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ON PARA-SASAKIAN MANIFOLDS SATISFYING CERTAIN CURVATURE CONDITIONS WITH CANONICAL PARACONTACT CONNECTION

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Abstract. In this article, the aim is to introduce a para-Sasakian manifold with a canonical paracontact connection. It is shown that $\varphi-$*conharmonically flat*, $\varphi-W_2$ *flat* and $\varphi-$*pseudo projectively flat* para-Sasakian manifolds with respect to canonical paracontact connection are all $\eta-$Einstein manifolds. Also, we prove that *quasi-pseudo projectively* flat para-Sasakian manifolds are of constant scalar curvatures.

1. Introduction

The notion of the almost paracontact structure on a differentiable manifold defined by I. Sato [11] (see also [12]). The structure is an analogue of the almost contact structure [5, 10], and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). Every differentiable manifold with almost paracontact structure defined by I. Sato has a compatible Riemannian metric.

An almost paracontact structure on a pseudo-Riemannian manifold M of dimension $(2n+1)$ defined by S. Kaneyuki and M. Konzai [7] and they constructed the almost paracomplex structure on $M^{2n+1} \times \mathbb{R}$. Recently, S. Zamkovoy [16] has associated the almost paracontact structure given in [7] to a pseudo-Riemannian metric of signature $(n+1, n)$ and showed that any almost paracontact structure admits such a pseudo-Riemannian metric.

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As a generalization of the well-known connection defined by N. Tanaka [13] and, independently, by S. M. Webster [15], in context of CR-geometry, Tanaka-Webster connection was introduced by S. Tanno [14]. In a paracontact metric manifold S. Zamkovoy [16] defined a canonical connection which plays the same role of the (generalized) Tanaka-Webster connection [14] in paracontact geometry (see also [1–3]). In this article, we study a canonical paracontact connection on a para-Sasakian manifold which seems to be the paracontact analogue of the (generalized) Tanaka-Webster connection.

The present paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3, we give some relations between curvature tensor (resp. Ricci tensor) with respect to canonical paracontact connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection. In Section 4, it is given that a $\varphi-$*conharmonically flat* para-Sasakian manifold with respect to canonical paracontact connection is an $\eta-$Einstein manifold. In Section 5, the goal is to examine $\varphi - W_2$ *flat* para-Sasakian manifolds. In the last section, we obtain a characterization for $\varphi-$*pseudo projectively flat* para-Sasakian manifolds.

2. Preliminaries

Let M be a differentiable manifold of dimension $2n+1$. If there exists a triple (φ, ξ, η) of a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η on M^{2n+1} which satisfies the relations [7]:

$$\varphi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \qquad \varphi\xi = 0, \tag{2.2}$$

$$\eta \circ \varphi = 0, \qquad rank(\varphi) = 2n, \tag{2.3}$$

where I denotes the identity transformation, then we say the triple (φ, ξ, η) is an *almost paracontact structure* and the manifold is an *almost paracontact manifold*.

Moreover, the tensor field φ induces an almost paracomplex structure on the paracontact distribution $D = \ker \eta$, i.e. the eigendistributions $D^{\pm}$ corresponding to the eigenvalues ± 1 of φ are both n-dimensional.

If an almost paracontact manifold M with an almost paracontact structure (φ, ξ, η) admits a pseudo-Riemannian metric g such that [16]

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \qquad X,\, Y \in \Gamma(TM), \tag{2.4}$$

then we say that M is an *almost paracontact metric manifold* with an *almost paracontact metric structure* (φ, ξ, η, g) and such metric g is called *compatible metric*. Any compatible metric g is necessarily of signature $(n+1, n)$.

From (2.4), one can see that [16]

$$g(X, \varphi Y) = -g(\varphi X, Y), \tag{2.5}$$

$$g(X, \xi) = \eta(X), \tag{2.6}$$

for any $X,\, Y \in \Gamma(TM)$.

The fundamental 2-form of M is defined by

$$\Phi(X,Y) = g(X, \varphi Y).$$

An almost paracontact metric structure becomes a paracontact metric structure [16] if $g(X,\varphi Y)=d\eta(X,Y)$, for all vector fields X,Y, where $d\eta(X,Y)=\frac{1}{2}\{X\eta(Y)-Y\eta(X)-\eta([X,Y])\}$.

