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PERIODIC SOLUTIONS FOR THIRD-ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT. In this paper, the following third-order nonlinear delay differential equation with periodic coefficients

$$\begin{aligned} &x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) \\ &= f\left(t, x\left(t\right), x(t-\tau(t))\right) + \frac{d}{dt} g\left(t, x\left(t - \tau\left(t\right)\right)\right), \end{aligned}$$

is considered. By employing Green's function, Krasnoselskii's fixed point theorem and the contraction mapping principle, we state and prove the existence and uniqueness of periodic solutions to the third-order nonlinear delay differential equation.

1. Introduction

Third order differential equations arise from in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows and so on [19, 23].

Delay differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, see the monographs [8, 20] and the papers [1–18], [21–23], [25–28] and the references therein.

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The second order nonlinear delay differential equation with periodic coefficients

$$x''(t)+p(t)x'(t)+q(t)x(t)=f(t,x(t),x(t-\tau(t)))+\frac{d}{dt}g(t,x(t-\tau(t))),$$

has been investigated in [5]. By using Krasnoselskii's fixed point theorem and the contraction mapping principle, Ardjouni and Djoudi obtained existence and uniqueness of periodic solutions.

In [23], Ren, Siegmund and Chen discussed the existence of positive periodic solutions for the third-order differential equation

$$x'''(t)+p(t)x''(t)+q(t)x'(t)+r(t)x(t)=g(t,x(t)).$$

By employing the fixed point index, the authors obtained existence results for positive periodic solutions.

Inspired and motivated by the works mentioned above and the papers [1–18], [21–23], [25–28] and the references therein, we concentrate on the existence of periodic solutions for the third-order nonlinear delay differential equation

$$\begin{aligned}&x'''(t)+p(t)x''(t)+q(t)x'(t)+r(t)x(t)\\&=f(t,x(t),x(t-\tau(t)))+\frac{d}{dt}g(t,x(t-\tau(t))),\end{aligned}\tag{1.1}$$

where p, q, r, τ are continuous real-valued functions. The functions $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous in their respective arguments. To show the existence of periodic solutions, we transform (1.1) into an integral equation and then use Krasnoselskii's fixed point theorem. The obtained integral equation splits in the sum of two mappings, one is a contraction and the other is compact. We also obtain the existence of a unique periodic solution of (1.1) by employing the contraction mapping principle as the basic mathematical tool.

The organization of this paper is as follows. In Section 2, we introduce some notations and lemmas, and state some preliminary results needed in later section. Then we give the Green's function of (1.1) which plays an important role in this paper. In Section 3, we present our main results on existence and uniqueness.

We state Krasnoselskii's fixed point theorem which enables us to prove the existence of periodic solutions to(1.1). For its proof we refer the reader to [24].

Theorem 1.1 (Krasnoselskii)**.** *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B},\|.\|)$. Suppose that H_1 and H_2 map $\mathbb{M}$ into $\mathbb{B}$ such that*
(i) $x,y\in\mathbb{M}$, implies $H_1x+H_2y\in\mathbb{M}$,
(ii) H_1 is compact and continuous,
(iii) H_2 is a contraction mapping.
Then there exists $z\in\mathbb{M}$ with $z=H_1z+H_2z$.

In this paper, we give the assumptions as follows that will be used in the main results.

(h1) There exist differentiable positive T-periodic functions a_1 and a_2 and a positive real constant ρ such that

$$\begin{cases}a_1(t)+\rho=p(t),\\ a_1'(t)+a_2(t)+\rho a_1(t)=q(t),\\ a_2'(t)+\rho a_2(t)=r(t).\end{cases}$$

(h2) $p, q, r, \tau \in C(\mathbb{R}, \mathbb{R}^+)$ are T-periodic functions with $\tau(t) \geq \tau^* > 0$ and

$$\int_0^T p(s)ds > \rho T, \quad \int_0^T q(s)ds > 0.$$

(h3) The functions $g(t,x)$ and $f(t,x,y)$ are continuous T-periodic in t and globally Lipshitz continuous in x and in x and y, respectively. That is

$$g(t+T,x) = g(t,x), \; f(t+T,x,y) = f(t,x,y),$$

and there are positive constants k_1, k_2 and k_3 such that

$$|g(t,x) - g(t,y)| \leq k_1 |x-y|,$$

and

$$|f(t,x,y) - f(t,z,w)| \leq k_1 |x-z| + k_2 |y-w|.$$

2. Green's function of third-order differential equation

For $T > 0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

We consider

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = h(t), \tag{2.1}$$

where h is a continuous T-periodic function. Obviously, by the condition $(h1)$, (2.1) is transformed into

$$\begin{cases} y'(t) + \rho y(t) = h(t), \\ x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t). \end{cases}$$

