

SEQUENTIAL CONVERGENCES IN A VECTOR LATTICE

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Abstract. In the present paper we deal with sequential convergences on a vector lattice L which are compatible with the structure of L .

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In this paper we will investigate the system $\text{Conv } L$ of all sequential convergences in a vector lattice L . The analogously defined notions of sequential convergences in a lattice ordered group or in a Boolean algebra were studied in [3]–[12].

The following results will be established.

The set $\text{Conv } L$ is nonempty if and only if L is archimedean. Let L be archimedean. Then $\text{Conv } L$ has the least element (it need not have, in general, a greatest element). Each interval of $\text{Conv } L$ is a Brouwerian lattice. If L is $(\aleph_0, 2)$ -distributive, then $\text{Conv } L$ is a complete lattice. There is a convex vector sublattice L_1 of L such that (i) $\text{Conv } L_1$ is a complete lattice; (ii) if L_2 is a convex vector sublattice of L such that $\text{Conv } L_2$ is a complete lattice, then $L_2 \subseteq L_1$. Let X_i ($i = 1, 2$) be archimedean vector lattices; if X_1 and X_2 are isomorphic as lattices and if $\text{Conv } X_1$ is a complete lattice, then $\text{Conv } X_2$ is a complete lattice as well. If L is a direct sum of linearly ordered vector lattices, then $\text{Conv } L$ is a complete lattice and has no atom. Some further results (concerning orthogonal sequences and strong units) are also proved.

1. PRELIMINARIES

The notion of a vector lattice is applied here in the same sense as in [1], Chap. XV. (In [16], the term “Riesz space” is used; in [13] vector lattices are called K -lineals.)

Let L be a vector lattice and let \mathbb{N} be the set of all positive integers. The direct product $\prod_{n \in \mathbb{N}} L_n$, where $L_n = L$ for each $n \in \mathbb{N}$, will be denoted by $L^{\mathbb{N}}$. The elements of $L^{\mathbb{N}}$ are denoted, e.g., as $(x_n)_{n \in \mathbb{N}}$, or simply (x_n) ; instead of n , sometimes other indices will be applied. (x_n) is said to be a sequence in L . If $x \in L$ and $x_n = x$ for each $n \in \mathbb{N}$, then we denote $(x_n) = \text{const } x$. The notion of a subsequence has the usual meaning.

If $\alpha \subseteq L^{\mathbb{N}} \times L$, then instead of $((x_n), x) \in \alpha$ we also write $x_n \rightarrow_{\alpha} x$.

If the partial order (as defined in L) is not taken into account, then we obtain a linear space which will be denoted by $\ell(L)$; similarly, if we disregard the multiplication of elements of L by reals, then we get a lattice ordered group; we denote it by $G(L)$.

The set of all reals will be denoted by \mathbb{R} . The symbol 0 denotes both the real number zero and the neutral element of L ; the meaning of this symbol will be clear from the context. For $(a_n) \in \mathbb{R}^{\mathbb{N}}$ and $a \in \mathbb{R}$ the symbol $a_n \rightarrow a$ has the usual meaning.

1.1. Definition. (Cf., e.g., [15].) A nonempty subset α of $L^{\mathbb{N}} \times L$ will be said to be a *convergence in $\ell(L)$* if it satisfies the following conditions:

- (i) If $x_n \rightarrow_{\alpha} x$ and if (y_n) is a subsequence of (x_n) , then $y_n \rightarrow_{\alpha} x$.
- (ii) If $x_n \rightarrow_{\alpha} x$ and $x_n \rightarrow_{\alpha} y$, then $x = y$.
- (iii) If $x_n \rightarrow_{\alpha} x$ and $y_n \rightarrow_{\alpha} y$, then $x_n + y_n \rightarrow_{\alpha} x + y$.
- (iv) If $x_n \rightarrow_{\alpha} x$ and $a \in \mathbb{R}$, then $ax_n \rightarrow_{\alpha} ax$.
- (v) If $x \in L$, $(a_n) \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$ and $a_n \rightarrow a$, then $a_n x \rightarrow_{\alpha} ax$.

The system of all convergences in $\ell(L)$ will be denoted by $\text{Conv}_{\ell} L$.

