Khayyam J. Math. 1 (2015), no. 1, 115–124

Khayyam Journal of Mathematics

URL: www.emis.de/journals/KJM/

SOME INTEGRAL INEQUALITIES FOR $\alpha-, m-, (\alpha, m)$ −LOGARITHMICALLY CONVEX FUNCTIONS

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Communicated by S. Hejazian

Abstract. In this paper, the authors establish some Hermite-Hadamard type inequalities by using elementary inequalities for functions whose first derivative absolute values are α-, m-, (α, m)-logarithmically convex.

1. Introduction and preliminaries

In this section, we will present definitions and some results used in this paper.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

hold. This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions (see [1]-[8]).

Definition 1.1. Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \tag{1.1}$$

for all $x, y \in I$ and $t \in [0, 1]$.

The concepts of α-, m- and (α, m)-logarithmically convex functions were introduced as follows.

Date: Received: 12 November 2014; Revised: 15 December 2014; Accepted: 23 December 2014.

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2010 *Mathematics Subject Classification.* Primary 26A15; Secondary 26A51, 26D10.

Key words and phrases. $\alpha-, m-, (\alpha, m)$ −logarithmically convex, Hadamard's inequality, Hölder's inequality, power mean inequality, Cauchy's inequality.

Definition 1.2. [1] A function $f:[0,b]\to(0,\infty)$ is said to be m-logarithmically convex if the inequality

$$f\left(tx+m\left(1-t\right)y\right)\leq\left[f\left(x\right)\right]^{t}\left[f\left(y\right)\right]^{m(1-t)} \tag{1.2}$$

holds for all $x,y\in[0,b]$, $m\in(0,1]$, and $t\in[0,1]$.

Obviously, if putting $m=1$ in Definition 1.2, then f is just the ordinary logarithmically convex on $[0,b]$.

Definition 1.3. [8] A function $f:[0,b]\to(0,\infty)$ is said to be α-logarithmically convex if

$$f\left(tx+\left(1-t\right)y\right)\leq\left[f\left(x\right)\right]^{t^{\alpha}}\left[f\left(y\right)\right]^{(1-t^{\alpha})} \tag{1.3}$$

holds for all $x,y\in[0,b]$, $\alpha\in(0,1]$ and $t\in[0,1]$.

Clearly, when taking $\alpha=1$ in Definition 1.3, then f becomes the ordinary logarithmically convex on $[0,b]$.

Definition 1.4. [1] A function $f:[0,b]\to(0,\infty)$ is said to be (α,m)-logarithmically convex if

$$f\left(tx+m\left(1-t\right)y\right)\leq\left[f\left(x\right)\right]^{t^{\alpha}}\left[f\left(y\right)\right]^{m(1-t^{\alpha})} \tag{1.4}$$

holds for all $x,y\in[0,b]$, $(\alpha,m)\in(0,1]\times(0,1]$, and $t\in[0,1]$.

Clearly, when taking $\alpha=1$ in Definition 1.4, then f becomes the standard m-logarithmically convex function on $[0,b]$, and, when taking $m=1$ in Definition 1.4, then f becomes the α-logarithmically convex function on $[0,b]$.

In [3], the following theorem which was obtained by Dragomir and Agarwal contains the Hermite-Hadamard type integral inequality.

Theorem 1.5. [3, Theorem 2.2] *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, the interior of I, $a,b\in I^{\circ}$ with $a<b$. If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|\leq\frac{\left(b-a\right)\left(\left|f'\left(a\right)\right|+\left|f'\left(b\right)\right|\right)}{8}. \tag{1.5}$$

Theorem 1.6. [3, Theorem 2.3] *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$, and $p>1$. If the new mapping $|f'(x)|^{p/p-1}$ is convex on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right| \qquad (1.6)\\ \leq\;&\frac{b-a}{2\left(p+1\right)^{1/p}}\left[\frac{\left|f'\left(a\right)\right|^{p/(p-1)}+\left|f'\left(b\right)\right|^{p/(p-1)}}{2}\right]^{(p-1)/p}.\end{aligned}$$

The aim of this paper is to establish some integral inequalities of Hermite-Hadamard type for α-, m-, (α,m)-logarithmically convex functions.

2. Hadamard Type Inequalities

In order to prove our main theorems, we need the following lemma [7].

Lemma 2.1. [7] *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.1}$$
$$= \frac{b-a}{2}\int_0^1\int_0^1 \left[f'(ta+(1-t)b) - f'(sa+(1-s)b)\right](s-t)\,dtds.$$

A simple proof of this equality can be also done integrating by parts in the right hand side (see [7]).

