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ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE LIPSCHITZ CLASS BY $(E,q)(C,\alpha,\beta)$ MEANS

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Abstract. In this paper two generalized theorems on the degree of approximation of conjugate functions belonging to the Lipschitz classes of the type Lipα, $0<\alpha\leq 1$, and $W(L_p,\xi(t))$ are proved. The first one gives the degree of approximation with respect to the L_∞-norm, and the second one with respect to L_p-norm, $p\geq 1$. In addition, a correct condition in proving of the second mentioned theorem is employed.

1. Introduction and preliminaries

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with its partial sums s_n. We denote by $C_n^{(\theta,\beta)}$ the n-th Cesáro means of order (θ,β), with $\theta+\beta>-1$ of the sequence (s_n), i.e. (see [2])

$$C_n^{(\theta,\beta)}=\frac{1}{A_n^{\theta+\beta}}\sum_{v=0}^{n}A_{n-v}^{\theta-1}A_v^{\beta}s_v,$$

where $A_n^{\theta+\beta}=O(n^{\theta+\beta})$, $\theta+\beta>-1$ and $A_0^{\theta+\beta}=1$.

The series $\sum_{n=0}^{\infty} u_n$ is said to be (C,θ,β) summable to the definite number s if

$$C_n^{(\theta,\beta)}=\frac{1}{A_n^{\theta+\beta}}\sum_{v=0}^{n}A_{n-v}^{\theta-1}A_v^{\beta}s_v\to s,\quad \text{as}\quad n\to\infty.$$

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Then, for $q > 0$ a real number the Euler means (E, q) of the sequence (s_n) are defined to be (see for example [2])

$$E_n^q = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} s_v.$$

The series $\sum_{n=0}^{\infty} u_n$ is said to be (E, q) summable to the definite number s if

$$E_n^q = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} s_v \to s, \quad \text{as} \quad n \to \infty.$$

The (E, q) transform of the (C, θ, β) transform, defines $(E, q)(C, \theta, \beta)$ transform and we shall denote it by $(EC)_n^{q,\theta,\beta}$.

Moreover, if

$$(EC)_n^{q,\theta,\beta} = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} C_k^{(\theta,\beta)} \to s, \quad \text{as} \quad n \to \infty,$$

then we shall say that the infinite series $\sum_{n=0}^{\infty} u_n$ is $(E, q)(C, \theta, \beta)$ summable to the definite number s.

We note that for $q = 1$, $\theta = 1$ and $\beta = 0$ the concept of $(E, q)(C, \theta, \beta)$ summability reduces to the $(E, 1)(C, 1)$ summability introduced in [10].

Let $f(x)$ be a 2π periodic function and integrable in the sense of Lebesgue. Then, let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series with n-th partial sum $s_n(f; x)$.

The conjugate series of the above Fourier series is given by

$$\sum_{n=1}^{\infty} (a_n \cos nx - b_n \sin nx). \tag{1.1}$$

For a function $f : R \to R$ the equalities

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\}$$

and

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx \right)^{1/p}, \quad p \geq 1$$

denote the L_∞-norm and L_p-norm, respectively.

The degree of approximation of a function f by a trigonometric polynomial t_n of order n under the norm $\|\cdot\|_\infty$ is defined by Zygmund [16] with

$$\|f - t_n\|_\infty = \sup\{|f(x) - t_n(x)| : x \in R\}$$

and the best approximation $E_n(f)$ of a function $f \in L_p$ is defined by the equality

$$E_n(f) = \min_{t_n} \|f - t_n\|_p.$$

A function $f \in \text{Lip}\alpha$ or $f \in \text{Lip}(\alpha, p)$ if

$$|f(x+t) - f(x)| = O(|t|^\alpha) \quad \text{for} \quad 0 < \alpha \leq 1$$

or

$$\left(\int_0^{2\pi} |f(x+t)-f(x)|^p dx\right)^{1/p} = O(|t|^\alpha) \quad \text{for} \quad 0<\alpha\le 1 \quad \text{and} \quad p\ge 1,$$

respectively.