1.2. Definition. (Cf. [3].) A nonempty subset α of $L^{\mathbb{N}} \times L$ will be said to be a *convergence in $G(L)$* if it satisfies the conditions (i), (ii), (iii) from 1.1, and if also the following conditions are fulfilled:

- (i₁) If $((x_n), x) \in L^{\mathbb{N}} \times L$ and if each subsequence (y_n) of (x_n) has a subsequence (z_n) such that $z_n \rightarrow_{\alpha} x$, then $x_n \rightarrow_{\alpha} x$.
- (ii₁) If $x \in L$ and $(x_n) = \text{const } x$, then $x_n \rightarrow_{\alpha} x$.
- (iii₁) If $x_n \rightarrow_{\alpha} x$, then $-x_n \rightarrow_{\alpha} -x$.
- (iv₁) If $x_n \rightarrow_{\alpha} x$ and $y_n \rightarrow_{\alpha} y$, then $x_n \wedge y_n \rightarrow_{\alpha} x \wedge y$ and $x_n \vee y_n \rightarrow_{\alpha} x \vee y$.
- (v₁) If $x_n \rightarrow_{\alpha} x$, $y_n \rightarrow_{\alpha} x$, $(z_n) \in L^{\mathbb{N}}$ and $x_n \leq z_n \leq y_n$ for each $n \in \mathbb{N}$, then $z_n \rightarrow_{\alpha} x$.

The system of all convergences in $G(L)$ will be denoted by $\text{Conv}_g L$.

Let us remark that in the paper [14] the Urysohn property (i₁) (which will be systematically applied below) was not assumed to be valid when investigating a sequential convergence in a lattice ordered group.

We denote by d the system of all elements $((x_n), x) \in L^{\mathbb{N}} \times L$ having the property that there is $m \in \mathbb{N}$ such that $x_n = x$ for each $n \geq m$. It is easy to verify that d belongs to $\text{Conv}_g L$, hence $\text{Conv}_g L$ is nonempty. The system $\text{Conv}_g L$ will be considered to be partially ordered by inclusion. It is obvious that d is the least element of $\text{Conv}_g L$.

Let us remark that the conditions (i), (ii), (iii), (i₁), (ii₁) and (iii₁) define a convergence group in the sense of [18] or a FLUSH convergence on the corresponding group (cf. [17]).

1.3. Definition. A nonempty subset α of $L^{\mathbb{N}} \times L$ will be said to be a *convergence in L* if $\alpha \in \text{Conv}_\ell L \cap \text{Conv}_g L$. The system of all convergences in L will be denoted by $\text{Conv} L$. If $\text{Conv} L \neq \emptyset$, then the set $\text{Conv} L$ will be partially ordered by inclusion.

The vector lattice L is said to be *archimedean* if, whenever $x, y \in L$ and $0 \leq nx \leq y$ for each $n \in \mathbb{N}$, then $x = 0$.

1.4. Lemma. *Let L be non-archimedean. Then $\text{Conv} L = \emptyset$.*

Proof. There exist $x, y \in L$ such that $0 < nx \leq y$ for each $n \in \mathbb{N}$. By way of contradiction, assume that $\alpha \in \text{Conv} L$. Because $\frac{1}{n} \rightarrow 0$ in \mathbb{R} , in view of 1.1, (v) we infer that $\frac{1}{n}y \rightarrow_\alpha 0$. Since $0 < x \leq \frac{1}{n}y$ for each $n \in \mathbb{N}$, according to (ii₁) and (v₁) of 1.2 the relation $x_n \rightarrow_\alpha x$ is valid, where $(x_n) = \text{const } x$. Thus in view of (ii₁) and (ii) we have arrived at a contradiction. \square

1.5. Lemma. *Let $\alpha \in \text{Conv}_g L$. Then α satisfies the condition (iv) from 1.1.*

Proof. Let $x_n \rightarrow_\alpha x$ and let $a \in \mathbb{R}$. There is $m \in \mathbb{N}$ with $|a| \leq m$. We have

$$x_n \rightarrow_\alpha x \Rightarrow |x_n - x| \rightarrow_\alpha 0,$$

whence in view of (iii) and by induction we get $m|x_n - x| \rightarrow_\alpha 0$. Since

$$0 \leq |ax_n - ax| = |a| |x_n - x| \leq m|x_n - x|,$$

according to (v₁) we obtain $|ax_n - ax| \rightarrow_\alpha 0$, thus $ax_n \rightarrow_\alpha ax$. \square

1.6. Corollary. *Let $\alpha \in \text{Conv}_g L$. Then $\alpha \in \text{Conv} L$ if and only if α satisfies the condition (v) from 1.1.*

If $L \neq \{0\}$, then the element d of $\text{Conv}_g L$ does not satisfy the condition (v) of 1.1. Hence if $L \neq \{0\}$, then $\text{Conv}_g L$ fails to be a subset of $\text{Conv} L$.

