, 44, 51, 56, 69]).

Evidently, every (a)-space is selectively (a). So, every monotonically normal space, in particular every GO-space, is selectively (a), being an (a)-space (see [51, Theorem 1]). For the same reason every selectively paracompact space is selectively (a). It was observed in [57] that every T_1, σ-compact space is selectively (a). The Tychonoff plank is an example of a selectively (a)-space which is not an (a)-space ([63, Example 2.6]).

Notice that every countably compact selectively a-space is SSM, and every selectively $(\mathcal{O},\Gamma)$-$(a)_{\text{closed discrete}}$ space is SSH.

We will demonstrate similarities and differences between (a)-spaces and selectively (a)-spaces; in particular, we will show that there are many similarities between them.

We begin with the following result which was stated in [11] without proof and which may be obtained by small changes in the proof of Lemma 1 and its corollary in [44].

Theorem 5.4. *Let X be a separable space. Then:*

(1) *If X is selectively (a), then every closed discrete subset of X has cardinality $< 2^\omega$;*
(2) *If X contains a discrete subspace having cardinality $\geq 2^\omega$, then X^2 is not hereditarily selectively (a).*

In [57], the item (2) of this result was proved for a general case.

Theorem 5.5. ([57, Theorem 3.1]) *If X is a selectively (a)-space, then X cannot contain closed and discrete subsets of size $\geq 2^{d(X)}$.*

The following theorem is a nice strengthening of a result established in [44] by Matveev for (a)-spaces.

Theorem 5.6. ([57, Theorem 3.4]) *Under* CH*, separable, Moore, selectively (a)-spaces are metrizable.*

It is shown in [69, Theorem 3] that there are Ψ-spaces which are (a)-spaces, hence selectively (a), and those which are not (a)-spaces. It was observed in [11] that there are also Ψ-spaces which are not selectively (a).

For Ψ-spaces we have the following (Propositions 4.1 and 4.2, in [57]).

Theorem 5.7. *Let $\mathcal{A}$ be an almost disjoint family of subsets of $\mathbb{N}$. Then:*

(1) *If $|\mathcal{A}| < \mathfrak{d}$, then $\Psi(\mathcal{A})$ is selectively (a);*
(2) *If $\mathcal{A}$ is maximal, then $\Psi(\mathcal{A})$ is selectively (a) if and only if $|\mathcal{A}| < \mathfrak{d}$;*
(3) *If $\mathfrak{p} = \mathfrak{c}$, then a Ψ-space satisfies property (a) if and only if satisfies selectively (a).*

(Here, $\mathfrak{p}$ is the pseudointersection number [75].)

It follows from this results that it is consistent that there are Ψ-spaces which are selectively (a)-spaces but not (a)-spaces.

Problem 5.8. (1) ([57, Question 5.3]) Is it consistent that there is an almost disjoint family $\mathcal{A}$ of size $\mathfrak{d}$ such that $\Psi(\mathcal{A})$ is selectively (a)

(2) ([57, Question 5.4]) If $\Psi(\mathcal{A})$ is normal, is it a selectively (a)-space

(3) ([57, Question 5.5]) If $\Psi(\mathcal{A})$ is countably paracompact, is it a selectively (a)- space

Let us notice that in [63] it was proved that assuming $2^{\aleph_0} = 2^{\aleph_1}$ there exists a normal space X that is not selectively (a).

Generalizing a result of Szeptycki and Vaughan regarding characterization of property (a) in Ψ-spaces, da Silva gave in [57] the following combinatorial characterization of selectively (a) Ψ-spaces.

Theorem 5.9. *Let $\mathcal{A} = \{A_\alpha : \alpha < \kappa\} \subset \omega^\omega$ be an almost disjoint family of size κ. The corresponding space $\Psi(\mathcal{A})$ is selectively (a) if and only if the following property holds: for every sequence $\{f_n : n < \omega\}$ in ω^ω there is a sequence $\{P_n : n < \omega\}$ of subsets of ω satisfying the following two conditions:*

(i) $|P_n \cap A_n| < \omega$ for all $n \in \omega$ and all $\alpha < \kappa$;

(ii) for every $\alpha < \kappa$ there is $n \in \omega$ such that $P_n cap A_\alpha \nsubseteq f_n(\alpha)$.

In [49] the authors proved that a certain effective parametrized weak diamond principle is enough to ensure countability of the almost disjoint family in this setting.

In [49, Corollary 3.3] it was observed that selectively (a)-spaces from almost disjoint families are necessarily countable under some additional set-theoretic assumptions, and concluded that it follows that the statement "all selectively (a)-spaces are countable" is consistent with CH.

These authors also noticed that there are no selectively (a) almost disjoint families of size $\mathfrak{c}$; on the other hand, countable almost disjoint families are associated to metrizable Ψ-spaces, so if $\mathcal{A}$ is countable, then $\Psi(\mathcal{A})$ is paracompact and therefore it is (a) (thus, selectively (a)).

The following results show the behaviour of selectively (a)-type spaces under mappings and basic operations with spaces.

It is trivial that the selective (a) property is not a hereditary property. It is also true in case of some special subspaces, for example, regular closed subspaces.