For a given positive increasing function $\xi(t)$ and an integer $p \ge 1$, $f \in \mathrm{Lip}(\xi(t),p)$ (see [14]) if

$$\left(\int_0^{2\pi} |f(x+t)-f(x)|^p dx\right)^{1/p} = O(\xi(t))$$

and $f \in W(L_p,\xi(t))$ if

$$\left(\int_0^{2\pi} |[f(x+t)-f(x)]\sin^\gamma x|^p dx\right)^{1/p} = O(\xi(t)), \quad \gamma\ge 0, \quad p\ge 1.$$

We note here in these definitions that for $\beta=0$ the class $W(L_p,\xi(t))$ reduces to the class $\mathrm{Lip}(\xi(t),p)$ and if $\xi(t)=t^\alpha$ then the class $W(L_p,\xi(t))$ reduces to the class $\mathrm{Lip}(\alpha,p)$, and if $p\to\infty$ then the class $\mathrm{Lip}(\alpha,p)$ reduces to the class $\mathrm{Lip}\alpha$.

A lot of authors have determined the degree of approximation of functions from above mentioned classes, using Cesáro and generalized Nörlund means (we refer the reader for details to the papers [1], and [3]–[15]. Very recently H. K. Nigam and K. Sharma [10] have established two theorems on determining the degree of approximation of conjugate functions using $(E,1)(C,1)$ means. The condition

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t|\phi_x(t)|}{\xi(t)}\right)^p \sin^{\gamma p} t dt\right\}^{1/p} = \mathcal{O}\left(\frac{1}{n+1}\right)$$

assumed in Theorem 2, of their result, is not sufficient for the validity of it. This condition leads to the divergent integral of type (see for details [6], page 14)

$$\left(\int_0^{\frac{\pi}{n}} t^{-(2+\gamma)p} dt\right)^{1/p}.$$

Here in this paper we shall generalize their theorems using $(E,q)(C,\theta,\beta)$ means instead of $(E,1)(C,1)$ means that are obviously particular cases of them. Moreover, we employ a correct condition in our result. To verify the main results we need first to prove some helpful statements given in the next section. Everywhere in this paper, we write $u=\mathcal{O}(v)$ if there exists a positive constant C such that $u\le Cv$.

2. Auxiliary Lemma

Throughout this paper we shall use notations

$$\phi_x(t) := f(x+t)+f(x-t),$$

$$\widetilde{D}_n^{q;\theta,\beta}(t) := \frac{1}{\pi(1+q)^n}\sum_{k=0}^n \binom{n}{k} q^{n-k}\frac{1}{A_k^{\theta+\beta}}\sum_{v=0}^k \frac{A_{k-v}^{\theta-1}A_v^\beta \cos\left(v+\frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

and we prove a lemma which plays a key role in the proof of the main results.

Lemma 2.1. *The estimate* $|\widetilde{D}_n^{q;\theta,\beta}(t)| = \mathcal{O}\left(\frac{1}{t}\right)$ *holds true for* $0 \le t \le \pi$.

Proof. **First proof:** Since for $0 \le t \le \pi$, $\sin(t/2) \ge t/\pi$, then

$$\begin{aligned} |\widetilde{D}_n^{q;\theta,\beta}(t)| &= \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\sum_{v=0}^{k}\frac{A_{k-v}^{\theta-1}A_v^{\beta}\cos\left(v+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right| \\ &\le \frac{1}{\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta}\left|\frac{\cos\left(v+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right| \\ &\le \frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta} \\ &= \frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}1^k q^{n-k} \\ &= \mathcal{O}\left(\frac{1}{t}\right), \end{aligned}$$

because of

$$\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta} = A_k^{\theta+\beta} \quad \text{and} \quad \sum_{k=0}^{n}\binom{n}{k}q^{n-k} = (1+q)^n.$$

The first proof of this lemma is completed.