For a $(2n+1)$ dimensional manifold M with the structure (φ,ξ,η,g), one can also construct a local orthonormal basis which is called a $\varphi-$*basis* $(X_i,\varphi X_i,\xi)$, $(i=1,2,...,n)$ [16].

An almost paracontact metric structure (φ,ξ,η,g) on M is a *para-Sasakian manifold* if and only if [16]

$$(\nabla_X\varphi)Y=-g(X,Y)\xi+\eta(Y)X, \tag{2.7}$$

where $X,Y\in\Gamma(TM)$ and ∇ is Levi-Civita connection of M.

From (2.7), it can be seen that

$$\nabla_X\xi=-\varphi X. \tag{2.8}$$

Example 2.1 ([4]). Let $M=\mathbb{R}^{2n+1}$ be the $(2n+1)-$dimensional real number space with $(x_1,y_1,x_2,y_2,...,x_n,y_n,z)$ standard coordinate system. Defining

$$\begin{aligned}
\varphi\frac{\partial}{\partial x_\alpha} &= \frac{\partial}{\partial y_\alpha}, \qquad \varphi\frac{\partial}{\partial y_\alpha}=\frac{\partial}{\partial x_\alpha}, \qquad \varphi\frac{\partial}{\partial z}=0,\\
\xi &= \frac{\partial}{\partial z}, \qquad \eta=dz,\\
g &= \eta\otimes\eta+\sum_{\alpha=1}^{n}dx_\alpha\otimes dx_\alpha-\sum_{\alpha=1}^{n}dy_\alpha\otimes dy_\alpha,
\end{aligned}$$

where $\alpha=1,2,...,n$, then the set (M,φ,ξ,η,g) is an almost paracontact metric manifold.

In a para-Sasakian manifold M, the following relations hold [16]:

$$\begin{aligned}
g(R(X,Y)Z,\xi) &= \eta(R(X,Y)Z)\\
&= g(X,Z)\eta(Y)-g(Y,Z)\eta(X), \qquad (2.9)
\end{aligned}$$

$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X, \tag{2.10}$$

$$R(\xi,X)Y=-g(X,Y)\xi+\eta(Y)X, \tag{2.11}$$

$$R(\xi,X)\xi=X-\eta(X)\xi, \tag{2.12}$$

$$S(X,\xi)=-2n\eta(X), \tag{2.13}$$

for any vector fields $X,Y,Z\in\Gamma(TM)$. Here, R is Riemannian curvature tensor and S is Ricci tensor defined by $S(X,Y)=g(QX,Y)$, where Q is Ricci operator.

Now we consider the connection $\bar{\nabla}$ defined by [14],

$$\bar{\nabla}_XY=\nabla_XY+\eta(X)\varphi Y-\eta(Y)\nabla_X\xi+(\nabla_X\eta)Y\cdot\xi, \tag{2.14}$$

where $X,Y\in\Gamma(TM)$ and ∇ denotes Levi-Civita connection on M.

In view of (2.8) in (2.14), we arrive at

$$\bar{\nabla}_XY=\nabla_XY+\eta(X)\varphi Y+\eta(Y)\varphi X+g(X,\varphi Y)\xi. \tag{2.15}$$

Definition 2.2. On a para-Sasakian manifold, the connection $\bar{\nabla}$ given by (2.15) is called a canonical paracontact connection.

On a para-Sasakian manifold, canonical paracontact connection $\bar{\nabla}$ has the following properties:

$$\begin{aligned} \bar{\nabla}\eta &= 0, \quad \bar{\nabla}g = 0, \quad \bar{\nabla}\xi = 0, && (2.16)\\ (\bar{\nabla}_X\varphi)Y &= (\nabla_X\varphi)Y + g(X,Y)\xi - \eta(Y)X. && (2.17) \end{aligned}$$

3. Curvature Tensor

The curvature tensor $\bar{R}$ of a para-Sasakian manifold M with respect to the canonical paracontact connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{3.1}$$

If we use equation (2.15) in (3.1) we get

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + 2g(X,\varphi Y)\varphi Z \\ &\quad +g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X, \end{aligned} \tag{3.2}$$

where R is the curvature tensor of M with respect to Levi-Civita connection ∇.

Assume that T and $\bar{T}$ are curvature tensors of type $(0,4)$ defined by

$$T(X,Y,Z,W) = g(R(X,Y)Z,W),$$

and

$$\bar{T}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W),$$

respectively.