Lemma 2.1 ([3]). *If $y, h \in P_T$, then y is a solution of equation*

$$y'(t) + \rho y(t) = h(t),$$

if and only if

$$y(t) = \int_t^{t+T} G_1(t,s)h(s)ds, \tag{2.2}$$

where

$$G_1(t,s) = \frac{\exp(\rho(s-t))}{\exp(\rho T) - 1}. \tag{2.3}$$

Corollary 2.2. *Green function G_1 satisfies the following properties*

$$G_1(t+T, s+T) = G_1(t,s), \; G_1(t, t+T) = G_1(t,t)\exp(\rho T),$$
$$G_1(t+T, s) = G_1(t,s)\exp(-\rho T), \; G_1(t, s+T) = G_1(t,s)\exp(\rho T),$$
$$\frac{\partial}{\partial t}G_1(t,s) = -\rho G_1(t,s),$$
$$\frac{\partial}{\partial s}G_1(t,s) = \rho G_1(t,s),$$

and

$$m_1 \le G_1(t,s) \le M_1,$$

where

$$m_1 = \frac{1}{\exp(\rho T) - 1}, \quad M_1 = \frac{\exp(\rho T)}{\exp(\rho T) - 1}.$$

Lemma 2.3 ([22]). *Suppose that (h1) and (h2) hold and*

$$\frac{R_1\left[\exp\left(\int_0^T a_1(v)dv\right) - 1\right]}{Q_1 T} \ge 1, \tag{2.4}$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\exp\left(\int_0^T a_1(v)dv\right)}{\exp\left(\int_0^T a_1(v)dv\right) - 1} a_2(s)\, ds \right|,$$
$$Q_1 = \left(1 + \exp\left(\int_0^T a_1(v)dv\right)\right)^2 R_1^2.$$

Then there are continuous T-periodic functions a and b such that

$$b(t) > 0, \quad \int_0^T a(v)dv > 0,$$

and

$$a(t) + b(t) = a_1(t), \ b'(t) + a(t)b(t) = a_2(t), \ \text{for } t \in \mathbb{R}.$$

Lemma 2.4 ([26]). *Suppose the conditions of Lemma 2.3 hold and $y \in P_T$. Then the equation*

$$x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int_t^{t+T} G_2(t,s)y(s)ds, \tag{2.5}$$

where

$$G_2(t,s) = \frac{\int_t^s \exp\left[\int_t^v b(u)du + \int_v^s a(u)du\right]dv + \int_s^{t+T} \exp\left[\int_t^v b(u)du + \int_v^{s+T} a(u)du\right]dv}{\left[\exp\left(\int_0^T a(v)dv\right) - 1\right]\left[\exp\left(\int_0^T b(v)dv\right) - 1\right]}. \tag{2.6}$$

Corollary 2.5. *Green's function G_2 satisfies the following proprieties*

$$G_2(t+T,s+T)=G_2(t,s),\ G_2(t,t+T)=G_2(t,t),$$
$$G_2(t+T,s)=\exp\left(-\int_0^T b(v)dv\right)\left[G_2(t,s)+\int_t^{t+T}E(t,u)F(u,s)\,du\right],$$
$$\frac{\partial}{\partial t}G_2(t,s)=-b(t)G_2(t,s)+F(t,s),$$
$$\frac{\partial}{\partial s}G_2(t,s)=a(t)G_2(t,s)-E(t,s),$$

where

$$E(t,s)=\frac{\exp\left(\int_t^s b(v)dv\right)}{\exp\left(\int_0^T b(v)dv\right)-1},\ F(t,s)=\frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right)-1}.$$

Lemma 2.6 ([22]). *Let $A=\int_0^T a_1(v)dv$ and $B=T^2\exp\left(\frac{1}{T}\int_0^T \ln(a_2(v))\,dv\right)$. If*

$$A^2\geq 4B, \tag{2.7}$$

then

$$\min\left\{\int_0^T a(v)dv,\ \int_0^T b(v)dv\right\}\geq\frac{1}{2}\left(A-\sqrt{A^2-4B}\right)=l,$$
$$\max\left\{\int_0^T a(v)dv,\ \int_0^T b(v)dv\right\}\leq\frac{1}{2}\left(A+\sqrt{A^2-4B}\right)=L.$$