Second proof: Applying the well-known inequality $\sin(t/2) \ge t/\pi$ for $0 \le t \le \pi$, we obtain

$$\begin{aligned} |\widetilde{D}_n^{q;\theta,\beta}(t)| &= \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\sum_{v=0}^{k}\frac{A_{k-v}^{\theta-1}A_v^{\beta}\cos\left(v+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right| \\ &\le \frac{1}{2t(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\Re\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta}e^{i\left(v+\frac{1}{2}\right)t}\right| \\ &\le \frac{1}{2t(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\Re\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta}e^{ivt}\right|\left|e^{\frac{it}{2}}\right| \\ &\le \frac{1}{2t(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\Re\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta}e^{ivt}\right| \\ &\le \frac{1}{2t(1+q)^n}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\Re\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta}e^{ivt}\right| \\ &\quad +\frac{1}{2t(1+q)^n}\left|\sum_{k=\tau}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\Re\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta}e^{ivt}\right| \\ &=: J_1 + J_2. \end{aligned}$$

For the quantity J_1, we have

$$\begin{aligned} J_1 &\leq \frac{1}{2t(1+q)^n}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\Re\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta}e^{ivt}\right| \\ &\leq \frac{1}{2t(1+q)^n}\sum_{k=0}^{\tau-1}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta}\left|e^{ivt}\right| \\ &\leq \frac{1}{2t(1+q)^n}\sum_{k=0}^{\tau-1}\binom{n}{k}q^{n-k}. \end{aligned} \tag{2.1}$$

The use of Abel's lemma leads to

$$\begin{aligned} J_2 &\leq \frac{1}{2t(1+q)^n}\left|\sum_{k=\tau}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\Re\sum_{v=0}^{k}A_{k-v}^{\theta-1}A_v^{\beta}e^{ivt}\right| \\ &\leq \frac{1}{2t(1+q)^n}\sum_{k=\tau}^{n}\binom{n}{k}\frac{q^{n-k}}{A_k^{\theta+\beta}}\max_{0\leq j\leq k}\left|\sum_{v=0}^{j}A_{k-v}^{\theta-1}A_v^{\beta}e^{ivt}\right| \\ &\leq \frac{1}{2t(1+q)^n}\sum_{k=\tau}^{n}\binom{n}{k}1^k q^{n-k}. \end{aligned} \tag{2.2}$$

Thus, the estimations (2.1) and (2.2) give

$$|\widetilde{D}_n^{q;\theta,\beta}(t)| \leq \frac{1}{2t(1+q)^n}\sum_{k=0}^{\tau-1}\binom{n}{k}q^{n-k} + \frac{1}{2t(1+q)^n}\sum_{k=\tau}^{n}\binom{n}{k}1^k q^{n-k} = \mathcal{O}\left(\frac{1}{t}\right),$$

which as well verifies the statement of the lemma. □

3. Main Results

At first, we prove the following theorem.

Theorem 3.1. *If a function $\overline{f}$, conjugate to a 2π periodic function f, belongs to Lipα class, then its degree of approximation by $(E,q)(C,\theta,\beta)$ means of conjugate Fourier series is given by*

$$\sup_{0<x<2\pi}\left|\overline{(EC)_n^{q;\theta,\beta}}(\overline{f}(x)) - \overline{f}(x)\right| = \left\|\overline{(EC)_n^{q;\theta,\beta}}(\overline{f}) - \overline{f}\right\|_\infty = \mathcal{O}\left(\frac{1}{(n+1)^\alpha}\right),\ 0<\alpha<1,$$

where $\overline{(EC)_n^{q;\theta,\beta}}(\overline{f}(x))$ denotes the $(E,q)(C,\theta,\beta)$ transform of partial sums of the series (1.1).