The positive cone $\{x \in L: x \geq 0\}$ of L will be denoted by L^+ . Under the inherited partial order and the operation $+$, L^+ is a lattice ordered semigroup.

1.7. Definition. A convex subsemigroup β of $(L^+)^{\mathbb{N}}$ will be said to be a *0-convergence in $G(L)$* if the following conditions are satisfied:

- (I) If $(g_n) \in \beta$, then each subsequence of (g_n) belongs to β .
- (II) If $(g_n) \in (L^+)^{\mathbb{N}}$ and if each subsequence of (g_n) has a subsequence belonging to (β) , then (g_n) belongs to β .
- (III) Let $x \in L^+$. Then $\text{const } x$ belongs to β if and only if $x = 0$.

The system of all 0-convergences in $G(L)$ will be denoted by $0\text{-Conv}_g L$. Let d_0 be the set of all $(x_n) \in (L^+)^{\mathbb{N}}$ such that $((x_n), 0) \in d$. Then $d_0 \in 0\text{-Conv}_g L$. Hence $0\text{-Conv}_g G \neq \emptyset$. The system $0\text{-Conv}_g L$ is partially ordered by inclusion.

Let $\alpha \in \text{Conv}_g L$. Put

$$(1) \quad \varphi_1(\alpha) = \{(|x_n - x|) : x_n \rightarrow_{\alpha} x\}.$$

Conversely, let $\beta \in 0\text{-Conv}_g L$. Denote

$$(2) \quad \varphi_2(\beta) = \{((x_n), x) : (|x_n - x|) \in \beta\}.$$

1.8. Lemma. (Cf. [4], Lemma 1.4 and Theorem 1.6.) φ_1 and φ_2 are inverse isomorphisms of $\text{Conv}_g L$ onto $0\text{-Conv}_g L$, or of $0\text{-Conv}_g L$ onto $\text{Conv}_g L$, respectively.

1.9. Definition. A nonempty subset β of $(L^+)^{\mathbb{N}}$ will be said to be a *0-convergence in L* if $\beta \in 0\text{-Conv}_g L$ and if, moreover, the following condition is satisfied:

- (IV) If $x \in L$ and $a_n \rightarrow 0$ in \mathbb{R} , then $(a_n x) \in \beta$.

Let $0\text{-Conv } L$ be the set of all 0-convergences in L . If this set is nonempty, then it will be considered to be partially ordered by inclusion.

Now let α and β run over the set $\text{Conv } L$ or $0\text{-Conv } L$, respectively, and let φ_1 and φ_2 be defined as in (1) and (2). Then by a routine proof and by using 1.5 we obtain the following result which is analogous to 1.8:

1.10. Lemma. (i) $\text{Conv } L = \emptyset \Leftrightarrow 0\text{-Conv } L = \emptyset$. (ii) If $\text{Conv } L \neq \emptyset$, then φ_1 and φ_2 are inverse isomorphisms of $\text{Conv } L$ onto $0\text{-Conv } L$, or of $0\text{-Conv } L$ onto $\text{Conv } L$, respectively.

As we remarked in the introduction, we are interested in studying the partially ordered system $\text{Conv } L$. Now, in view of 1.10, it suffices to investigate the system $0\text{-Conv } L$. Next, according to 1.4, it suffices to consider the case when L is archimedean.

2. REGULAR SETS

In what follows we assume that L is an archimedean vector lattice.

Let $\emptyset \neq A \subseteq (L^+)^{\mathbb{N}}$. The set A will be said to be regular with respect to $G(L)$ (or L , respectively) if there is $\alpha \in 0\text{-Conv}_g L$ (or $\alpha \in 0\text{-Conv} L$) such that $A \subseteq \alpha$.