Corollary 2.7. *Functions G_2, E and F satisfy*

$$m_2\leq G_2(t,s)\leq M_2,$$
$$E(t,s)\leq\frac{e^L}{e^l-1},$$
$$F(t,s)\leq e^L,$$

where

$$m_2=\frac{T}{\left(\exp(L)-1\right)^2},\ M_2=\frac{T\exp\left(\int_0^T a_1(v)\,dv\right)}{\left(\exp(l)-1\right)^2}.$$

Lemma 2.8 ([11]). *Suppose the conditions of Lemma 2.3 hold and $h\in P_T$. Then the equation*

$$x'''(t)+p(t)x''(t)+q(t)x'(t)+r(t)x(t)=h(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed by

$$x(t)=\int_t^{t+T}G(t,s)h(s)ds, \tag{2.8}$$

where

$$G(t,s)=\int_t^{t+T}G_2(t,\sigma)\,G_1(\sigma,s)\,d\sigma. \tag{2.9}$$

Corollary 2.9. *Green's function G satisfies the following properties*

$$G(t+T,s+T)=G(t,s),\ \ G(t,t+T)=G(t,t)\exp(\rho T),$$

$$\frac{\partial}{\partial t}G(t,s)=(\exp(-\rho T)-1)\,G_1(t,t)\,G_2(t,s)-b(t)\,G(t,s)$$
$$+\int_t^{t+T}F(t,\sigma)G_1(\sigma,s)\,d\sigma,$$

$$\frac{\partial}{\partial s}G(t,s)=\rho G(t,s),$$

and

$$m\le G(t,s)\le M,$$

where

$$m=\frac{T^2}{(\exp(l)-1)^2(\exp(\rho T)-1)},\ \ M=\frac{T^2\exp\left(\rho T+\int_0^T a(v)\,dv\right)}{(\exp(l)-1)^2(\exp(\rho T)-1)}.$$

3. Main Results

In this section, we will study the existence and uniqueness of periodic solutions of (1.1).

Lemma 3.1. *Suppose* $(h1)-(h3)$ *and (2.4) hold. The function $x\in P_T$ is a solution of (1.1) if and only if*

$$x(t)=(\exp(\rho T)-1)\,G(t,t)\,g(t,x(t-\tau(t)))$$
$$+\int_t^{t+T}G(t,s)\left\{-\rho g(s,x(s-\tau(s)))+f(s,x(s),x(s-\tau(s)))\right\}ds. \quad (3.1)$$

Proof. Let $x\in P_T$ be a solution of (1.1). From Lemma 2.8, we have

$$x(t)=\int_t^{t+T}G(t,s)\left[f(s,x(s),x(s-\tau(s)))+\frac{\partial}{\partial s}g(s,x(s-\tau(s)))\right]ds$$
$$=\int_t^{t+T}G(t,s)\,f(s,x(s),x(s-\tau(s)))\,ds$$
$$+\int_t^{t+T}G(t,s)\,\frac{\partial}{\partial s}g(s,x(s-\tau(s)))\,ds. \quad (3.2)$$

Performing an integration by parts, we get

$$\int_t^{t+T}G(t,s)\,\frac{\partial}{\partial s}g(s,x(s-\tau(s)))\,ds$$
$$=G(t,s)\,g(s,x(s-\tau(s)))|_t^{t+T}-\int_t^{t+T}\left[\frac{\partial}{\partial s}G(t,s)\right]g(s,x(s-\tau(s)))\,ds$$
$$=(\exp(\rho T)-1)\,G(t,t)\,g(t,x(t-\tau(t)))-\rho\int_t^{t+T}G(t,s)\,g(s,x(s-\tau(s)))\,ds.$$
$$(3.3)$$

We obtain (3.1) by substituting (3.3) in (3.2). Since each step is reversible, the converse follows easily. This completes the proof. □

Define the mapping $H : P_T \to P_T$ by

$$\begin{aligned}&\left(H\varphi\right)(t)\\&=\int_t^{t+T} G(t,s)\left\{-\rho g\left(s,\varphi\left(s-\tau\left(s\right)\right)\right)+f\left(s,\varphi\left(s\right),\varphi\left(s-\tau\left(s\right)\right)\right)\right\}ds\\&+\left(\exp\left(\rho T\right)-1\right)G\left(t,t\right)g\left(t,\varphi\left(t-\tau\left(t\right)\right)\right).\end{aligned} \tag{3.4}$$