Proof. Let $\overline{s}_v(x)$ be the partial sums of the series (1.1). Then, in [7] it is verified that

$$\overline{s}_v(x) - \overline{f}(x) = \frac{1}{2\pi}\int_0^\pi \phi_x(t)\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt.$$

Thus, the (C,θ,β) transform $\overline{C_k^{\theta,\beta}}(x)$ of $\overline{s}_v(x)$ is

$$\overline{C_k^{\theta,\beta}}(x) - \overline{f}(x) = \frac{1}{2\pi A_k^{\theta+\beta}}\int_0^\pi \phi_x(t)\sum_{v=0}^{k}\frac{A_{k-v}^{\theta-1}A_v^{\beta}\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt. \tag{3.1}$$

Further, denoting the $(E,q)(C,\theta,\beta)$ transform of $\overline{s}_v(x)$ by $\overline{(EC)_n^{q;\theta,\beta}}$, we have

$$\begin{aligned}
&\overline{(EC)_n^{q;\theta,\beta}}(\overline{f}(x)) - \overline{f}(x) = \\
&\quad = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\int_0^\pi \frac{\phi_x(t)}{\pi A_k^{\theta+\beta}}\sum_{v=0}^{k}\frac{A_{k-v}^{\theta-1}A_v^{\beta}\cos\left(v+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}dt \\
&\quad = \int_0^\pi \phi_x(t)\widetilde{D}_n^{q;\theta,\beta}(t)dt \\
&\quad = \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right)\phi_x(t)\widetilde{D}_n^{q;\theta,\beta}(t)dt \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{3.2}$$

We apply Lemma 2.1 in order to estimate I_1:

$$\begin{aligned}
|I_1| &\leq \int_0^{\frac{1}{n+1}} |\phi_x(t)||\widetilde{D}_n^{q;\theta,\beta}(t)|dt \\
&= \mathcal{O}\left(\int_0^{\frac{1}{n+1}} t^{\alpha-1}dt\right) = \mathcal{O}\left(\frac{1}{(n+1)^\alpha}\right).
\end{aligned} \tag{3.3}$$

Also, applying again Lemma 2.1, we have

$$\begin{aligned}
|I_2| &\leq \int_{\frac{1}{n+1}}^{\pi} |\phi_x(t)||\widetilde{D}_n^{q;\theta,\beta}(t)|dt \\
&= \mathcal{O}\left(\int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1}dt\right) = \mathcal{O}\left(\frac{1}{(n+1)^\alpha}\right).
\end{aligned} \tag{3.4}$$

Based on (3.3), (3.4), and (3.2), the required estimation is an immediate result. The proof of the theorem is completed. □

The following result gives the degree of approximation of conjugate functions with respect to L_p-norm, $1 \leq p < \infty$.

Theorem 3.2. *If $\overline{f}$, conjugate to a 2π periodic function f, belongs to $W(L_p, \xi(t))$ class, then its degree of approximation by $(E,q)(C,\theta,\beta)$ means of conjugate Fourier series is given by*

$$\|\overline{(EC)_n^{q;\theta,\beta}}(\overline{f}) - \overline{f}\|_p = \mathcal{O}\left((n+1)^{\gamma+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right)$$

provided that $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \quad \textit{is a decreasing sequence,} \tag{3.5}$$

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{|\phi_x(t)|}{\xi(t)}\right)^p \sin^{\gamma p}\frac{t}{2}dt\right\}^{1/p} = \mathcal{O}\left(\frac{1}{(n+1)^{1/p}}\right), \tag{3.6}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi_x(t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = \mathcal{O}\left((n+1)^{\delta}\right) \tag{3.7}$$

where δ *is an arbitrary number such that* $s(1-\delta)-1>0$, $1/p+1/s=1$, $1\le p<\infty$, *conditions (3.6) and (3.7) hold uniformly in* x *and* $(EC)_n^{q;\theta,\beta}$ *are* $(E,q)(C_n^{\theta,\beta})$ *means of the series (1.1), and*

$$\overline{f}(x) = -\frac{1}{2\pi}\int_0^{2\pi}\phi_x(t)\cot\left(\frac{t}{2}\right)dt.$$

Proof. We shall use the equality

$$\overline{(EC)_n^{q;\theta,\beta}}(\overline{f}(x)) - \overline{f}(x) = \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right)\phi_x(t)\widetilde{D}_n^{q;\theta,\beta}(t)dt =: J_1 + J_2, \tag{3.8}$$

obtained earlier in the proof of theorem 3.1.