Theorem 5.10. ([11]) *A closed-and-open image $Y = f(X)$ of a selectively (a)-space X is also selectively (a).*

The product of two selectively (a)-spaces need not be selectively (a); the Sorgenfrey line S and its square S^2 can serve as an example (by Theorem 5.4 S^2 is not selectively (a)).

It would be interesting to answer the following question posed in [11] (compare with [26, Theorem 16]):

Problem 5.11. Is the product of a selectively (a)-space X and a metrizable compact space Y selectively (a)

We have the following

Theorem 5.12. ([11]) *If the product $X \times Y$ of a space X and a compact space Y is selectively (a), then X is selectively $(\mathcal{O}, \mathcal{O})$-$(a)_{\text{closed}}$.*

Now we consider when AD spaces have some of properties under consideration.

Theorem 5.13. *If $X \in \mathsf{Sel}(\mathcal{O}, \mathcal{O})$-$(a)_{\text{discrete}}$ and $e(\mathsf{AD}(X)) < \omega_1$, then $\mathsf{AD}(X)$ is also in $\mathsf{Sel}(\mathcal{O}, \mathcal{O})$-$(a)_{\text{discrete}}$.*

Another result of the same sort was proved in [63]: If X is a normal selectively (a)-space with $e(X) < \omega_1$, then $\mathsf{AD}(X)$ is selectively (a).

Similarly, in [11] it was proved:

Theorem 5.14. *If the Alexandroff duplicate $\mathsf{AD}(X)$ of a space X is selectively $(\mathcal{O}, \mathcal{O})$-$(a)_{\text{countable}}$, then $e(X) < \omega_1$.*

In [11, Question 2.11], the authors asked if a space X is selectively (a) provided the space $\mathsf{AD}(X)$ is selectively (a)

This question was answered in [63]: there exists a Tychonoff countably compact space X such that $\mathsf{AD}(X)$ is selectively (a), but X is not selectively (a).

We close this subsection by one more natural question of this kind: when subspaces of the Alexandroff duplicate $\mathsf{AD}(X)$ of a space X have properties of selectively (a)-type. We have the following:

Theorem 5.15. ([11]) *Let A and B be subspaces of a space X such that $\overline{A} \cap B = \emptyset$ and $Z = (A \times \{1\}) \cup (B \times \{0\})$. If $e(Z) < \omega_1$ and B is selectively $(\mathcal{O}, \mathcal{O})$-$(a)_{\text{discrete}}$, then Z is selectively $(\mathcal{O}, \mathcal{O})$-$(a)_{\text{discrete}}$.*

6. Uniform selection principles

In [32] we have defined selection properties in uniform spaces and demonstrated that selection principles in uniform spaces are a good application of star selection principles to concrete special classes of spaces. The exposition in this section is based mainly on the paper [32], although the approach in this article is different from (but equivalent to) the approach in [32].

Recall two equivalent approaches to the definition of uniform spaces; one is to define a uniformity on a set X in terms of uniform covers, and the second to define it by using entourages of the diagonal [18]. The first approach allows to define uniform selection principles similarly to definitions of the usual topological selection principles. By using this way we showed in [32] that uniform selection principles are a kind of star selection properties as well as a kind of strongly star selection properties. Then we passed to description of uniform selection principles in terms of entourages of the diagonal.

Recall some definitions and facts about uniform spaces.

A *quasi-uniformity* on a set X is a filter $\mathbb{U}$ on $X \times X$ satisfying the following two conditions:

$(QU1)$ $\Delta_X \subset U$ for each $U \in \mathbb{U}$;

$(QU2)$ For each $U \in \mathbb{U}$ there is $V \in \mathbb{U}$ such that $V \circ V \subset U$,

where $\Delta_X = \{(x,x) : x \in X\}$ is the diagonal of X, and $V \circ V = \{(x,y) \in X \times X : \exists z \in X \text{ with } (x,z) \in V, (z,y) \in V\}$.

The pair $(X, \mathbb{U})$ is called a *quasi-uniform space*.

A quasi-uniformity $\mathbb{U}$ is a *uniformity* on X, and $(X, \mathbb{U})$ is a uniform space, if $\mathbb{U}$ satisfies also the condition

$(QU3)$ $U \in \mathbb{U}$ implies $U^{-1} \in \mathbb{U}$,

where $U^{-1} = \{(x,y) \in X \times X : (y,x) \in U\}$.

For a subset A of a (quasi-)uniform space $(X, \mathbb{U})$ and $U \in \mathbb{U}$ we write

$$U[A] := \{y \in X : (x,y) \in U \text{ for some } x \in A\}.$$

We define uniform selection principles as follows. If $(X, \mathbb{U})$ is a uniform space, then it is said to be:

- UM: *uniformly Menger* or M-*bounded* if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of entourages of the diagonal of X there is a sequence $\langle A_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that $X = \bigcup_{n\in\mathbb{N}} U_n[A_n]$.
- $\mathsf{U}\omega\mathsf{M}$: ω-M-*bounded* if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of entourages of the diagonal of X there is a sequence $\langle A_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that each finite subset of X is contained in some $U_n[A_n]$.
- UH: *uniformly Hurewicz* or H-*bounded* if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of elements of $\mathbb{U}$ there is a sequence $\langle A_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that each $x \in X$ belongs to all but finitely many sets $U_n[A_n]$.
- UR: *uniformly Rothberger* or R-*bounded* (resp. ω-R-*bounded*) if for each sequence $\langle U_n : n \in \mathbb{N}\langle$ of entourages of the diagonal of X there is a sequence $\langle x_n : n \in \mathbb{N}\langle$ of points in X such that $X = \bigcup_{n\in\mathbb{N}} U_n[x_n]$ (resp. each finite subset of X is contained in some $U_n[x_n]$.