The next theorems gives a new result of the upper Hermite-Hadamard inequality for α-, m-, (α, m)-logarithmically convex functions.

Theorem 2.2. *Let $I \supset [0,\infty)$ be an open interval and let $f: I \to (0,\infty)$ be a differentiable function on I such that $f' \in L(a,b)$ for $0 \le a < b < \infty$. If $|f'(x)|$ is (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $(\alpha, m) \in (0,1]^2$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.2}$$
$$\le \begin{cases} \frac{(b-a)}{3}\left|f'\left(\frac{b}{m}\right)\right|^m, & \eta = 1 \\ \frac{(b-a)}{2}\left|f'\left(\frac{b}{m}\right)\right|^m \frac{-\alpha^2\ln^2\eta - 2\alpha\ln\eta + 2\eta^\alpha - 2}{\alpha^3\ln^3\eta}, & \eta < 1 \end{cases}$$

where $\eta = |f'(a)| / \left|f'\left(\frac{b}{m}\right)\right|^m$.

Proof. By Lemma 2.1 and since $|f'|$ is an (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$, then we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le \frac{b-a}{2}\int_0^1\int_0^1 \left|(f'(ta+(1-t)b)) - (f'(sa+(1-s)b))\right| |s-t|\,dtds$$
$$\le \frac{b-a}{2}\left[\int_0^1\int_0^1 |s-t|\,|f'(a)|^{t^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-t^\alpha)} dtds\right]$$
$$+\frac{b-a}{2}\left[\int_0^1\int_0^1 |s-t|\,|f'(a)|^{s^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-s^\alpha)} dtds\right]$$

Let $0 < k \le 1$, $0 \le m \le 1$, and $0 < n \le 1$. Then

$$k^{m^n} \le k^{nm}. \tag{2.3}$$

When $\eta = 1$, by (2.3), we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{b-a}{2}\left|f'\left(\frac{b}{m}\right)\right|^m\left[\int_0^1\int_0^1|s-t|\,dtds+\int_0^1\int_0^1|s-t|\,dtds\right]$$
$$= \frac{b-a}{3}\left|f'\left(\frac{b}{m}\right)\right|^m$$

When $0<\eta<1$, by (2.3), we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{2}\left|f'\left(\frac{b}{m}\right)\right|^m\left[\int_0^1\int_0^1|s-t|\,\eta^{\alpha t}dtds+\int_0^1\int_0^1|s-t|\,\eta^{\alpha s}dtds\right]$$
$$= \frac{b-a}{2}\left|f'\left(\frac{b}{m}\right)\right|^m\left[\frac{-\alpha^2\ln^2\eta-2\alpha\ln\eta+4\eta^\alpha+\alpha^2\eta^\alpha\ln^2\eta-2\alpha\eta^\alpha\ln\eta-4}{2\alpha^3\ln^3\eta}\right.$$
$$\left.+\frac{-\alpha\ln\eta+2\eta^\alpha-\alpha\eta^\alpha\ln\eta-2}{2\alpha^2\ln^2\eta}\right]$$

which completes the proof. □

Corollary 2.3. *Let $I\supset[0,\infty)$ be an open interval and let $f: I\to(0,\infty)$ be a differentiable function on I such that $f'\in L(a,b)$ for $0\leq a<b<\infty$. If $|f'(x)|$ is m-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $m\in(0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\begin{cases}\frac{(b-a)}{3}\left|f'\left(\frac{b}{m}\right)\right|^m, & \eta=1\\ \frac{(b-a)}{2}\left|f'\left(\frac{b}{m}\right)\right|^m\frac{-\ln^2\eta-2\ln\eta+2\eta-2}{\ln^3\eta}, & \eta<1\end{cases}$$

where η is same as Theorem 2.2.

Corollary 2.4. *Let $I\supset[0,\infty)$ be an open interval and let $f: I\to(0,\infty)$ be a differentiable function on I such that $f'\in L(a,b)$ for $0\leq a<b<\infty$. If $|f'(x)|$is α-logarithmically convex on $[0,b]$ for $\alpha\in(0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\begin{cases}\frac{(b-a)}{3}\left|f'(b)\right|, & \eta=1\\ \frac{(b-a)}{2}\left|f'(b)\right|\frac{4\eta^\alpha-4\alpha\ln\eta-2\alpha^2\ln^2\eta-4}{2\alpha^3\ln^3\eta}, & \eta<1\end{cases}$$

where $\eta=|f'(a)|/|f'(b)|$.