2.1. Lemma. *Let $\emptyset \neq A \subseteq (L^+)^{\mathbb{N}}$. Then the following conditions are equivalent:*

- (i) *A fails to be regular with respect to $G(L)$.*
- (ii) *There exist $0 < z \in L$, positive integers m, k , elements $(y_n^1), \dots, (y_n^k)$ of A and subsequences (x_n^1) of $(y_n^1), \dots, (x_n^k)$ of (y_n^k) such that*

$$z \leq m(x_n^1 \vee x_n^2 \vee \dots \vee x_n^k) \quad \text{for each } n \in \mathbb{N}.$$

Proof. The implication (ii) \Rightarrow (i) is obvious. Let (i) be valid. In view of the results of [4] (cf. also [10], Proposition 2.1) there exist $0 < z \in L$, positive integers m_1, k , elements $(y_n^1), \dots, (y_n^k)$ of A and subsequences (x_n^1) of $(y_n^1), \dots, (x_n^k)$ of (y_n^k) such that

$$z \leq m_1(x_n^1 + x_n^2 + \dots + x_n^k) \quad \text{for each } n \in \mathbb{N}.$$

Hence according to Lemma 2.4, [10] there is $m \in \mathbb{N}$ with

$$z \leq m(x_n^1 \vee x_n^2 \vee \dots \vee x_n^k) \quad \text{for each } n \in \mathbb{N}.$$

□

Let A_0 be the set of all sequences (x_n) in L having the property that there are $0 \leq x \in L$ and $(a_n) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $a_n \rightarrow 0$ in \mathbb{R} and $x_n = a_n x$ for each $n \in \mathbb{N}$.

2.2. Lemma. *The set A_0 is regular with respect to $G(L)$ and also with respect to L .*

Proof. By way of contradiction, assume that A_0 fails to be regular with respect to $G(L)$. Then the condition (ii) from 2.1. holds for A_0 .

For each $i \in \{1, 2, \dots, k\}$ there are $0 < x^i \in L$ and $(a_n^i) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $a_n^i \rightarrow 0$ in \mathbb{R} and

$$x_n^i = a_n^i x^i \quad \text{for each } n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ we put $a_n = \max\{a_n^1, a_n^2, \dots, a_n^k\}$. Then $a_n \rightarrow 0$ in \mathbb{R} and

$$\begin{aligned} 0 < z &\leq m(x_n^1 \vee x_n^2 \vee \dots \vee x_n^k) = m(a_n^1 x^1 \vee \dots \vee a_n^k x^k) \\ &\leq m a_n (x^1 \vee \dots \vee x^k) \quad \text{for each } n \in \mathbb{N}. \end{aligned}$$

Next, for each $n \in \mathbb{N}$ there is $n(1) \in \mathbb{N}$ such that $ma_{n(1)} < \frac{1}{n}$, hence

$$0 < z < \frac{1}{n}(x^1 \vee \dots \vee x^k) \quad \text{for each } n \in \mathbb{N}.$$

Thus $nz < x^1 \vee \dots \vee x^k$ for each $n \in \mathbb{N}$, which is impossible, because L is archimedean. Thus there is $\alpha \in 0\text{-Conv}_g L$ with $A_0 \subseteq \alpha$. Then α fulfils the condition (IV), hence $\alpha \in 0\text{-Conv } L$. \square

2.3. Theorem. *Let L be an archimedean vector lattice. Then $\text{Conv } L \neq \emptyset$.*

Proof. In view of 2.2 there is $\alpha \in 0\text{-Conv } L$ with $A_0 \subseteq \alpha$. Hence $0\text{-Conv } L \neq \emptyset$. Thus according to 1.10 we have $\text{Conv } L \neq \emptyset$. \square

2.4. Lemma. *Let $\alpha \in 0\text{-Conv } L$. Then $A_0 \subseteq \alpha$.*

Proof. This follows immediately from the fact that α satisfies the condition (IV) of 1.9. \square

2.5. Corollary. *Let I be a nonempty set and for each $i \in I$ let $\alpha_i \in 0\text{-Conv } L$. Then $\emptyset \neq \bigcap_{i \in I} \alpha_i \in 0\text{-Conv } L$.*