Remark 6.1. It is evident that if a uniform space X has the Menger property with respect to topology generated by the uniformity, then X is M-bounded. However, the converse need not be true: a non-Lindelöf Tychonoff space is an example of M-bounded space (with respect to the generated uniformity) which has no the Menger property. (Similar remarks hold for the R-boundedness and H-boundedness.) But a regular topological space X has the Menger (Hurewicz, Rothberger) property if and only if its fine uniformity is M-bounded (H-bounded, R-bounded).

M-bounded and especially H-bounded uniform spaces have some properties which are similar to the corresponding properties of totally bounded uniform spaces.

Recall that a uniform space $(X, \mathbb{U})$ is said to be *totally bounded* or *precompact* (resp. *pre-Lindelöf* or *ω-bounded* if for each $U \in \mathbb{U}$ there is a finite (resp. countable) $A \subset X$ such that $U[A] = X$. It is understood that totally bounded uniform spaces are H-bounded and thus M-bounded and that M-boundedness implies pre-Lindelöfness.

The difference between uniform and topological selection principles is big enough [32]. Here we point out some of differences on the example of Hurewicz properties.

(1) Every subspace of an H-bounded uniform space is H-bounded. (M- boundedness is also a hereditary property.)
(2) A uniform space X is H-bounded if and only if its completion $\tilde{X}$ is H-bounded.
(3) The product of two H-bounded uniform spaces is also H-bounded.

Let us mention that the product of two M-bounded uniform spaces need not be M-bounded (see the case of topological groups in the next subsection).

We states the following two results from [32]

Theorem 6.2. *For a uniform space $(X, \mathbb{U})$ the following are equivalent:*

1. *X is ω-M-bounded;*
2. *For each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of elements of $\mathbb{U}$ there is a sequence $\langle F_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that there is a sequence $n_1 < n_2 < \cdots$ such that each finite $A \subset X$ is contained in $\bigcup\{U_i[F_i] : n_k \leq i < n_{k+1}\}$ for some $k \in \mathbb{N}$.*

Theorem 6.3. *For a uniform space $(X, \mathbb{U})$ the following are equivalent:*

1. *X is H-bounded;*
2. *For each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of elements of $\mathbb{U}$ there is a sequence $\langle F_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that there is a sequence $n_1 < n_2 < \cdots$ such that each $x \in X$ belongs $\bigcup\{U_i[F_i] : n_k \leq i < n_{k+1}\}$ for all but finitely many $k \in \mathbb{N}$.*

6.1. **Topological groups.** In this subsection we discuss selection principles in topological groups to illustrate the general theory of uniform selection properties on a specific topological structure. The book [1] is an excellent source concerning topological groups.

Definitions of selection properties in topological groups are as follows.

Definition 6.4. A topological group $(G, \cdot)$ is said to be

1. *Menger-bounded* (shortly, M-*bounded*) if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of neighborhoods of the neutral element $e \in G$ there is a sequence $\langle A_n : n \in \mathbb{N}\rangle$ of finite subsets of G such that $X = \bigcup_{n\in\mathbb{N}} A_n \cdot U_n$;
2. *ω-Menger-bounded* (shortly, ω-M-*bounded*), called also *Scheepers-bounded*, if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of neighborhoods of the neutral element $e \in G$ there is a sequence $\langle A_n : n \in \mathbb{N}\rangle$ of finite subsets of G such that each finite subset of G is contained in some $A_n \cdot U_n$;

(3) *Hurewicz-bounded* (shortly, H-*bounded*) if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of neighborhoods of the neutral element $e \in G$ there is a sequence $\langle A_n : n \in \mathbb{N}\rangle$ of finite subsets of G such that each $x \in G$ belongs to all but finitely many $A_n \cdot U_n$;
(4) *Rothberger-bounded* (shortly, R-*bounded*) if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of neighborhoods of the neutral element $e \in G$ there is a sequence $\langle x_n : n \in \mathbb{N}\rangle$ of elements of G such that $X = \bigcup_{n\in\mathbb{N}} x_n \cdot U_n$;

These classes of groups have been introduced by the author of this article in 1998 (see [3, p. 1269]), and the class of M-bounded groups was introduced independently by Okunev and Tkachenko under the name *o-bounded groups*.

The class of M-bounded groups is the most investigated and there is a big list of papers on this topic. More information on M-bounded topological groups the reader can find in [23], [70], [3], [4], [5], [6], [42], [41], [72], [76], [77]; see also [22].

There are two-person infinite games naturally associated to each of mentioned classes of groups. For example, the game associated to M-bounded groups was introduced in [70] as follows. Two players, ONE and TWO, play a round for each $n \in \mathbb{N}$. In the n-th round ONE chooses a neighborhood U_n of the neutral element of G and then TWO chooses a finite set $F_n \subset G$. Two wins a play $U_1, F_1; U_2, F_2; ...$ if and only if $\{F_n \cdot U_n : n \in \mathbb{N}\}$ covers G. A topological group G is called *strictly o-bounded* or *strictly* M-*bounded* if TWO has a winning strategy in the above game. It is easy to see that each strictly M-bounded group is M-bounded. Also, each group having the Menger property is M-bounded. Every subgroup of a σ-compact group is strictly M-bounded [23].