Theorem 2.5. *Let $I\supset[0,\infty)$ be an open interval and let $f: I\to(0,\infty)$ be a differentiable function on I such that $f'\in L(a,b)$ for $0\leq a<b<\infty$. If $|f'(x)|^q$ is an (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]^2$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.4}$$
$$\leq\begin{cases}(b-a)\left|f'\left(\frac{b}{m}\right)\right|^m\left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}}, & \eta=1\\ (b-a)\left|f'\left(\frac{b}{m}\right)\right|^m\left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\times\left(\frac{\eta(\alpha q,\alpha q)-1}{\ln\eta(\alpha q,\alpha q)}\right)^{\frac{1}{q}}, & \eta<1\end{cases}$$

where $\eta(\alpha,\alpha)$ is same as Theorem 2.2.

Proof. Since $|f'|^q$ is an (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$, from Lemma 2.1 and the well known Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{2}\int_0^1\int_0^1 \left|\left(f'(ta+(1-t)b)\right)-\left(f'(sa+(1-s)b)\right)\right||s-t|\,dtds \\
\leq\ & \frac{b-a}{2}\int_0^1\int_0^1 |s-t|\,|f'(a)|^{t^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-t^\alpha)} dtds \\
& +\frac{b-a}{2}\int_0^1\int_0^1 |s-t|\,|f'(a)|^{s^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-s^\alpha)} dtds \\
\leq\ & \frac{b-a}{2}\left|f'\left(\frac{b}{m}\right)\right|^m\left(\int_0^1\int_0^1 |s-t|^p\,dtds\right)^{\frac{1}{p}} \\
& \times\left[\left(\int_0^1\int_0^1 \eta^{qt^\alpha}dtds\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1 \eta^{qs^\alpha}dtds\right)^{\frac{1}{q}}\right]
\end{aligned}
\tag{2.5}
$$

If $\eta = 1$, by (2.3), we obtain

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & (b-a)\left|f'\left(\frac{b}{m}\right)\right|^m\left(\int_0^1\int_0^1 |s-t|^p\,dtds\right)^{\frac{1}{p}} \\
=\ & (b-a)\left|f'\left(\frac{b}{m}\right)\right|^m\left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}}
\end{aligned}
$$

If $\eta < 1$, by (2.3), we obtain

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{2}\left|f'\left(\frac{b}{m}\right)\right|^m\left(\int_0^1\int_0^1 |s-t|^p\,dtds\right)^{\frac{1}{p}} \\
& \times\left[\left(\int_0^1\int_0^1 \eta^{qt^\alpha}dtds\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1 \eta^{qs^\alpha}dtds\right)^{\frac{1}{q}}\right] \\
=\ & (b-a)\left|f'\left(\frac{b}{m}\right)\right|^m\left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\times\left(\frac{\eta(\alpha q,\alpha q)-1}{\ln\eta(\alpha q,\alpha q)}\right)^{\frac{1}{q}}
\end{aligned}
\tag{2.6}
$$

which completes the proof. □

Corollary 2.6. *Let $I \supset [0, \infty)$ be an open interval and let $f : I \to (0, \infty)$ be a differentiable function on I such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|^q$ is* *an m-logarithmically convex on* $\left[0, \frac{b}{m}\right]$ *for* $m \in (0,1]$ *and* $p = q = 2$, *then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\ & (b-a)\left|f'\left(\frac{b}{m}\right)\right|^m \sqrt{\frac{1}{6}} \times \begin{cases} 1, & \eta = 1 \\ \left(\frac{\eta(2,2)-1}{\ln \eta(2,2)}\right)^{\frac{1}{2}}, & \eta < 1 \end{cases} \end{aligned}$$

Corollary 2.7. *Let* $I \supset [0,\infty)$ *be an open interval and let* $f : I \to (0,\infty)$ *be a differentiable function on* I *such that* $f' \in L(a,b)$ *for* $0 \leq a < b < \infty$. *If* $|f'(x)|$*is* α*-logarithmically convex on* $[0,b]$ *for* $\alpha \in (0,1]$, *then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\ & (b-a)\left|f'(b)\right| \left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}} \times \begin{cases} 1, & \eta = 1 \\ \left(\frac{\eta(\alpha q,\alpha q)-1}{\ln \eta(\alpha q,\alpha q)}\right)^{\frac{1}{q}}, & \eta < 1 \end{cases} \end{aligned}$$

where $\eta = |f'(a)| / |f'(b)|$.