Theorem 3.1. *In a para-Sasakian manifold the following relations hold:*

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0, \tag{3.3}$$

$$\bar{T}(X,Y,Z,W) + \bar{T}(Y,X,Z,W) = 0, \tag{3.4}$$

$$\bar{T}(X,Y,Z,W) + \bar{T}(X,Y,W,Z) = 0, \tag{3.5}$$

$$\bar{T}(X,Y,Z,W) - \bar{T}(Z,W,X,Y) = 0. \tag{3.6}$$

Suppose that $E_i = \{e_i, \varphi e_i, \xi\}$ $(i = 1,2,...,n)$ is a local orthonormal $\varphi-$basis of a para-Sasakian manifold M. Then the Ricci tensor $\bar{S}$ and the scalar curvature $\bar{\tau}$ of M with respect to canonical paracontact connection $\bar{\nabla}$ are defined by

$$\begin{aligned} \bar{S}(X,Y) &= \sum_{i=1}^{n} g(\bar{R}(e_i,X)Y,e_i) - \sum_{i=1}^{n} g(\bar{R}(\varphi e_i,X)Y,\varphi e_i) \\ &\quad +g(\bar{R}(\xi,X)Y,\xi) \end{aligned} \tag{3.7}$$

and

$$\bar{\tau} = \sum_{j=1}^{n} \bar{S}(e_j,e_j) - \sum_{j=1}^{n} \bar{S}(\varphi e_j,\varphi e_j) + \bar{S}(\xi,\xi), \tag{3.8}$$

respectively.

Theorem 3.2. *In a para-Sasakian manifold M, the Ricci tensor $\bar{S}$ and scalar curvature $\bar{\tau}$ of canonical paracontact connection $\bar{\nabla}$ are defined by*

$$\bar{S}(X,Y) = S(X,Y) - 2g(X,Y) + (2n+2)\eta(X)\eta(Y), \tag{3.9}$$

$$\bar{\tau} = \tau - 2n, \tag{3.10}$$

where S and τ denote the Ricci tensor and scalar curvature of Levi-Civita connection ∇, respectively. Consequently, $\bar{S}$ is symmetric.

Lemma 3.3. *If M is a para-Sasakian manifold with canonical paracontact connection $\bar{\nabla}$, then*

$$g(\bar{R}(X,Y)Z,\xi) = \eta(\bar{R}(X,Y)Z) = 0, \tag{3.11}$$

$$\bar{R}(X,Y)\xi = \bar{R}(\xi,X)Y = \bar{R}(\xi,X)\xi = 0, \tag{3.12}$$

$$\bar{S}(X,\xi) = 0, \tag{3.13}$$

for all $X,Y,Z \in \Gamma(TM)$.

4. Conharmonical Curvature Tensor With Canonical Paracontact Connection

The conharmonic curvature tensor (see [6]) of M with a canonical paracontact connection is given by;

$$\bar{K}(X,Y)V = \bar{R}(X,Y)V - \frac{1}{2n-1}\left(\begin{array}{c} \bar{S}(Y,V)X - \bar{S}(X,V)Y \\ +g(Y,V)\bar{Q}X - g(X,V)\bar{Q}Y \end{array}\right). \tag{4.1}$$

By using (3.2) and (3.9), we obtain from (4.1)

$$\begin{aligned} \bar{K}(X,Y)V \;=\; & R(X,Y)V + g(Y,V)\eta(X)\xi - g(X,V)\eta(Y)\xi \\ & +\eta(Y)\eta(V)X - \eta(X)\eta(V)Y + 2g(X,\varphi Y)\varphi V \\ & +g(X,\varphi V)\varphi Y - g(Y,\varphi V)\varphi X \\ & -\frac{1}{2n-1}\left(\begin{array}{c} S(Y,V)X - S(X,V)Y \\ -4g(Y,V)X + 4g(X,V)Y \\ +g(Y,V)QX - g(X,V)QY \\ +(2n+2)\left[\begin{array}{c} \eta(Y)\eta(V)X - \eta(X)\eta(V)Y \\ +g(Y,V)\eta(X)\xi - g(X,V)\eta(Y)\xi \end{array}\right] \end{array}\right). \end{aligned} \tag{4.2}$$

Definition 4.1. A differentiable manifold M satisfying the condition

$$\varphi^2\bar{K}(\varphi X,\varphi Y)\varphi V = 0, \tag{4.3}$$

is called $\varphi-$conharmonically flat.