In [2] it is proved that in metrizable case strictly M-bounded groups are exactly H-bounded groups.

Theorem 6.5. ([2, Theorem 5]) *For a metrizable group G the following statements are equivalent:*

(1) *G is strictly M-bounded;*
(2) *G is H-bounded.*

Many selection principles in topological spaces can be characterized game-theoretically. For example, it is a classical result by Hurewicz that a topological space X has the Menger property if and only if ONE does not have a winning strategy in the corresponding game (see [54]). However, for topological groups (and, more general, for star selection principles) it is not the case.

In [23], Hernandez has constructed an M-bounded subgroup G of $\mathbb{R}^\omega$ that is not strictly M-bounded. In [42, Theorem 8.5] it is proved that assuming $\mathsf{cov}(\mathcal{M}) = \mathfrak{d} = \mathfrak{b}$ there is a group G (a subgroup of $\mathbb{Z}^{\mathbb{N}}$) which is R-bounded and H-bounded (in all finite powers) but G does not have the Menger property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$. Tsaban [72], Tsaban constructed strictly M-bounded groups which have the Menger and Hurewicz covering properties, but are not σ-compact.

However, in [3] the following game-theoretic characterization for metrizable R-bounded groups has been obtained:

Theorem 6.6. ([3, Theorem 22]) *Let $(G, \cdot)$ be a σ-compact metrizable group. The following are equivalent:*

(1) *G is R-bounded;*
(2) ONE *has no winning strategy in the game naturally corresponded to* R-*boundedness.*

Metrizable M-bounded groups and R-bounded groups can be also characterized measure-theoretically. Recall that a metric space (X, d) has *strong measure zero* if for each sequence $\langle \varepsilon_n : n \in \mathbb{N} \rangle$ of positive real numbers there is a sequence $\langle A_n : n \in \mathbb{N} \rangle$ of subsets of X such that for each n, $\mathrm{diam}_d(A_n) < \varepsilon_n$ and $X = \bigcup_{n\in\mathbb{N}} A_n$. (X, d) has M-*measure zero* if for each sequence $\langle \varepsilon_n : n \in \mathbb{N} \rangle$ of positive real numbers there is a sequence $\langle \mathcal{A}_n : n \in \mathbb{N} \rangle$ of such that for each n, $\mathcal{A}_n$ is a finite family of subsets of X, $\mathrm{diam}_d(A) < \varepsilon_n$ for each $A \in \mathcal{A}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is an open cover of X.

Theorem 6.7. ([3, Theorem 12]) *For a metrizable group G the following are equivalent :*

(1) *G is M-bounded;*
(2) *G has M-measure zero in each left-invariant metrization of G.*

Theorem 6.8. ([3, Theorem 19]) *For a metrizable group G the following are equivalent :*

(1) *G is R-bounded;*
(2) *G has strong measure zero in each left-invariant metrization of G.*

Let us mention that H-bounded metrizable groups can be characterize measure-theoretically (see [2]; H. Michalewski has obtained independently a similar result in his PhD dissertation in 2003).

An interesting result proved by Scheepers in [55, Th. 3, Cor. 4] states that σ-compact topological groups can be characterized Ramsey-theoretically. (Many selection (covering) properties in topological spaces can be characterized in this manner.)

Machura and Tsaban estimated minimal cardinalities of subgroups G of $\mathbb{Z}^{\mathbb{N}}$ which does not have boundedness properties: for M-boundedness and ω-M- boundedness it is $\mathfrak{d}$, for H-boundedness it is $\mathfrak{b}$, and for R-boundedness it is $\mathsf{cov}(\mathcal{M})$.

We will discuss now preservation of boundedness properties of groups by products of groups. Tkachenko [70] and Hernandez [23] asked if the product of two M-bounded groups also M-bounded.

Several authors answered this question in negative. In [38] (see also [39]), it was given (assuming CH) an example of two linear metric spaces with Menger property such that their product is not M-bounded. Tsaban proved in [72] that there are M-bounded subgroups of $\mathbb{R}^{\mathbb{N}}$ whose product is not M-bounded. The paper [41] contains the result stating that under CH there is a Menger-bounded group $G \leq \mathbb{Z}^{\mathbb{N}}$ whose square is not Menger-bounded.

Another question was asked in [24]: (i) is the product of two strictly M-bounded groups G and H also strictly M-bounded; (ii) is the product of an M-bounded group with a strictly M-bounded group again an M-bounded group. In [2] these questions were answered by the following: (a) the product of two metrizable strictly M-bounded groups is strictly M-bounded; (b) If G an M-bounded group and H is a metrizable strictly M-bounded group, then $G \times H$ is an $\mathsf{M4o}$-bounded group.

At the end of this subsection we list some other interesting results concerning boundedness properties of products of groups.