Theorem 2.8. *Let* $I \supset [0,\infty)$ *be an open interval and let* $f : I \to (0,\infty)$ *be a differentiable function on* I *such that* $f' \in L(a,b)$ *for* $0 \leq a < b < \infty$. *If* $|f'(x)|^q$ *is* (α,m)*-logarithmically convex on* $\left[0, \frac{b}{m}\right]$ *for* $(\alpha,m) \in (0,1]^2$, *and then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{3}\left|f'\left(\frac{b}{m}\right)\right|^m$$

$$\times \begin{cases} 1, & \eta = 1 \\ \frac{3}{2}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{2\varphi-2}{[\ln\varphi]^3} - \frac{\varphi+1}{[\ln\varphi]^2} - \frac{1-\varphi}{2\ln\varphi}\right)^{\frac{1}{q}} + \left(\frac{\varphi-1}{[\ln\varphi]^2} - \frac{\varphi+1}{2\ln\varphi}\right)^{\frac{1}{q}}\right] & \eta < 1 \end{cases}$$

where $\eta(\alpha,\alpha)$ *is same as Theorem 2.2, and* $\varphi = \eta(\alpha q, \alpha q)$.

Proof. Since $|f'|^q$ is an (α,m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$, for $q \geq 1$, from Lemma 2.1 and the well known power mean integral inequality, we get

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\ & \frac{b-a}{2}\int_0^1\int_0^1 \left|\left(f'(ta+(1-t)b)\right) - \left(f'(sa+(1-s)b)\right)\right| |s-t|\,dtds \\ \leq\ & \frac{b-a}{2}\left(\int_0^1\int_0^1 |s-t|\,dtds\right)^{1-\frac{1}{q}} \left(\int_0^1\int_0^1 |s-t|\left|f'(ta+(1-t)b)\right|^q dtds\right)^{\frac{1}{q}} \\ &+\frac{b-a}{2}\left(\int_0^1\int_0^1 |s-t|\,dtds\right)^{1-\frac{1}{q}} \left(\int_0^1\int_0^1 |s-t|\left|f'(sa+(1-s)b)\right|^q dtds\right)^{\frac{1}{q}} \\ \leq\ & \frac{b-a}{2}\left|f'\left(\frac{b}{m}\right)\right|^m \left(\int_0^1\int_0^1 |s-t|\,dtds\right)^{1-\frac{1}{q}} \left(\int_0^1\int_0^1 |s-t|\,\eta^{qt^\alpha} dtds\right)^{\frac{1}{q}} \\ &+\frac{b-a}{2}\left|f'\left(\frac{b}{m}\right)\right|^m \left(\int_0^1\int_0^1 |s-t|\,dtds\right)^{1-\frac{1}{q}} \left(\int_0^1\int_0^1 |s-t|\,\eta^{qs^\alpha} dtds\right)^{\frac{1}{q}} \end{aligned}$$

When $\eta = 1$, by (2.3), we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|^m\left(\int_0^1\int_0^1 |s-t|\,dtds\right)^{\frac{1}{q}} \\
&+\frac{b-a}{2}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|^m\left(\int_0^1\int_0^1 |s-t|\,dtds\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{3}\left|f'\left(\frac{b}{m}\right)\right|^m
\end{aligned}$$

When $\eta < 1$, by (2.3), we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|^m\left(\int_0^1\int_0^1 |s-t|\,\eta^{\alpha qt}dtds\right)^{\frac{1}{q}} \\
&+\frac{b-a}{2}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|^m\left(\int_0^1\int_0^1 |s-t|\,\eta^{\alpha qs}dtds\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|^m \\
&\times\left\{\left[\frac{2\eta(\alpha q,\alpha q)-2}{[\ln(\eta(\alpha q,\alpha q))]^3}-\frac{\eta(\alpha q,\alpha q)+1}{[\ln(\eta(\alpha q,\alpha q))]^2}-\frac{1-\eta(\alpha q,\alpha q)}{2\ln(\eta(\alpha q,\alpha q))}\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{\eta(\alpha q,\alpha q)-1}{[\ln(\eta(\alpha q,\alpha q))]^2}-\frac{\eta(\alpha q,\alpha q)+1}{2\ln(\eta(\alpha q,\alpha q))}\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

