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**FUZZY INTERIOR IDEALS
IN ORDERED SEMIGROUPS**

(submitted by M. M. Arslanov)

ABSTRACT. In regular and in intra-regular ordered semigroups the ideals and the interior ideals coincide. In regular and in intra-regular *poe*-semigroups the ideal elements and the interior ideal elements coincide. In an attempt to show the similarity between the theory of ordered semigroups and the theory of fuzzy ordered semigroups, we prove here that in regular and in intra-regular ordered semigroups the fuzzy ideals and the fuzzy interior ideals coincide. We also prove that A is an interior ideal of an ordered semigroup S if and only if the characteristic function f_A is a fuzzy interior ideal of S . We finally introduce the concept of a fuzzy simple ordered semigroup, we prove that an ordered semigroup is simple if and only if it is fuzzy simple, and we characterize the simple ordered semigroups in terms of fuzzy interior ideals.

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1. INTRODUCTION-PREREQUISITES

In regular ordered semigroups the ideals and the interior ideals coincide. This is the case for intra-regular ordered semigroups as well: In intra-regular ordered semigroups the ideals and the interior ideals are the same. Suppose that the ordered semigroup possesses a greatest element, that is it is a *poe*-semigroup. In regular *poe*-semigroups the ideal elements and the interior ideal elements coincide. In intra-regular *poe*-semigroups the ideal elements and the interior ideal elements coincide as well [7]. An ordered semigroup S with a fuzzy subset defined on S , is called a *fuzzy ordered semigroup*. The following question is natural: What happens in case of fuzzy ordered semigroups? The theories of ordered semigroups and of fuzzy ordered semigroups are parallel to each other. In this paper we first introduce the concept of a fuzzy interior ideal in an ordered semigroup. Then, in an attempt to show the similarity between the theory of ordered semigroups and the theory of fuzzy ordered semigroups, we prove here that in regular and in intra-regular ordered semigroups the concepts of fuzzy ideals and of fuzzy interior ideals are the same concepts. Moreover we prove that for an ordered semigroup S , a set A is an interior ideal of S if and only if the characteristic function f_A is a fuzzy interior ideal of S . We introduce the concept of a fuzzy simple ordered semigroup and we prove that an ordered semigroup is simple if and only if it is fuzzy simple. Finally, we characterize the simple ordered semigroups in terms of fuzzy interior ideals. So in addition to the characterization of simple ordered semigroups by means of ideals we already have, we obtain characterizations of simple ordered semigroups in terms of fuzzy interior ideals. Fuzzy interior ideals of semigroups (without order) and fuzzy simple semigroups (without order) have been considered by Kuroki in [9].

Given an ordered semigroup S , a fuzzy subset of S (or a fuzzy set in S) is, by definition, an arbitrary mapping $f : S \rightarrow [0, 1]$ where $[0, 1]$ is the usual closed interval of real numbers. For each subset A of S , the characteristic function f_A is the fuzzy subset on S defined as follows:

$$f_A : S \rightarrow [0, 1] \mid a \rightarrow f_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

A fuzzy subset f of S is called a *fuzzy right ideal* of S if

- (1) $f(xy) \geq f(x)$ for all $x, y \in S$ and
- (2) If $x \leq y$, then $f(x) \geq f(y)$ for all $x, y \in S$.

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A fuzzy subset f of S is called a *fuzzy ideal* of S if it is both a fuzzy right and a fuzzy left ideal of S [8].

By a *poe-semigroup* we mean an ordered semigroup (*po-semigroup* [1]) S having a greatest element " e " (i.e. $e \geq a \forall a \in S$). If (S, \cdot, \leq) is an ordered semigroup, and A a subset of S , we denote by $[A]$ the subset of S defined as follows:

$$[A] := \{t \in S \mid t \leq a \text{ for some } a \in A\}.$$

An ordered semigroup (S, \cdot, \leq) is called *regular* if for each $a \in S$ there exists $x \in S$ such that $a \leq axa$. Equivalent Definitions: (1) $A \subseteq ASA$ for each $A \subseteq S$. (2) $a \in (aSa)$ for each $a \in S$ [3].

An ordered semigroup (S, \cdot, \leq) is called *intra-regular* if for each $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. Equivalent Definitions: (1) $A \subseteq SA^2S$ for each $A \subseteq S$. (2) $a \in (Sa^2S)$ for each $a \in S$ [4].

2. FUZZY INTERIOR IDEALS

We prove here that in regular and in intra-regular ordered semigroups the ideals and the interior ideals coincide.

If (S, \cdot, \leq) is an ordered semigroup, a nonempty subset A of S is called an *interior ideal* of S if

- (1) $SAS \subseteq A$ and
- (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$ [7].

Condition (2) is equivalent to the condition $[A] = A$.