Moreover, using Hölder's inequality and the fact that $\phi\in W(L_p,\xi(t))$, condition (3.6), $\sin t\ge\frac{2t}{\pi}$, Lemma 2.1, and second mean value theorem for integrals, we have

$$\begin{aligned}
|J_1| &\le \left\{\int_0^{\frac{1}{n+1}}\left(\frac{|\phi(t)|}{\xi(t)}\right)^p \sin^{\gamma p}\frac{t}{2}dt\right\}^{1/p}\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)|\widetilde{D}_n^{q;\theta,\beta}(t)|}{\sin^{\gamma}\frac{t}{2}}\right)^s dt\right\}^{1/s}\\
&= \mathcal{O}\left(\frac{1}{(n+1)^{1/p}}\right)\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{t^{1+\gamma}}\right)^s dt\right\}^{1/s}\\
&= \mathcal{O}\left(\frac{1}{(n+1)^{1/p}}\xi\left(\frac{1}{n+1}\right)\right)\left\{\int_{\varepsilon}^{\frac{1}{n+1}}\frac{dt}{t^{(1+\gamma)s}}dt\right\}^{1/s}, \qquad \left(0<\varepsilon<\frac{1}{n+1}\right)\\
&= \mathcal{O}\left((n+1)^{-\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)(n+1)^{1+\gamma}\right)\\
&= \mathcal{O}\left((n+1)^{\gamma}\,\xi\left(\frac{1}{n+1}\right)\right), \quad \text{because of} \quad 1/p+1/s=1.
\end{aligned} \tag{3.9}$$

Again, using Hölder's inequality, $|\sin t| \le 1$, $\sin t \ge \frac{2t}{\pi}$, conditions (3.5) and (3.7), Lemma 2.1, and second mean value theorem for integrals, we obtain

$$\begin{aligned}
|J_2| &\le \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p \sin^{\gamma p}\frac{t}{2}dt\right\}^{1/p}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)|\widetilde{D}_n^{q;\theta,\beta}(t)|}{t^{-\delta}\sin^{\gamma}\frac{t}{2}}\right)^s dt\right\}^{1/s}\\
&= \mathcal{O}\left((n+1)^{\delta}\right)\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{\gamma+1-\delta}}\right)^s dt\right\}^{1/s}\\
&= \mathcal{O}\left((n+1)^{\delta}\right)\left\{\int_{\frac{1}{\pi}}^{n+1}\left(\frac{\xi(1/u)}{u^{\delta-1-\gamma}}\right)^s \frac{du}{u^2}\right\}^{1/s}\\
&= \mathcal{O}\left((n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right)\left\{\int_{\frac{1}{\pi}}^{n+1}\frac{du}{u^{s(\delta-1-\gamma)+2}}\right\}^{1/s}\\
&= \mathcal{O}\left((n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right)\left\{\frac{(n+1)^{s(\gamma+1-\delta)-1}-\pi^{s(\delta-1-\gamma)+1}}{s(\gamma+1-\delta)-1}\right\}^{1/s}\\
&= \mathcal{O}\left((n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right)\left\{(n+1)^{\gamma+1-\delta-1/s}\right\}\\
&= \mathcal{O}\left((n+1)^{\gamma+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right), \quad \text{where} \quad 1/p+1/s=1. \qquad (3.10)
\end{aligned}$$