which completes the proof. □

Corollary 2.9. *Let $I \supset [0,\infty)$ be an open interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L(a,b)$ for $0 \leq a < b < \infty$. If $|f'(x)|^q$is m-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $m \in (0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \begin{cases} \frac{b-a}{3}\left|f'\left(\frac{b}{m}\right)\right|^m, & \eta = 1 \\ \frac{(b-a)}{2}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|^m\left\{\left[\frac{2\eta(q,q)-2}{[\ln\eta(q,q)]^3}-\frac{\eta(q,q)+1}{[\ln\eta(q,q)]^2}-\frac{1-\eta(q,q)}{2\ln\eta(q,q)}\right]^{\frac{1}{q}}\right. \\ \qquad\left.+\left[\frac{\eta(q,q)-1}{[\ln\eta(q,q)]^2}-\frac{\eta(q,q)+1}{2\ln\eta(q,q)}\right]^{\frac{1}{q}}\right\}, & \eta < 1 \end{cases}$$

Corollary 2.10. *LetI $\supset [0,\infty)$ be an open interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L(a,b)$ for $0 \leq a < b < \infty$. If $|f'(x)|$*

is α-logarithmically convex on $[0,b]$ for $\alpha\in(0,1]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{3}\left|f'(b)\right|$$
$$\leq\left\{\begin{array}{ll}1, & \eta=1\\ \frac{3}{2}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left\{\left[\frac{2\eta(\alpha q,\alpha q)-2}{[\ln(\eta(\alpha q,\alpha q))]^3}-\frac{\eta(\alpha q,\alpha q)+1}{[\ln(\eta(\alpha q,\alpha q))]^2}-\frac{1-\eta(\alpha q,\alpha q)}{2\ln(\eta(\alpha q,\alpha q))}\right]^{\frac{1}{q}}\right. & \\ \left.+\left[\frac{\eta(\alpha q,\alpha q)-1}{[\ln(\eta(\alpha q,\alpha q))]^2}-\frac{\eta(\alpha q,\alpha q)+1}{2\ln(\eta(\alpha q,\alpha q))}\right]^{\frac{1}{q}}\right\} & ,\quad \eta<1\end{array}\right.$$

where $\eta=|f'(a)|/|f'(b)|$.

Theorem 2.11. *Let $f:I\subset\mathbb{R}_+\to\mathbb{R}_+$ be differentiable on I°, $a,b\in I$, with $a<b$ and $f'\in L([a,b])$. If $|f'|$ is an (α,m)-logarithmically convex $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]^2$ and $\mu_1,\mu_2,\tau_1,\tau_2>0$ with $\mu_1+\tau_1=1$ and $\mu_2+\tau_2=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{2}\left|f'\left(\frac{b}{m}\right)\right|^m \tag{2.7}$$
$$\times\left\{\begin{array}{ll}\frac{2\mu_1^3}{(2\mu_1+1)(\mu_1+1)}+\frac{2\mu_2^3}{(2\mu_2+1)(\mu_2+1)}+\tau_1+\tau_2, & \eta=1\\ \frac{2\mu_1^3}{(2\mu_1+1)(\mu_1+1)}+\frac{2\mu_2^3}{(2\mu_2+1)(\mu_2+1)}+\tau_1\frac{\eta\left(\frac{\alpha}{\tau_1},\frac{\alpha}{\tau_1}\right)-1}{\ln\eta\left(\frac{\alpha}{\tau_1},\frac{\alpha}{\tau_1}\right)}+\tau_2\frac{\eta\left(\frac{\alpha}{\tau_2},\frac{\alpha}{\tau_2}\right)-1}{\ln\eta\left(\frac{\alpha}{\tau_2},\frac{\alpha}{\tau_2}\right)}, & \eta<1\end{array}\right.$$

where $\eta(\alpha,\alpha)$ is same as Theorem 2.2.