1. ([23]) The product of a σ-compact and an M-bounded group is M-bounded.
2. ([3]) A group G is ω-M-bounded if and only if G^n is M-bounded for all $n \in \mathbb{N}$.
3. ([41]) Under some additional assumptions (weaker than CH) there is for each $k \in \mathbb{N}$ a metrizable group G such that G^k is Menger-bounded but G^{k+1} is not.
4. ([48]) Under some additional cardinal restrictions there are subgroups of $\mathbb{Z}^{\mathbb{N}}$ whose kth power is Menger-bounded and whose $(k+1)$st power is not.
5. (Mildenberger-Shelah and, independently, Banakh-Zdomskyy) Consistently, every topological group G such that G^2 is Menger-bounded has Menger-bounded all finite powers.

6.2. **Asymmetric cases.** In this short subsection, reporting some results from [37], we demonstrate that for quasi-uniform spaces the situation with boundedness properties may be quite different from ones in uniform spaces. Necessary information about quasi-uniform spaces the interested reader can find in [20] and [40].

Here are basic fact we need in the sequel.

If $(X, \mathbb{U})$ is a quasi-uniform space, then $(X, \mathbb{U}^{-1})$ is also a quasi-uniform space. Here

$$\mathbb{U}^{-1} = \{U^{-1} : U \in \mathbb{U}\}$$

is called the *conjugate* of $\mathbb{U}$.

The supremum of $\mathbb{U}$ and $\mathbb{U}^{-1}$, denoted by $\mathbb{U}^s$, is a uniformity on X called the *symmetrization* of $\mathbb{U}$.

Recall that a quasi-uniform space $(X, \mathbb{U})$ is said to be:

(1) *precompact* (resp. *pre-Lindelöf*) if for each $U \in \mathbb{U}$ there is a finite (resp. countable) set $F \subset X$ such that $U[F] = X$;
(2) *totally bounded* if for each $U \in \mathbb{U}$ there is a finite cover $\mathcal{C}$ of X such that $C \times C \subset U$ for each $C \in \mathcal{C}$.

In uniform spaces precompactness and total boundedness coincide. Evidently, total boundedness of a quasi-uniform space implies its precompactness. It is known that there are precompact (even compact) quasi-uniform spaces which are not totally bounded.

Having in mind the previous note we define now selective versions of precompactness and total boundedness in quasi-uniform spaces.

Definition 6.9. A quasi-uniform space $(X, \mathbb{U})$ is:

(pre-M) *pre-Menger* if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of elements of $\mathbb{U}$ there is a sequence $\langle F_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that $X = \bigcup_{n\in\mathbb{N}} U_n[F_n]$;

(pre-ωM) *pre-ω-Menger* if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of elements of $\mathbb{U}$ there is a sequence $\langle F_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that each finite subset $A \subset X$ is contained in $U_n[F_n]$ for some $n \in \mathbb{N}$;

(pre-H) *pre-Hurewicz* if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of elements of $\mathbb{U}$ there is a sequence $\langle F_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that each $x \in X$ belongs to all but finitely many sets $U_n[F_n]$;

(pre-R) *pre-Rothberger* if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of elements of $\mathbb{U}$ there is a sequence $\langle x_n : n \in \mathbb{N}\rangle$ of elements of X such that $X = \bigcup_{n\in\mathbb{N}} U_n[x_n]$;

(pre-GN) *pre-Gerlits-Nagy* if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of elements of $\mathbb{U}$ there is a sequence $\langle x_n : n \in \mathbb{N}\rangle$ of elements of X such that each $x \in X$ belongs to all but finitely many $U_n[x_n]$.

These selection properties can analogously defined for quasi-metric spaces (see [40]) by replacing entourages of the diagonal for quasi-uniform spaces by open balls for quasi-metric spaces.

To each selection property of a quasi-uniform space $(X, \mathbb{U})$ defined above one can correspond an infinitely long game similarly to definitions in uniform spaces and topological groups, but we do not consider this.

Definition 6.10. Let $(X, \mathbb{U})$ be a quasi-uniform space and let $\mathcal{P}$ be an element of $\{M, \omega M, H, R, GN\}$. X is said to be *$\mathcal{P}$-bounded* if the uniform space $(X, \mathbb{U}^s)$ is $\mathcal{P}$-bounded.

It would be interesting to know that a quasi-uniform space $(X, \mathbb{U})$ is Menger-bounded if and only if for each sequence $(U_n : n \in \mathbb{N})$ there is a sequence $(\mathcal{C}_n : n \in \mathbb{N})$ of finite collections of subsets of X such that $\bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ covers X and for each $n \in \mathbb{N}$, $C \times C \subset U_n$ for each $C \in \mathcal{C}_n$.

Remark 6.11. Let $(X, \mathbb{U})$ be a quasi-uniform space. A cover $\mathcal{C}$ of X is a *quasi-uniform cover* of X if there is $U \in \mathbb{U}$ such that for each $x \in X$ there exists $C \in \mathcal{C}$ with $U[x] \subset C$. $\mathbb{U}$ is a *Lebesgue quasi-uniformity* if each open cover of $(X, \tau_{\mathbb{U}})$ is a quasi-uniform cover of $(X, \mathbb{U})$ (see [20, p. 97]). It is easy to check that pre-Menger Lebesgue quasi-uniformities are Menger (i.e. the topological space $(X, \tau_{\mathbb{U}})$ is Menger). Observe also that a σ-precompact quasi-uniform space is pre-Hurewicz.