Let us denote by d^0 the intersection of all $\alpha_i \in 0\text{-Conv } L$ with $A_0 \subseteq \alpha_i$ (such α_i do exist in view of 2.2). According to 2.4 and 2.5 we obtain:

2.6. Corollary. *d^0 is the least element of $0\text{-Conv } L$. If $\alpha \in 0\text{-Conv } L$, then the interval $[d^0, \alpha]$ of the partially ordered set $0\text{-Conv } L$ is a complete lattice.*

2.7. Proposition. *$d^0 = A_0$.*

Proof. In view of the definition of d^0 we have $A_0 \subseteq d^0$. Let $(z_n) \in d^0$. Then in view of [10], Proposition 2.1, and according to 2.4 there are $m, k \in \mathbb{N}$, elements $(y_n^1), \dots, (y_n^k)$ of A_0 and subsequences (x_n^1) of $(y_n^1), \dots, (x_n^k)$ of (y_n^k) such that

$$z_n \leq m(x_n^1 \vee \dots \vee x_n^k).$$

For each $i \in \{1, 2, \dots, k\}$ there are $x^i \in L^+$ and $(a_n^i) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $a_n^i \rightarrow 0$ in \mathbb{R} and $x_n^i = a_n^i x^i$ for each $n \in \mathbb{N}$. Put $a_n = \max\{a_n^1, \dots, a_n^k\}$. Hence $a_n \rightarrow 0$ in \mathbb{R} and

$$z_n \leq a_n(mx^1 \vee \dots \vee mx^k).$$

Thus $(z_n) \in A_0$ and therefore $d^0 \subseteq A_0$. \square

For each $X \subseteq (L^+)^{\mathbb{N}}$ let us denote by X^* the set of all $(x_n) \in (L^+)^{\mathbb{N}}$ such that each subsequence of (x_n) has a subsequence which belongs to X .

Let A_1 be the set of all $(x_n) \in (L^+)^{\mathbb{N}}$ which have the following property: there exist $0 \leq x \in L$ and $m \in \mathbb{N}$ such that $x_n \leq \frac{1}{n}x$ for each $n \geq m$.

Another constructive characterization of d^0 is given by the following lemma.

2.8. Lemma. $d^0 = A_1^*$.

Proof. Since $A_1 \subseteq A_0$, we clearly have $A_1^* \subseteq d^0$. Let $(x_n) \in d^0$. In view of 2.7 there are $x \in L^+$ and $(a_n) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $x_n = a_n x$ for each $n \in \mathbb{N}$. Let (y_n) be a subsequence of (x_n) and let (b_n) be the corresponding subsequence of (a_n) ; hence $y_n = b_n x$ for each $n \in \mathbb{N}$. There exists a subsequence (c_n) of (b_n) such that $c_n \leq \frac{1}{n}$ for each $n \in \mathbb{N}$. Put $z_n = c_n x$ for each $n \in \mathbb{N}$. Then $(c_n x)$ is a subsequence of (y_n) and $(c_n x) \in A_1$. Hence $(x_n) \in A_1^*$ and thus $d^0 \subseteq A_1^*$. \square

2.9. Proposition. *There exists an archimedean vector lattice L such that $0\text{-Conv } L$ has no greatest element.*

Proof. It suffices to apply an analogous example as in [3], Section 5 (with the distinction that the real functions under consideration in the example are not assumed to be integer valued). \square

2.10. Theorem. *Let L be an archimedean vector lattice. Suppose that L is $(\mathbb{N}_0, 2)$ -distributive. Then $0\text{-Conv } L$ possesses a greatest element.*

Proof. This is a consequence of 2.6 and of the fact that $0\text{-Conv}_g L$ has a greatest element (cf. [12]). \square

Lemma 1.10 and Lemma 2.6 yield that each interval of the partially ordered set $0\text{-Conv } L$ is, at the same time, an interval of $0\text{-Conv}_g L$. Hence in view of [5], Theorem 2.5 we obtain:

2.11. Proposition. *Each interval of $0\text{-Conv } L$ is a Brouwerian lattice.*

3. THE SETS OF THE FORM $\alpha \cup A_0$

Let $\emptyset \neq \alpha \subseteq (L^+)^{\mathbb{N}}$ be such that α is regular with respect to $G(L)$. We shall investigate the problem whether the set $\alpha \cup A_0$ is regular with respect to L .