Proof. Since $|f'|^q$ is an (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$, from Lemma 2.1, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.8}$$
$$\leq\frac{(b-a)}{2}\int_0^1\int_0^1\left|\left(f'(ta+(1-t)b)\right)-\left(f'(sa+(1-s)b)\right)\right||s-t|\,dtds$$
$$\leq\frac{(b-a)}{2}\left[\int_0^1\int_0^1|s-t|\left|f'(a)\right|^{t^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-t^\alpha)}dtds\right]$$
$$+\frac{(b-a)}{2}\left[\int_0^1\int_0^1|s-t|\left|f'(a)\right|^{s^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-s^\alpha)}dtds\right]$$
$$=\frac{(b-a)}{2}\left|f'\left(\frac{b}{m}\right)\right|^m\left[\int_0^1\int_0^1|s-t|\,\eta^{t^\alpha}dtds+\int_0^1\int_0^1|s-t|\,\eta^{s^\alpha}dtds\right]$$

for all $t\in[0,1]$. Using the well known inequality $rt\leq\mu r^{\frac{1}{\mu}}+\tau t^{\frac{1}{\tau}}$, on the right side of (2.8), we get

$$\int_0^1\int_0^1|s-t|\,\eta^{t^\alpha}dtds+\int_0^1\int_0^1|s-t|\,\eta^{s^\alpha}dtds \tag{2.9}$$
$$\leq\mu_1\int_0^1\int_0^1|s-t|^{\frac{1}{\mu_1}}dtds+\tau_1\int_0^1\int_0^1\eta^{\frac{t^\alpha}{\tau_1}}dtds$$

$$+\mu_2\int_0^1\int_0^1 |s-t|^{\frac{1}{\mu_2}}\,dtds+\tau_2\int_0^1\int_0^1 \eta^{\frac{s^\alpha}{\tau_2}}dtds$$

When $\eta=1$, by (2.3), we get

$$\int_0^1\int_0^1 |s-t|\,\eta^{t^\alpha}dtds+\int_0^1\int_0^1 |s-t|\,\eta^{s^\alpha}dtds \tag{2.10}$$
$$\leq \frac{2\mu_1^3}{(2\mu_1+1)(\mu_1+1)}+\frac{2\mu_2^3}{(2\mu_2+1)(\mu_2+1)}+\tau_1+\tau_2$$

When $\eta<1$, by (2.3), we get

$$\begin{aligned}
&\int_0^1\int_0^1 |s-t|\,\eta^{t^\alpha}dtds+\int_0^1\int_0^1 |s-t|\,\eta^{s^\alpha}dtds \\
\leq\ & \mu_1\int_0^1\int_0^1 |s-t|^{\frac{1}{\mu_1}}\,dtds+\tau_1\int_0^1\int_0^1 \eta^{\frac{t^\alpha}{\tau_1}}dtds \\
&+\mu_2\int_0^1\int_0^1 |s-t|^{\frac{1}{\mu_2}}\,dtds+\tau_2\int_0^1\int_0^1 \eta^{\frac{s^\alpha}{\tau_2}}dtds \\
\leq\ & \mu_1\int_0^1\int_0^1 |s-t|^{\frac{1}{\mu_1}}\,dtds+\mu_2\int_0^1\int_0^1 |s-t|^{\frac{1}{\mu_2}}\,dtds \\
&+\tau_1\int_0^1\int_0^1 \eta^{\frac{\alpha t}{\tau_1}}dtds+\tau_2\int_0^1\int_0^1 \eta^{\frac{\alpha s}{\tau_2}}dtds \\
=\ & \frac{2\mu_1^3}{(2\mu_1+1)(\mu_1+1)}+\frac{2\mu_2^3}{(2\mu_2+1)(\mu_2+1)} \\
&+\tau_1\frac{\eta\left(\frac{\alpha}{\tau_1},\frac{\alpha}{\tau_1}\right)-1}{\ln\eta\left(\frac{\alpha}{\tau_1},\frac{\alpha}{\tau_1}\right)}+\tau_2\frac{\eta\left(\frac{\alpha}{\tau_2},\frac{\alpha}{\tau_2}\right)-1}{\ln\eta\left(\frac{\alpha}{\tau_2},\frac{\alpha}{\tau_2}\right)}
\end{aligned} \tag{2.11}$$

from (2.8)-(2.11), which completes the proof. □

Corollary 2.12. *Under the assumptions of Theorem 2.11, and $\mu=\mu_1=\mu_2>0$, $\tau=\tau_1=\tau_2>0$ with $\mu+\tau=1$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{2}\left|f'\left(\frac{b}{m}\right)\right|^m\times\begin{cases}\frac{4\mu^3}{(2\mu+1)(\mu+1)}+2\tau, & \eta=1\\ \frac{4\mu^3}{(2\mu+1)(\mu+1)}+2\tau\frac{\eta\left(\frac{\alpha}{\tau},\frac{\alpha}{\tau}\right)-1}{\ln\eta\left(\frac{\alpha}{\tau},\frac{\alpha}{\tau}\right)}, & \eta<1\end{cases}$$

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