In the diagram below we give relationships among the covering properties of quasi-uniform spaces. The Menger (Hurewicz, Rothberger) property concerns the topology $\tau_{\mathbb{U}}$ generated by $\mathbb{U}$. In [37] we showed that the arrows in this diagram are not reversible.

$$
\begin{array}{ccccccccc}
\text{compact} & \Rightarrow & \text{Hurewicz} & \Rightarrow & \text{Menger} & \Leftarrow & \text{Rothberger} & \Leftarrow & \text{GN} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{pre}-\text{compact} & \Rightarrow & \text{pre}-\text{Hurewicz} & \Rightarrow & \text{pre}-\text{Menger} & \Leftarrow & \text{pre}-\text{Rothberger} & \Leftarrow & \text{pre}-\text{GN} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{totally bounded} & \Rightarrow & \text{H}-\text{bounded} & \Rightarrow & \text{M}-\text{bounded} & \Leftarrow & \text{R}-\text{bounded} & \Leftarrow & \text{GN}-\text{bounded}
\end{array}
$$

Diagram 2: Quasi-uniform case

6.3. **Paratopological groups.** Here we give a specific illustration for nonsymmetric cases in paratopological groups. A group $(G, \cdot)$ with a topology τ is a *paratopological group* if the group operation is jointly continuous mapping from $G \times G$ to G. For more details on paratopological groups see [71].

Let $\eta(e_G)$ denote the system of neighbourhoods of the identity element e_G of G. Then $(G, \cdot, \tau^{-1})$ denotes the paratopological group such that $\{U^{-1} : U \in \eta(e_G)\}$ is a neighbourhood system at e_G, and $(G^*, \cdot, \tau^*)$ is the topological group $(G, \cdot, \tau \vee \tau^{-1})$.

A paratopological group $(G, \cdot, \tau)$ is *pre-Menger* if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ in $\eta(e_G)$ there are finite sets $A_n \subset G$, $n \in \mathbb{N}$, such that $G = \bigcup_{n\in\mathbb{N}} A_n U_n$. G is *totally Menger* if the group $(G^*, \cdot, \tau^*)$ is M-bounded. Similarly we define pre-Rothberger, pre-Hurewicz, pre-Gerlits-Nagy paratopological groups.

The Sorgenfrey line $\mathbb{S}$ is an example of a pre-Menger paratopological group which is not Menger. This group is not pre-Rothberger, too.

We saw that subgroups of M-bounded topological groups are also M-bounded. But it is not the case in paratopological groups.

We quote only four results, without proofs, following [36], to illustrate differences between topological and paratopological case.

Theorem 6.12. *If a paratopological group $(G, \cdot, \tau)$ is pre-Menger and H is a dense subgroup of $(G, \cdot, \tau^{-1})$, then H is pre-Menger.*

Theorem 6.13. *For a paratopological group $(G, \cdot, \tau)$ the following are equivalent:*

(1) *All finite powers of G are pre-Menger;*
(2) *G is pre-ω-Menger.*

Theorem 6.14. *Let $(G, \cdot, \tau)$ be a pre-Menger paratopological group and (H, σ) a precompact paratopological group. Then $G \times H$ is a paratopological group.*

Theorem 6.15. *If $(G, \cdot, \tau^*)$ is a pre-Menger topological group, and (H, σ) a hereditarily precompact paratopological group, then the product $(G \times H, \tau^* \times \sigma)$ is hereditarily pre-Menger.*

Problem 6.16. If paratopological groups G and H are such that G is hereditarily pre-Menger and H is hereditarily precompact, is then the product $G \times H$ hereditarily pre-Menger

Acknowledgement: The author is grateful to the journal Director-in-Chief, Professor Mohammad Sal Moslehian, who invited me to write this survey paper. I also thank Editor-in-Chief, Professor Hamid Reza Ebrahimi Vishki for his kind cooperation during preparation of this article.

References

1. A.V. Arhangelskii, M. Tkachenko, *Topological Groups and Related Structures*, Atlantis Studies in Mathematics, Vol. 1, Atlantis Press/World Scientific, AmsterdamParis, 2008.
2. L. Babinkostova, *Metrizable groups and strict o-boundedness*, Mat. Vesnik **58** (2006), no. 3-4, 131–138.
3. L. Babinkostova, Lj.D.R. Kočinac, M. Scheepers, *Combinatorics of open covers* (XI): *Menger- and Rothberger-bounded groups*, Topology Appl. **154** (2007), no. 7, 1269–1280.