It can be easily seen that $\varphi^2\bar{K}(\varphi X,\varphi Y)\varphi V = 0$ holds if and only if

$$g(\bar{K}(\varphi X,\varphi Y)\varphi V,\varphi U) = 0, \tag{4.4}$$

for any $X,Y,U,V \in \Gamma(TM)$.

In view of (4.2), $\varphi-$conharmonically flatness gives

$$\begin{aligned} & g(R(\varphi X,\varphi Y)\varphi V,\varphi U)+2g(\varphi X,Y)g(V,\varphi U) \\ & +g(\varphi X,V)g(Y,\varphi U)-g(\varphi Y,V)g(X,\varphi U) \\ = & \frac{1}{2n-1}\left(\begin{array}{c} S(\varphi Y,\varphi V)g(\varphi X,\varphi U)-S(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ -2g(\varphi Y,\varphi V)g(\varphi X,\varphi U)+2g(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ +S(\varphi X,\varphi U)g(\varphi Y,\varphi V)-S(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -2g(\varphi Y,\varphi V)g(\varphi X,\varphi U)+2g(\varphi X,\varphi V)g(\varphi Y,\varphi U) \end{array}\right). \end{aligned} \tag{4.5}$$

Choosing $\{e_i, \varphi e_i, \xi\}$ as an orthonormal basis of vector fields in M, so by suitable contraction of (4.5) with respect to X and U we obtain

$$S(\varphi Y,\varphi V)-2g(\varphi Y,\varphi V)=\frac{1}{2n-1}\left(\begin{array}{c} (2n-2)S(\varphi Y,\varphi V) \\ +(\tau+4-6n)g(\varphi Y,\varphi V) \end{array}\right),$$

for any vector fields Y and V on M. From above equation, we get

$$S(Y,V)=(2n-\tau-2)g(Y,V)+(\tau-4n+2)\eta(Y)\eta(V),$$

which implies that M is an $\eta-$Einstein manifold. This leads us to state the following:

Theorem 4.2. *Let M be a $\varphi-$conharmonically flat para-Sasakian manifold with respect to canonical paracontact connection. Then M is an $\eta-$Einstein manifold.*

Definition 4.3. A differentiable manifold M satisfying the condition

$$g(\bar{K}(X,Y)V,\varphi U)=0, \tag{4.6}$$

is called quasi-conharmonically flat.

From (4.2) we can write

$$\begin{aligned} & g(R(X,Y)V,\varphi U)+g(X,\varphi U)\eta(Y)\eta(V) \\ & -g(Y,\varphi U)\eta(X)\eta(V)+2g(X,\varphi Y)g(\varphi V,\varphi U) \\ & +g(X,\varphi V)g(\varphi Y,\varphi U)-g(Y,\varphi V)g(\varphi X,\varphi U) \\ = & \frac{1}{2n-1}\left(\begin{array}{c} S(Y,V)g(X,\varphi U)-S(X,V)g(Y,\varphi U) \\ -4g(Y,V)g(X,\varphi U)+4g(X,V)g(Y,\varphi U) \\ +S(X,\varphi U)g(Y,V)-S(Y,\varphi U)g(X,V) \\ +(2n+2)\left[\begin{array}{c} g(X,\varphi U)\eta(Y)\eta(V) \\ -g(Y,\varphi U)\eta(X)\eta(V) \end{array}\right] \end{array}\right). \end{aligned} \tag{4.7}$$

Putting $Y=V=\xi$ in (4.7) and by using (2.1) we get

$$g(R(X,\xi)\xi,\varphi U)+g(X,\varphi U)=\frac{1}{2n-1}\left(S(X,\varphi U)-2g(X,\varphi U)\right), \tag{4.8}$$

for any vector fields X and U on M. In view of (2.12) in (4.8), we obtain

$$S(X,\varphi U)=-2g(X,\varphi U).$$

Replacing U by φU in the above equation, we have

$$S(X,U)=-2g(X,U)+(2-2n)\eta(X)\eta(U).$$

Therefore we conclude the following:

Theorem 4.4. *Let M be a quasi-conharmonically flat para-Sasakian manifold with respect to canonical paracontact connection. Then M is an $\eta-$Einstein manifold.*

5. W_2-Curvature Tensor With Canonical Paracontact Connection

In [8] Pokhariyal and Mishra have introduced new tensor field, called W_2 and $E-$tensor field, in a Riemannian manifold and studied their properties.