First we shall deal with the case when L is a projectable vector lattice. (Projectable lattice ordered groups and vector lattices were studied by several authors; cf. e.g., [2] and [16].)

For the sake of completeness we recall the following notions.

Let L be a vector lattice and $X \subseteq L$. We put

$$X^d = \{y \in L : |y| \wedge |x| = 0 \text{ for each } x \in X\}.$$

Then X^d is said to be a polar of L . The vector lattice L is called projectable if for each $x \in L$, the set $\{x\}^d$ is a direct factor of L .

An element $e \in L$ is called a strong unit of L if for each $x \in L$ there is $n \in \mathbb{N}$ such that $x \leq ne$.

Since each strong unit of an archimedean vector lattice L_1 is, at the same time, a strong unit of the Dedekind completion of L_1 , we have

3.1. Proposition. (Cf., e.g., [19], Theorem V.3.1.) *Let L_1 be an archimedean vector lattice having a strong unit. Then there is a set I such that there exists an isomorphism of L_1 into the vector lattice $\prod_{i \in I} R_i$, where $R_i = \mathbb{R}$ for each $i \in I$.*

3.2. Lemma. *Let $\alpha \in \text{Conv}_g L$. Then the following conditions are equivalent:*

- (i) *The set $\alpha \cup A_0$ fails to be regular with respect to $G(L)$.*
- (ii) *There are $t, z \in L$ and $(z_n) \in \alpha$ such that $0 < z \leq t$ and*

$$z = z_n \vee (z \wedge \frac{1}{n}t) \text{ for each } n \in \mathbb{N}.$$

Proof. According to 2.1, (ii) \Rightarrow (i). Suppose that (i) is valid. Thus in view of 2.7 and 2.8, the set $\alpha \cup A_1$ fails to be regular with respect to $G(L)$. Hence the condition (ii) from 2.1 holds, where $A = \alpha \cup A_1$.

If $(x_n^1), \dots, (x_n^k) \in \alpha$, then α would not be regular with respect to $G(L)$, which is a contradiction. If $(x_n^1), \dots, (x_n^k) \in A_1$, then we obtain a contradiction with respect to 2.2. Hence without loss of generality we can suppose that there is $k(1) \in \mathbb{N}$ with $1 < k(1) < k$ such that

$$(x_n^1), \dots, (x_n^{k(1)}) \in \alpha \text{ and } (x_n^{k(1)+1}), \dots, (x_n^k) \in A_1.$$

Put $z_n = m(x_n^1 \vee \dots \vee x_n^{k(1)})$ for each $n \in \mathbb{N}$. Then $(z_n) \in \alpha$.

For each $j \in \{k(1)+1, \dots, k\}$ there are $0 < y^j \in L$ and $(a_n^j) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $a_n^j \rightarrow 0$ in \mathbb{R} and $y_n^j = a_n^j y^j$ for each $n \in \mathbb{N}$. Denote

$$a_n = \max\{a_n^{k(1)+1}, \dots, a_n^k\}, \quad t = y^{k(1)+1} \vee \dots \vee y^k.$$

There is a subsequence $(n(1))$ of the sequence (n) such that

$$ma_{n(1)} < \frac{1}{n} \text{ for each } n \in \mathbb{N}.$$

Hence we have

$$m(x_{n(1)}^{k(1)+1} \vee \dots \vee x_{n(1)}^k) \leq \frac{1}{n}t \quad \text{for each } n \in \mathbb{N}.$$

Therefore

$$0 < z \leq z_{n(1)} \vee \frac{1}{n}t \quad \text{for each } n \in \mathbb{N}.$$

Because $(z_{n(1)}) \in \alpha$, it suffices to write z_n instead of $z_{n(1)}$. Thus

$$(1) \quad z = z \wedge (z_n \vee \frac{1}{n}t) = (z \wedge z_n) \vee (z \wedge \frac{1}{n}t) \quad \text{for each } n \in \mathbb{N}.$$