Note that to apply Krasnoselskii's fixed point theorem we need to construct two mappings, one is a contraction and the other is compact. Therefore, we express (3.4) as

$$\left(H\varphi\right)(t)=\left(H_1\varphi\right)(t)+\left(H_2\varphi\right)(t),$$

where $H_1, H_2 : P_T \to P_T$ are given by

$$\left(H_1\varphi\right)(t)=\int_t^{t+T} G(t,s)\left\{-\rho g\left(s,\varphi\left(s-\tau\left(s\right)\right)\right)+f\left(s,\varphi\left(s\right),\varphi\left(s-\tau\left(s\right)\right)\right)\right\}ds, \tag{3.5}$$

and

$$\left(H_2\varphi\right)(t)=\left(\exp\left(\rho T\right)-1\right)G\left(t,t\right)g\left(t,\varphi\left(t-\tau\left(t\right)\right)\right). \tag{3.6}$$

To simplify notation, we introduce the constants

$$\beta=\max_{t\in[0,T]}\left\{b(t)\right\},\ \delta=\frac{\exp\left(L\right)}{\exp\left(l\right)-1}. \tag{3.7}$$

Lemma 3.2. *Suppose* $(h1)-(h3)$, *(2.4) and (2.7) hold. Then* $H_1 : P_T \to P_T$ *is compact.*

Proof. Let H_1 be defined by (3.5). Obviously, $H_1\varphi$ is continuous and it is easy to show that $\left(H_1\varphi\right)(t+T)=\left(H_1\varphi\right)(t)$. To see that H_1 is continuous, we let $\varphi,\psi\in P_T$. Given $\varepsilon>0$, take $\theta=\varepsilon/N$ with $N=MT\left(\rho k_1+k_2+k_3\right)$ where k_1, k_2 and k_3 are given by $(h3)$. Now, for $\|\varphi-\psi\|<\theta$, we obtain

$$\|H_1\varphi-H_1\psi\|\le M\int_t^{t+T}\left[\rho k_1\|\varphi-\psi\|+\left(k_2+k_3\right)\|\varphi-\psi\|\,ds\right]\le N\|\varphi-\psi\|<\varepsilon.$$

This proves that H_1 is continuous. To show that the image of H_1 is contained in a compact set, we consider $\mathbb{D}=\{\varphi\in P_T:\|\varphi\|\le\mathfrak{L}\}$, where $\mathfrak{L}$ is a fixed positive constant. Let $\varphi_n\in\mathbb{D}$, where n is a positive integer. Observe that in view of $(h3)$ we have

$$\begin{aligned}|g(t,x)|&=|g(t,x)-g(t,0)+g(t,0)|\\&\le|g(t,x)-g(t,0)|+|g(t,0)|\\&\le k_1\|x\|+\mu_1,\end{aligned}$$

and

$$\begin{aligned}|f(t,x,y)|&=|f(t,x,y)-f(t,0,0)+f(t,0,0)|\\&\le|f(t,x,y)-f(t,0,0)|+|f(t,0,0)|\\&\le k_2\|x\|+k_3\|y\|+\mu_2,\end{aligned}$$

where $\mu_1 = \max_{t\in[0,T]} |g(t,0)|$ and $\mu_2 = \max_{t\in[0,T]} |f(t,0,0)|$. Hence if H_1 is given by (3.5) we obtain

$$\|H_1\varphi_n\| \leq D,$$

for some positive D. Next we calculate $\frac{d}{dt}(H_1\varphi_n)(t)$ and show that it is uniformly bounded. By making use of $(h1)$, $(h2)$ and $(h3)$ we obtain by taking the derivative in (3.5) that

$$\begin{aligned}
&\frac{d}{dt}(H_1\varphi_n)(t) \\
&= \int_t^{t+T} \left[(\exp(-\rho T) - 1) G_1(t,t) G_2(t,s) - b(t) G(t,s) \right. \\
&\left. + \int_t^{t+T} F(t,\sigma) G_1(\sigma,s)\, d\sigma \right] \left[-\rho g(s,\varphi(s-\tau(s))) + f(s,\varphi(s),\varphi(s-\tau(s))) \right] ds.
\end{aligned}$$