The curvature tensor W_2 is defined

$$W_2(X,Y,Z,V) = R(X,Y,Z,V) + \frac{1}{n-1}\left(g(X,Z)S(Y,V) - g(Y,Z)S(X,V)\right),$$

where S is a Ricci tensor of type $(0,2)$.

The $\bar{W}_2-$curvature tensor of a para-Sasakian manifold M with respect to canonical paracontact connection is defined by;

$$\bar{W}_2(X,Y)V = \bar{R}(X,Y)V - \frac{1}{2n}\left(g(Y,V)\bar{Q}X - g(X,V)\bar{Q}Y\right). \tag{5.1}$$

By using (3.2) and (3.9) from (5.1), we obtain

$$\begin{aligned}\bar{W}_2(X,Y)V &= R(X,Y)V + g(Y,V)\eta(X)\xi - g(X,V)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(V)X - \eta(X)\eta(V)Y + 2g(X,\varphi Y)\varphi V \\ &\quad +g(X,\varphi V)\varphi Y - g(Y,\varphi V)\varphi X \\ &\quad -\frac{1}{2n}\begin{pmatrix} +g(Y,V)QX - g(X,V)QY \\ -2g(Y,V)X + 2g(X,V)Y \\ +(2n+2)\begin{bmatrix} g(Y,V)\eta(X)\xi \\ -g(X,V)\eta(Y)\xi \end{bmatrix}\end{pmatrix}.\end{aligned} \tag{5.2}$$

Definition 5.1. A differentiable manifold M satisfying the condition

$$\varphi^2\bar{W}_2(\varphi X,\varphi Y)\varphi V = 0, \tag{5.3}$$

is called $\varphi-$W$_2$ flat.

It can be easily seen that $\varphi^2\bar{W}_2(\varphi X,\varphi Y)\varphi V = 0$ holds if and only if

$$g(\bar{W}_2(\varphi X,\varphi Y)\varphi V,\varphi U) = 0, \tag{5.4}$$

for any $X,Y,U,V \in \Gamma(TM)$.

In view of (5.2), we can write

$$\begin{aligned}& g(R(\varphi X,\varphi Y)\varphi V,\varphi U) + 2g(\varphi X,Y)g(V,\varphi U) \\ & +g(\varphi X,V)g(Y,\varphi U) - g(\varphi Y,V)g(X,\varphi U) \\ =\ & \frac{1}{2n}\begin{pmatrix} S(\varphi X,\varphi U)g(\varphi Y,\varphi V) - S(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -2g(\varphi Y,\varphi V)g(\varphi X,\varphi U) + 2g(\varphi X,\varphi V)g(\varphi Y,\varphi U)\end{pmatrix}.\end{aligned} \tag{5.5}$$

Choosing $\{e_i,\varphi e_i,\xi\}$ as an orthonormal basis of vector fields in M, so by suitable contraction of (5.5) with respect to X and U we obtain

$$S(\varphi Y,\varphi V) - 2g(\varphi Y,\varphi V) = \frac{1}{2n}\begin{pmatrix} -S(\varphi Y,\varphi V) \\ +(\tau+2-2n)g(\varphi Y,\varphi V)\end{pmatrix},$$

for any vector fields Y and V on M. From above equation, we get

$$S(Y,V)=-\left(\frac{\tau+2n+2}{2n+1}\right)g(Y,V)+\left(\frac{\tau-4n^2+2}{2n+1}\right)\eta(Y)\eta(V),$$

which implies that M is an $\eta-$Einstein manifold. So we have:

Theorem 5.2. *Let M be a $\varphi-W_2$ flat para-Sasakian manifold with respect to canonical paracontact connection. Then it is an $\eta-$Einstein manifold.*

Definition 5.3. A differentiable manifold M satisfying the condition

$$g(\bar{W}_2(X,Y)V,\varphi U)=0, \tag{5.6}$$

is called quasi$-\mathrm{W}_2$ flat.