Inserting (3.9) and (3.10) into (3.8), we obtain

$$\left|\overline{(EC)_n^{q;\theta,\beta}}(\overline{f}(x))-\overline{f}(x)\right| = \mathcal{O}\left((n+1)^{\gamma+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right),$$

and whence,

$$\|\overline{(EC)_n^{q;\theta,\beta}}(\overline{f})-\overline{f}\|_p = \mathcal{O}\left((n+1)^{\gamma+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right).$$

The proof is completed. □

4. Corollaries

In this section we give some direct consequences of the main results. First it is clear that $(E,q)(C,\theta,\beta)$ means can be reduced to the following means:

1. If $\beta=0$ then we obtain $(E,q)(C,\theta,\beta)\equiv(E,q)(C,\theta,0)\equiv(E,q)(C,\theta)$ means.
2. If $\theta=1$ then we obtain $(E,q)(C,\theta,\beta)\equiv(E,q)(C,1,\beta)$ means.
3. If $\beta=0$, $q=1$ then we obtain $(E,q)(C,\theta,\beta)\equiv(E,1)(C,\theta,0)\equiv(E,1)(C,\theta)$ means.
4. If $\theta=1$, $q=1$ then we obtain $(E,q)(C,\theta,\beta)\equiv(E,1)(C,1,\beta)$ means.
5. If $\theta=q=1$, $\beta=0$ then we obtain $(E,q)(C,\theta,\beta)\equiv(E,1)(C,1)$ means.

Denoting $(E,q)(C,\theta)$, $(E,q)(C,1,\beta)$, $(E,1)(C,\theta)$, $(E,1)(C,1,\beta)$, $(E,1)(C,1)$ means of $\overline{s}_n(f;x)$, respectively, by $\overline{(EC)}_n^{(q;\theta,0)}(f;x)$, $\overline{(EC)}_n^{(q;1,\beta)}(f;x)$, $\overline{(EC)}_n^{(1;\theta,0)}(f;x)$, $\overline{(EC)}_n^{(1;1,\beta)}(f;x)$ and $\overline{(EC)}_n^{(1;1,0)}(f;x)$, then from theorems 3.1 and 3.2 lots of corollaries can be derived.

We shall formulate below only some of them.

Corollary 4.1 ([10]). *If $\theta = q = 1$, $\beta = 0$ and all conditions of Theorem 3.1 are satisfied, then*

$$\|\overline{(EC)_n^{1;1,0}}(\overline{f}) - \overline{f}\|_\infty = \mathcal{O}\left(\frac{1}{(n+1)^\alpha}\right), \quad 0 < \alpha < 1.$$

Corollary 4.2 ([10]). *If $\theta = q = 1$, $\beta = 0$ and all conditions of Theorem 3.2 are satisfied (with corrected condition (3.6)), then*

$$\|\overline{(EC)_n^{1;1,0}}(\overline{f}) - \overline{f}\|_p = \mathcal{O}\left((n+1)^{\gamma+\frac{1}{p}}\ \xi\left(\frac{1}{n+1}\right)\right).$$

Corollary 4.3 ([10]). *If $\gamma = \beta = 0$, $\theta = q = 1$, and $\xi(t) = t^\alpha$, then the degree of approximation of a function $\overline{f}$, conjugate to a 2π-periodic function $f \in Lip(\alpha, p)$, $1/p \le \alpha \le 1$, is given by*

$$\|\overline{(EC)_n^{1;1,0}}(\overline{f}) - \overline{f}\|_p = \mathcal{O}\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right).$$

Corollary 4.4 ([10]). *Let $\gamma = \beta = 0$, $\theta = q = 1$, and $\xi(t) = t^\alpha$. If $p \to \infty$ in Corollary 4.3, then $f \in Lip(\alpha, p)$ reduces to $Lip\alpha$ for $0 < \alpha < 1$, and we have*

$$\|\overline{(EC)_n^{1;1,0}}(\overline{f}) - \overline{f}\|_\infty = \mathcal{O}\left(\frac{1}{(n+1)^\alpha}\right).$$

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