Consequently, by invoking $(h3)$ and (3.7), we obtain

$$\begin{aligned}
&\left| \frac{d}{dt}(H_1\varphi_n)(t) \right| \\
&\leq \left[(1-\exp(-\rho T)) M_1 M_2 + M\beta + M_1\delta T \right] (\rho(k_1\mathfrak{L} + \mu_1) + (k_2+k_3)\mathfrak{L} + \mu_2) T \\
&\leq K,
\end{aligned}$$

for some positive K. Hence the sequence $(H_1\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(H_1\varphi_{n_k})$ of $(H_1\varphi_n)$ converges uniformly to continuous T-periodic function. Thus H_1 is continuous and $H_1(\mathbb{D})$ is contained in a compact subset of P_T. □

Lemma 3.3. *If H_2 is given by (3.6) with*

$$k_1(\exp(\rho T) - 1) M < 1, \tag{3.8}$$

then $H_2 : P_T \to P_T$ is a contraction.

Proof. Let H_2 be defined by (3.6). It is easy to show that $(H_2\varphi)(t+T) = (H_2\varphi)(t)$. To see that H_2 is a contraction, let $\varphi, \psi \in P_T$. Then we have

$$\|H_2\varphi - H_2\psi\| = \sup_{t\in[0,T]} |(H_2\varphi)(t) - (H_2\psi)(t)| \leq k_1(\exp(\rho T) - 1) M \|\varphi - \psi\|.$$

Hence $H_2 : P_T \to P_T$ is a contraction. □

Theorem 3.4. *Suppose that conditions $(h1)-(h3)$, (2.4), (2.7) and (3.8) hold. Suppose there exists a positive constant J satisfying the inequality*

$$(\exp(\rho T) - 1) M (k_1 J + \mu_1) + (\rho(k_1 J + \mu_1) + (k_2+k_3) J + \mu_2) T \leq J.$$

Then (1.1) has a solution $x \in P_T$ such that $\|x\| \leq J$.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. By Lemma 3.2, the operator $H_1 : \mathbb{M} \to P_T$ is compact and continuous. Also, from Lemma 3.3, the operator $H_2 : \mathbb{M} \to P_T$

is a contraction. Conditions (ii) and (iii) of Krasnoselskii theorem are satisfied. We need to show that condition (i) is fulfilled. To this end, let $\varphi, \psi \in \mathbb{M}$. Then

$$
\begin{aligned}
&\left|\left(H_1\varphi\right)(t)+\left(H_2\psi\right)(t)\right| \\
&\leq M\int_t^{t+T}\left[\rho\left(k_1\|\varphi\|+\mu_1\right)+\left(k_2+k_3\right)\|\varphi\|+\mu_2\right]ds \\
&+\left(\exp\left(\rho T\right)-1\right)M\left(k_1\|\psi\|+\mu_1\right) \\
&\leq\left(\exp\left(\rho T\right)-1\right)M\left(k_1J+\mu_1\right)+\left(\rho\left(k_1J+\mu_1\right)+\left(k_2+k_3\right)J+\mu_2\right)T\leq J.
\end{aligned}
$$

Thus $\|H_1\varphi+H_2\psi\|\leq J$ and so $H_1\varphi+H_2\psi\in\mathbb{M}$. All the conditions of Krasnoselskii theorem are satisfied and consequently the operator H defined in (3.4) has a fixed point in $\mathbb{M}$. By Lemma 3.1 this fixed point is a solution of (1.1) and the proof is complete. □

Theorem 3.5. *Suppose that conditions $(h1)-(h3)$, (2.4) and (2.7) hold. If*

$$
\left(\exp\left(\rho T\right)-1\right)Mk_1+\left(\rho k_1+k_2+k_3\right)T<1,
$$

then (1.1) has a unique T-periodic solution.

Proof. Let the mapping H be given by (3.4). For $\varphi,\psi\in P_T$, we have

$$
\begin{aligned}
&\left|\left(H\varphi\right)(t)-\left(H\psi\right)(t)\right| \\
&\leq M\int_t^{t+T}\left[\rho k_1\|\varphi-\psi\|+\left(k_2+k_3\right)\|\varphi-\psi\|\right]ds+\left(\exp\left(\rho T\right)-1\right)Mk_1\|\varphi-\psi\|.
\end{aligned}
$$

Hence

$$
\|H\varphi-H\psi\|\leq\left[\left(\exp\left(\rho T\right)-1\right)Mk_1+\left(\rho k_1+k_2+k_3\right)T\right]\|\varphi-\psi\|.
$$

By the contraction mapping principle, H has a fixed point in P_T and by Lemma 3.1, this fixed point is a solution of (1.1). The proof is complete. □

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