From (5.2), we can write

$$\begin{aligned} & g(R(X,Y)V,\varphi U)+g(X,\varphi U)\eta(Y)\eta(V) \\ & -g(Y,\varphi U)\eta(X)\eta(V)+2g(X,\varphi Y)g(\varphi V,\varphi U) \\ & +g(X,\varphi V)g(\varphi Y,\varphi U)-g(Y,\varphi V)g(\varphi X,\varphi U) \\ = & \frac{1}{2n}\left(\begin{array}{c} S(X,\varphi U)g(Y,V)-S(Y,\varphi U)g(X,V) \\ -2g(X,\varphi U)g(Y,V)+2g(Y,\varphi U)g(X,V) \\ +(2n+2)\left[\begin{array}{c} g(X,\varphi U)\eta(Y)\eta(V) \\ -g(Y,\varphi U)\eta(X)\eta(V)\end{array}\right]\end{array}\right).\end{aligned} \tag{5.7}$$

Putting $Y=V=\xi$ in (5.7) and by using (2.1) we get

$$g(R(X,\xi)\xi,\varphi U)+g(X,\varphi U)=\frac{1}{2n}\left(S(X,\varphi U)+2ng(X,\varphi U)\right), \tag{5.8}$$

for any vector fields X and U on M.

In view of (2.12) we obtain

$$S(X,\varphi U)=-2ng(X,\varphi U).$$

Replacing U by φU in the above equation we have

$$S(X,U)=-2ng(X,U).$$

Therefore we get the following:

Theorem 5.4. *Let M be a quasi$-W_2$ flat para-Sasakian manifold with respect to canonical paracontact connection. Then M is an Einstein manifold.*

6. Pseudo-Projective Curvature Tensor With Canonical Paracontact Connection

Prasad [9] defined and studied a tensor field $\bar{P}$ on a Riemannian manifold of dimension n, which includes projective curvature tensor P. This tensor field $\bar{P}$ is known as *pseudo-projective curvature tensor*.

In this section, we study pseudo-projective curvature tensor in a para-Sasakian manifold with respect to canonical paracontact connection $\bar{\nabla}$ and we denote this curvature tensor with $\bar{P}\bar{P}$.

Let M be a para-Sasakian manifold with canonical paracontact connection. *The pseudo-projective curvature tensor* $\bar{P}\bar{P}$ of M with respect to canonical paracontact connection $\bar{\nabla}$ is defined by;

$$\begin{aligned} \bar{P}\bar{P}(X,Y)V &= a\bar{R}(X,Y)V + b\left(\bar{S}(Y,V)X - \bar{S}(X,V)Y\right) \\ &\quad - \frac{\bar{\tau}}{(2n+1)}\left(\frac{a}{2n} + b\right)\left(g(Y,V)X - g(X,V)Y\right), \end{aligned} \tag{6.1}$$

where a and b are constants such that a, $b \neq 0$. If $a = 1$ and $b = \frac{1}{2n+2}$, then (6.1) takes the form

$$\begin{aligned} \bar{P}\bar{P}(X,Y)V &= \bar{R}(X,Y)V + \frac{1}{2n+2}\left(\bar{S}(Y,V)X - \bar{S}(X,V)Y\right) \\ &\quad - \frac{\bar{\tau}}{(2n+2)n}\left(g(Y,V)X - g(X,V)Y\right). \end{aligned} \tag{6.2}$$

By using (3.2), (3.9) and (3.10), from (6.2) we obtain

$$\begin{aligned} \bar{P}\bar{P}(X,Y)V &= R(X,Y)V + g(Y,V)\eta(X)\xi - g(X,V)\eta(Y)\xi \\ &\quad + \eta(Y)\eta(V)X - \eta(X)\eta(V)Y + 2g(X,\varphi Y)\varphi V \\ &\quad + g(X,\varphi V)\varphi Y - g(Y,\varphi V)\varphi X \\ &\quad + \frac{1}{2n+2}\left(\begin{array}{c} S(Y,V)X - S(X,V)Y \\ -2g(Y,V)X + 2g(X,V)Y \\ +(2n+2)\left[\begin{array}{c} \eta(Y)\eta(V)X \\ -\eta(X)\eta(V)Y \end{array}\right] \end{array}\right) \\ &\quad - \frac{\tau - 2n}{(2n+2)n}\left(g(Y,V)X - g(X,V)Y\right). \end{aligned} \tag{6.3}$$

Definition 6.1. A differentiable manifold M satisfying the condition

$$\varphi^2\bar{P}\bar{P}(\varphi X,\varphi Y)\varphi V = 0, \tag{6.4}$$

is called $\varphi-$pseudo projectively flat.

It can be easily seen that $\varphi^2\bar{P}\bar{P}(\varphi X,\varphi Y)\varphi V = 0$ holds if and only if

$$g(\bar{P}\bar{P}(\varphi X,\varphi Y)\varphi V,\varphi U) = 0, \tag{6.5}$$

for any $X,Y,U,V \in \Gamma(TM)$.