Definition 2.1. Let (S, \cdot, \leq) be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy interior ideal* of S , if the following assertions are satisfied:

- (1) $f(xay) \geq f(a)$ for all $x, a, y \in S$ and
- (2) If $x \leq y$, then $f(x) \geq f(y)$.

Lemma 2.2. [8] Let (S, \cdot, \leq) is an ordered semigroup and $\emptyset \neq A \subseteq S$. Then $A = [A]$ if and only if the fuzzy subset f_A of S has the property:

$$x, y \in S, x \leq y \implies f_A(x) \geq f_A(y).$$

Proposition 2.3. Let (S, \cdot, \leq) be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is an interior ideal of S if and only if the characteristic function f_A is a fuzzy interior ideal of S .

Proof. \Rightarrow . Let $x, a, y \in S$. If $a \in A$, then $f_A(a) := 1$. Since A is an interior ideal of S , we have $xy \in SAS \subseteq A$. Since $xy \in A$, we have $f_A(xy) := 1$. Then we have $f_A(xy) \geq f_A(a)$, and condition (1) of Definition 2.1. is satisfied. If $a \notin A$, then $f_A(a) := 0$. Since $f_A(xy) \geq 0$, we have $f_A(xy) \geq f_A(a)$, and condition (1) of Definition 2.1. is satisfied. Now since A is an interior ideal of S , we have $A = (A]$. Then, by Lemma 2.2, condition (2) of Definition 2.1 holds true. Hence f_A is a fuzzy interior ideal of S .

\Leftarrow . Let $x, y \in S$, $a \in A$. Since f_A is a fuzzy interior ideal of S , we have $f_A(xy) \geq f_A(a)$. Since $a \in A$, we have $f_A(a) := 1$, so $f_A(xy) \geq 1$. Then $f_A(xy) = 1$, and $xy \in A$. Hence we have $SAS \subseteq A$. Since f_A is a fuzzy interior ideal of S , the following condition is satisfied:

$$x, y \in S, x \leq y \implies f_A(x) \geq f_A(y).$$

Then, by Lemma 2.2, we have $A = (A]$. Hence A is an interior ideal of S . \square

Proposition 2.4. *Let (S, \cdot, \leq) be an ordered semigroup and f a fuzzy ideal of S . Then f is a fuzzy interior ideal of S .*

Proof. Let $x, a, y \in S$. Since f is a fuzzy left ideal of S and $x, ay \in S$, we have $f(x(ay)) \geq f(ay)$. Since f is a fuzzy right ideal of S , we have $f(ay) \geq f(a)$. Then we have $f(xay) \geq f(a)$, and f is a fuzzy interior ideal of S . \square

Proposition 2.5. *Let (S, \cdot, \leq) be a regular ordered semigroup and f a fuzzy interior ideal of S . Then f is a fuzzy ideal of S .*

Proof. Let $x, y \in S$. Then $f(xy) \geq f(x)$. Indeed: Since S is regular and $x \in S$, there exists $z \in S$ such that $x \leq xzx$. Then we have $xy \leq (xzx)y = (xz)xy$. Then, since f is a fuzzy interior ideal of S and $xy, (xz)xy \in S$, we have $f(xy) \geq f((xz)xy)$. Again since f is a fuzzy interior ideal of S and $xz, x, y \in S$, we have $f((xz)xy) \geq f(x)$. Thus we have $f(xy) \geq f(x)$, and f is a fuzzy right ideal of S . In a similar way we prove that f is a fuzzy left ideal of S . Thus f is a fuzzy ideal of S . \square

By Propositions 2.4 and 2.5 we have the following:

Theorem 2.6. *In regular ordered semigroups the concepts of fuzzy ideals and fuzzy interior ideals coincide.*

Proposition 2.7. *Let (S, \cdot, \leq) be an intra-regular ordered semigroup and f a fuzzy interior ideal of S . Then f is a fuzzy ideal of S .*

Proof. Let $a, b \in S$. Then $f(ab) \geq f(a)$. Indeed: Since S is intra-regular and $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2y$. Then

$ab \leq (xa^2y)b$. Since f is a fuzzy interior ideal of S , we have $f(ab) \geq f(xa^2yb) = f(xa(ayb))$. Again since f is a fuzzy interior ideal of S , we have $f(xa(ayb)) \geq f(a)$. Thus we have $f(ab) \geq f(a)$, and f is a fuzzy right ideal of S . In a similar way we prove that f is fuzzy left ideal of S . Therefore f is a fuzzy ideal of S . \square

By Propositions 2.4 and 2.7 we have the following:

Theorem 2.8. *In intra-regular ordered semigroups the concepts of fuzzy ideals and fuzzy interior ideals coincide.*

3. FUZZY SIMPLE ORDERED SEMIGROUPS

In this paragraph we introduce the concept of fuzzy simple ordered semigroups, we prove that an ordered semigroup is simple if and only if it is fuzzy simple, and we characterize this type of ordered semigroups in terms of fuzzy interior ideals.