4. T. Banakh, *Locally minimal topological groups and their embeddings into products of o-bounded groups*, Comment. Math. Univ. Carolin. **41** (2000), no. 4, 811-815.
5. T. Banakh, *On index of total boundedness of (strictly) o-bounded groups*, Topology Appl. **120** (2002), no. 3, 427-439.
6. T. Banakh, D. Repovš, L. Zdomskyy, *o-boundedness of free topological groups*, Topology Appl. **157** (2010), no. 2, 466–481.
7. M. Bonanzinga, F. Cammaroto, Lj.D.R. Kočinac, *Star-Hurewicz and related properties*, Appl. Gen. Topol. **5** (2004), no. 1, 79–89.
8. M. Bonanzinga, F. Cammaroto, Lj.D.R. Kočinac, M.V. Matveev, *On weaker forms of Menger, Rothberger and Hurewicz properties*, Mat. Vesnik **61** (2009), no. 1, 13–23.
9. M. Bonanzinga, M. Matveev, *Some covering properties for Ψ-spaces*, Mat. Vesnik **61** (2009), no. 1, 3–11.
10. M. Bonanzinga, B.A. Pansera, *Relative versions of some star-selection principles*, Acta Math. Hungar. **117** (2007), no. 3, 231–243.
11. A. Caserta, G. Di Maio, Lj.D.R. Kočinac, *Versions of properties* (a) *and* (pp), Topology Appl. **158** (2011), no. 12, 1630–1638.
12. A. Caserta, S. Watson, *Alexandroff duplicate and its subspaces*, Appl. Gen. Topol. **8** (2007), no. 2, 187-205.
13. G.D. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. **32** (1970), no. 1, 47-54.
14. P. Daniels, *Pixley-Roy spaces over subsets of the reals*, Topology Appl. **29** (1988), no. 1, 93–106.
15. E.K. van Douwen, *The Pixley-Roy topology on spaces of subsets*, In: Set Theoretic Topology, Academic Press, 1977, pp. 111–134.
16. E.K. van Douwen, *The integers and topology*, In: K. Kunen, J. Vaughan (eds.), Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, pp. 111-167.
17. E.K. van Douwen, G.M. Reed, A.W. Roscoe and I.J. Tree, *Star covering properties*, Topology Appl. **39** (1991), no. 1, 71–103.
18. R. Engelking, *General Topology* (revised and completed edition), Heldermann, 1989.
19. W.M. Fleishman, *A new extension of countable compactness*, Fund. Math. **67** (1970), no. 1, 1–9.
20. P. Fletcher, W.F. Lindgren, *Quasi-Uniform Spaces*, Lecture Notes Pure Appl. Math., vol. 77, MarcelDekker, 1982.
21. D. Gauld, M.K. Vamanamurthy, *Covering properties and metrization of manifolds* 2, Topology Proc. **24** (1999), 173-186.
22. I.I. Guran, *On topological groups close to being Lindelöf*, Soviet Math. Dokl. **23** (1981), 173-175.
23. C. Hernández, *Topological groups close to being σ-compact*, Topology Appl. **102** (2000), no. 1, 101-111.
24. C. Hernández, D. Robbie, M. Tkachenko, *Some properties of o-bounded and strictly o-bounded groups*, Appl. Gen. Topol. **1** (2000), no. 1, 29-43.
25. W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Math. Z. **24** (1925), 401–421.
26. W. Just, M.V. Matveev, P.J. Szeptycki, *Some results on property* (a), Topology Appl. **101** (2000), no. 1, 103-111.
27. D. Kocev, *Selection principles in relator spaces*, Acta Math. Hungar. **126** (2010), no. 1-2, 78–93.
28. D. Kocev, *On weaker forms of relator Menger, relator Rothberger and relator Hurewicz properties*, Filomat **26** (2012), no. 3, 427–437.
29. D. Kocev, *Some selection properties of topological spaces and their generalizations*, Doctoral Dissertation, University of Niš, 2012, pp. v+65 (in Serbian).
30. Lj.D.R. Kočinac, *Star-Menger and related spaces*, Publ. Math. Debrecen **55** (1999), no. 3–4, 421–431.
31. Lj.D.R. Kočinac, *Star-Menger and related spaces*, II, Filomat **13** (1999), 129–140.