In view of (6.2) $\varphi-$pseudo projectively flatness means

$$\begin{aligned} & g(R(\varphi X,\varphi Y)\varphi V,\varphi U) + 2g(\varphi X,Y)g(V,\varphi U) \\ & + g(\varphi X,V)g(Y,\varphi U) - g(\varphi Y,V)g(X,\varphi U) \\ = \; & -\frac{1}{2n+2}\left(\begin{array}{c} S(\varphi Y,\varphi V)g(\varphi X,\varphi U) - S(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ -2g(\varphi Y,\varphi V)g(\varphi X,\varphi U) + 2g(\varphi X,\varphi V)g(\varphi Y,\varphi U) \end{array}\right) \\ & + \frac{\tau - 2n}{n(2n+2)}\left(\begin{array}{c} g(\varphi Y,\varphi V)g(\varphi X,\varphi U) \\ -g(\varphi X,\varphi V)g(\varphi Y,\varphi U) \end{array}\right). \end{aligned} \tag{6.6}$$

Choosing $\{e_i, \varphi e_i, \xi\}$ as an orthonormal basis of vector fields in M, so by suitable contraction of (6.6) with respect to X and U we obtain

$$\begin{aligned} S(\varphi Y,\varphi V)-2g(\varphi Y,\varphi V) &= -\frac{1}{2n+2}\begin{pmatrix} (2n-1)S(\varphi Y,\varphi V) \\ +(2-4n)g(\varphi Y,\varphi V) \end{pmatrix} \\ &\quad +\frac{\tau-2n}{n(2n+2)}\left((2n-1)g(\varphi Y,\varphi V)\right), \end{aligned}$$

for any vector fields Y and V on M. From above equation, we get

$$S(Y,V)=\left(\frac{\tau-4n^2-4n-2n\tau}{4n^2+n}\right)g(Y,V)+\left(\frac{2n\tau-\tau-8n^3+2n^2+4n}{4n^2+n}\right)\eta(Y)\eta(V),$$

which implies that M is an $\eta-$Einstein manifold. Therefore we have the following:

Theorem 6.2. *Let M be a $\varphi-$pseudo projectively flat para-Sasakian manifold with respect to canonical paracontact connection. Then M is an $\eta-$Einstein manifold.*

Definition 6.3. A differentiable manifold M satisfying the condition

$$g(\bar{P}\bar{P}(X,Y)V,\varphi U)=0, \tag{6.7}$$

is called quasi-pseudo projectively flat.

From (6.3), we can write

$$\begin{aligned} & g(R(X,Y)V,\varphi U)+g(X,\varphi U)\eta(Y)\eta(V) \\ & -g(Y,\varphi U)\eta(X)\eta(V)+2g(X,\varphi Y)g(\varphi V,\varphi U) \\ & +g(X,\varphi V)g(\varphi Y,\varphi U)-g(Y,\varphi V)g(\varphi X,\varphi U) \\ = & -\frac{1}{2n+2}\begin{pmatrix} S(Y,V)g(X,\varphi U)-S(X,V)g(Y,\varphi U) \\ -2g(Y,V)g(X,\varphi U)+2g(X,V)g(Y,\varphi U) \\ +(2n+2)\begin{bmatrix} g(X,\varphi U)\eta(Y)\eta(V) \\ -g(Y,\varphi U)\eta(X)\eta(V) \end{bmatrix} \end{pmatrix} \\ & +\frac{\tau-2n}{(2n+2)n}\begin{pmatrix} g(Y,V)g(X,\varphi U) \\ -g(X,V)g(Y,\varphi U) \end{pmatrix}. \end{aligned} \tag{6.8}$$

Putting $Y=V=\xi$ in (6.8) and by using (2.1), we get

$$g(R(X,\xi)\xi,\varphi U)+g(X,\varphi U)=\frac{\tau-2n}{(2n+2)n}g(X,\varphi U), \tag{6.9}$$

for any vector fields X and U on M.

In view of (2.12) we obtain

$$\left(\frac{\tau-2n}{(2n+2)n}\right)g(X,\varphi U)=0.$$

In this case we can state following:

Theorem 6.4. *If a para-Sasakian manifold M is quasi-pseudo projectively flat with respect to canonical paracontact connection, then it is of constant scalar curvature.*

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