An ordered semigroup S is called *simple* if does not contain proper ideals, that is, for any ideal A of S , we have $A = S$ [5,6].

Definition 3.1. An ordered semigroup S is called *fuzzy simple* if every fuzzy ideal of S is a constant function, that is, for every fuzzy ideal f of S , we have $f(a) = f(b)$ for all $a, b \in S$.

Notation 3.2. If S is an ordered semigroup and $a \in S$, we denote by I_a the subset of S defines as follows:

$$I_a := \{b \in S \mid f(b) \geq f(a)\}.$$

Proposition 3.3. *Let (S, \cdot, \leq) be an ordered semigroup and f a fuzzy right ideal of S . Then the set I_a is a right ideal of S for every $a \in S$.*

Proof. Let $a \in S$. First of all, $\emptyset \neq I_a \subseteq S$ (since $a \in I_a$). Let $b \in I_a$ and $s \in S$. Then $bs \in I_a$. Indeed: Since f is a fuzzy right ideal of S and $b, s \in S$, we have $f(bs) \geq f(b)$. Since $b \in I(a)$, we have $f(b) \geq f(a)$. Then $f(bs) \geq f(a)$, so $bs \in I_a$. Let $b \in I_a$ and $S \ni s \leq b$. Then $s \in I_a$. Indeed: Since f is a fuzzy right ideal of S , $s, b \in S$ and $s \leq b$, we have $f(s) \geq f(b)$. Since $b \in I_a$, we have $f(b) \geq f(a)$. Then $f(s) \geq f(a)$, so $s \in I_a$. \square

In a similar way we prove the following:

Proposition 3.4. *Let (S, \cdot, \leq) be an ordered semigroup and f a fuzzy left ideal of S . Then the set I_a is a left ideal of S for every $a \in S$.*

By Propositions 3.3 and 3.4 we have the following:

Proposition 3.5. *Let (S, \cdot, \leq) be an ordered semigroup and f a fuzzy ideal of S . Then the set I_a is an ideal of S for every $a \in S$.*

Lemma 3.6. [8] *Let S be an ordered semigroup and $\emptyset \neq I \subseteq S$. Then I is an ideal of S if and only if the characteristic function f_I is a fuzzy ideal of S .*

Theorem 3.7. *An ordered semigroup (S, \cdot, \leq) is simple if and only if it is fuzzy simple.*

Proof. \implies . Let f be a fuzzy ideal of S and $a, b \in S$. Since f is a fuzzy ideal of S and $a \in S$, by Proposition 3.5, the set I_a is an ideal of S . Since S is simple, we have $I_a = S$. Then $b \in I_a$, from which $f(b) \geq f(a)$. By symmetry, we get $f(a) \geq f(b)$. Thus we have $f(a) = f(b)$ and S is fuzzy simple.

\Leftarrow . Suppose S contains proper ideals and let I be an ideal of S such that $I \neq S$. By Lemma 3.6, f_I is a fuzzy ideal of S . We have $S \subseteq I$. Indeed: Let $x \in S$. Since S is fuzzy simple, the fuzzy ideal f_I is a constant function, that is, $f_I(x) = f_I(b)$ for every $b \in S$. Let now $a \in I$ ($I \neq \emptyset$). Then we have $f_I(x) = f_I(a) := 1$, hence $x \in I$. Thus we have $S \subseteq I$, and so $S = I$. We get a contradiction. \square

Lemma 3.8. [2] *An ordered semigroup S is simple if and only if for every $a \in S$, we have $S = (SaS]$.*

Theorem 3.9. *Let (S, \cdot, \leq) be an ordered semigroup. Then S is simple if and only if every fuzzy interior ideal of S is a constant function.*

Proof. \implies . Let f be a fuzzy interior ideal of S and $a, b \in S$. Since S is simple and $b \in S$, by Lemma 3.8, we have $S = (SbS]$. Since $a \in S$, we have $a \in (SbS]$. Then there exist $x, y \in S$ such that $a \leq xby$. Since $a, xby \in S$, $a \leq xby$ and f a fuzzy interior ideal of S , we have $f(a) \geq f(xby)$. Since $x, b, y \in S$ and f is a fuzzy interior ideal of S , we have $f(xby) \geq f(b)$. Then we have $f(a) \geq f(b)$. In a similar way we prove that $f(a) \leq f(b)$. Then $f(a) = f(b)$, and f is a constant function. \Leftarrow . Let f be a fuzzy ideal of S . By Proposition 2.4, f is a fuzzy interior ideal of S . By hypothesis, f is a constant function. Then S is fuzzy simple and, by Theorem 3.7, S is simple. \square

As a consequence we have the following:

Theorem 3.10. *For an ordered semigroup S , the following are equivalent:*

- (1) S is simple.
- (2) $S = (SaS]$ for every $a \in S$.

- (3) S is fuzzy simple.
- (4) Every fuzzy interior ideal of S is a constant function.

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