32. Lj.D.R. Kočinac, *Selection principles in uniform spaces*, Note Mat. **22** (2003/2004), no. 2, 127–139.
33. Lj.D.R. Kočinac, *Selected results on selection principles*, In: Proc. Third Seminar Geom. Topol., July 15–17, 2004, Tabriz, Iran, pp. 71–104.
34. Lj.D.R. Kočinac, *Some covering properties in topological and uniform spaces*, Proc. Steklov Inst. Math. **252** (2006), no. 1, 122–137.
35. Lj.D.R. Kočinac, *Selection properties in fuzzy metric spaces*, Filomat **26** (2012), no. 2, 305–312.
36. Lj.D.R. Kočinac, *Remarks on selection properties of paratopological groups*, in preparation.
37. Lj.D.R. Kočinac, H.-P.A. Künzi, *Selection properties of uniform and related structures*, Topology Appl. **160** (2013), no. 18, 2495–2504.
38. A. Krawczyk, H. Michalewski, *Linear metric spaces close to being σ-compact*, preprint, 2002, 10 pages.
39. A. Krawczyk, H. Michalewski, *An example of a topological group*, Topology Appl. **127** (2003), no. 3, 325–330.
40. H.-P.A. Künzi, *An introduction to quasi-uniform spaces*, Contemp. Math. **486** (2009), 239-304.
41. M. Machura, S. Shelah, B. Tsaban, *Squares of Menger-bounded groups*, Trans. Amer. Math. Soc. **362** (2010), no. 4, 1751–1764.
42. M. Machura, B. Tsaban, *The combinatorics of the Baer-Specker group*, Israel J. Math. **168** (2008), 125–152.
43. M.V. Matveev, *Absoutely countably compact spaces*, Topology Appl. **58** (1994), no. 1, 81–92.
44. M.V. Matveev, *Some questions on property* (a), Questions Answers Gen. Topology **15** (1997), no. 2, 103–111.
45. M.V. Matveev, *A survey on star covering properties*, Topology Atlas, Preprint No. 330 (1998).
46. M.V. Matveev, *On the extent of* SSM *spaces*, preprint, July 1998.
47. K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Stzungsberischte Abt. 3a, Mathematik, Astronomie, Pysik, Meteorologie und Mechanik (Wiener Akademie, Wien) **133** (1924), 421–444.
48. H. Mildenberger, *Cardinal characteristics for Menger-bounded subgroups*, Topology Appl. **156** (2008), no. 1, 130–137.
49. C.J.G. Morgan, S.G. da Silva, *Selectively* (a)*-spaces from almost disjoint families are necessarily countable under a certain parametrized weak diamond principle*, Houston J. Math., to appear.
50. F. Rothberger, *Eine Verschärfung der Eigenschaft* C, Fund. Math. **30** (1938), 50–55.
51. M.E. Rudin, I.S. Stares, J.E. Vaughan, *From countable compactness to absolute countable compactness*, Proc. Amer. Math. Soc. **125** (1997), no. 3, 927-934.
52. M. Sakai, *Star covering versions of the Menger propperty*, Topology Appl. **176** (2014), 22–34.
53. M. Sakai, M. Scheepers, *The combinatorics of open covers*, In: K.P. Hart, J. van Mill, P. Simon (eds.), Recent Progress in General Topology III, Atlantis Press, 2014, pp. 751–800.
54. M. Scheepers, *Combinatorics of open covers* (I): *Ramsey theory*, Topology Appl. **69** (1996), no. 1, 31–62.
55. M. Scheepers, *Rothberger bounded groups and Ramsey theory*, Topology Appl. **158** (2011), no. 13, 1575–1583.
56. S.G. da Silva, *On the presence of countable paracompactness, normality and property* (a) *in spaces from almost disjoint families*, Questions Answers Gen. Topology **25** (2007), no. 1, 1–18.
57. S.G. da Silva, (a)*-spaces and selectively* (a)*-spaces from almost disjoint families*, Acta Math. Hungar. **142** (2014), no. 2, 420–432.
58. Y.-K. Song, *Absolutely strongly star-Hurewicz spaces*, Open Math. **13** (2015), no. 1, 26–32.

59. Y.-K. Song, *Absolutely strongly star-Menger spaces*, Topology Appl. **160** (2013), no. 3, 475–481.
60. Y.-K. Song, *Remarks on strongly star-Hurewicz spaces*, Filomat **27** (2013), no. 6, 1127–1131.
61. Y.-K. Song, *Remarks on star-Hurewicz spaces*, Bull. Polish Acad. Sci. Math. **61** (2013), no. 3–4, 247–255.
62. Y.-K. Song, *Remarks on strongly star-Menger spaces*, Comment. Math. Univ. Carolin. **54** (2013), no. 1, 97–104.
63. Y.-K. Song, *Remarks on selectively* (a)*-spaces*, Topology Appl. **160** (2013), no. 6, 806–811.
64. Y.-K. Song, *A note on star-Rothberger spaces*, Houston J. Math. **40** (2014), no. 2, 645–653.
65. Y.-K. Song, *On star-*K*-Menger spaces*, Hacet. J. Math. Stat. **43** (2014), no. 5, 769–776.
66. Y.-K. Song, *Remarks on star-Menger spaces*, Houston J. Math. **40** (2014), no. 3, 917–925.
67. Y.-K. Song, R. Li, *A note on star-Hurewicz spaces*, Filomat **27** (2013), no. 6, 1091–1095.
68. Y.-K. Song, Z. Yin, *On star-*C*-Menger spaces*, Quaest. Math. **37** (2014), no. 3, 337–347..
69. P.J. Szeptycki, J.E. Vaughan, *Almost disjoint families and property* (a), Fund. Math. **158** (1998), no. 3, 229-240.
70. M. Tkachenko, *Introduction to topological groups*, Topology Appl. **86** (1998), no. 3, 179-231.
71. M. Tkachenko, *Paratopological and Semitopological Groups Versus Topological Groups*, In: K.P. Hart, J. van Mill, P. Simon (eds.), Recent Progress in General Topology III, Atlantis Press, 2014, pp. 825–882.
72. B. Tsaban, *o-bounded groups and other topological groups with strong combinatorial properties*, Proc. Amer. Math. Soc. **134** (2006), no. 3, 881–891.
73. B. Tsaban, *Selection principles and special sets of reals*, In: E. Pearl (ed.), Open Problems in Topology II, Elsevier Science, 2007, pp. 91–108.
74. B. Tsaban, *Combinatorial aspects of selective star covering properties in* $Psi-spaces$, Topology Appl. to appear.
75. J.E. Vaughan, *Small uncountable cardinals and topology*, In: J. van Mill, G.M. Reed (eds.), Open Problems in Topology, North-Holland, Amsterdam, 1990, pp. 195-216.
76. T. Weiss, *A note on unbounded strongly measure zero subgroups of the Baer-Specker group*, Topology Appl. **156** (2008), no. 1, 138–141.
77. L.S. Zdomskyy, *o-boundedness of free objects over a Tychonoff space*, Mat. Stud. **25** (2006), no. 1, 10-28.

University of Niš, Faculty of Sciences and Mathematics, 18000 Niš, Serbia
E-mail address: lkocinac@gmail.com