If $z \wedge t = 0$, then $z \wedge \frac{1}{n}t = 0$ for each $n \in \mathbb{N}$, whence $z \leq z_n$ for each $n \in \mathbb{N}$ and thus α fails to be regular, which is a contradiction. Therefore $z \wedge t > 0$ and then, without loss of generality, we can take $z \wedge t$ instead of z ; hence we have $z \leq t$. Next, $(z \wedge z_n) \in \alpha$, thus without loss of generality we can take $(z \wedge z_n)$ instead of (z_n) . Hence in view of (1) we infer that (ii) is valid. \square

3.3. Proposition. *Assume that L is projectable. Let $\alpha \in 0\text{-Conv}_g L$. Then $\alpha \cup A_0$ is regular with respect to L .*

Proof. In view of 2.7 it suffices to verify that $\alpha \cup A_0$ is regular with respect to $G(L)$.

By way of contradiction, suppose that $\alpha \cup A_0$ fails to be regular with respect to $G(L)$. Then the condition (ii) from 3.2 is valid. There exists $m \in \mathbb{N}$ such that $z \not\leq \frac{1}{m}t$. Thus

$$(1') \quad z^0 = (z - \frac{1}{m}t)^+ > 0.$$

Let us denote by P the polar of L generated by z^0 ; i.e., $P = \{z^0\}^{dd}$. Since L is projectable, P is a direct factor in L . For each $g \in L$ let $g(P)$ be the component of g in P . In view of the condition (ii) of 3.2 we have

$$(2) \quad z(P) = z_n(P) \vee (z(P) \wedge \frac{1}{n}t(P)) \quad \text{for each } n \in \mathbb{N}.$$

If $z(P) = 0$, then $z^0 = z^0(P) = 0$, which is a contradiction. Thus $z(P) > 0$. Next, from $z \leq t$ we infer that $z(P) \leq t(P)$.

Let L_1 be the convex ℓ -subgroup of $G(P)$ generated by the element $t(P)$. Then $t(P)$ is a strong unit of L_1 and L_1 is a linear subspace of L . Let I and φ be as in 3.2. For each $i \in I$ we have $\varphi(z(P))(i) \geq 0$. According to the definition of P we obtain

$$(z - \frac{1}{m}t)^- \in P^d$$

whence $(z - \frac{1}{m}t)(P) = z_0(P)$. In view of (1'),

$$(3) \quad 0 < z^0 = z^0(P) = z(P) - \frac{1}{m}t(P),$$

hence the set $I_1 = \{i \in I: \varphi(z(P))(i) > 0\}$ is nonempty.

Let $i \in I_1$ and $n > m$. According to (3),

$$(4) \quad \varphi(z(P))(i) \geq \frac{1}{n}\varphi(t(P))(i).$$

Also, in view of (2),

$$\begin{aligned} \varphi(z(P))(i) &= \varphi(z_n(P))(i) \vee (\varphi(z(P))(i) \wedge \frac{1}{n}\varphi(t(P))(i)) \\ &= \max\{\varphi(z_n(P))(i), \min\{\varphi(z(P))(i), \frac{1}{n}\varphi(t(P))(i)\}\}. \end{aligned}$$

Thus according to (4),

$$\varphi(z(P))(i) = \max\{\varphi(z_n(P))(i), \frac{1}{n}\varphi(t(P))(i)\}.$$

By applying (4) again we get

$$\varphi(z(P))(i) = \varphi(z_n(P))(i).$$

Therefore $\varphi(z(P))(i) = \varphi(z_n(P))(i)$ for each $i \in I$. Hence

$$(5) \quad 0 < z(P) = z_n(P) \quad \text{for each } n > m.$$

Since $z_n(P) \leq z_n$ for each $n \in \mathbb{N}$ and since (z_n) is regular with respect to L , we infer that $(z_n(P))$ is regular with respect to L . Thus in view of (5) we have arrived at a contradiction. \square

Now let us drop the assumption that L is projectable. We denote by L' the Dedekind completion of L . It is well-known that L' is projectable.

3.4. Lemma. *Let $\emptyset \neq \alpha \subseteq (L^+)^{\mathbb{N}}$. Assume that α is regular with respect to $G(L)$. Then α is regular with respect to $G(L')$.*

Proof. By way of contradiction, assume that α fails to be regular with respect to $G(L')$. Then the condition (ii) from 2.1 holds (with the distinction that $z \in L'$ and A is replaced by α). There exists $0 < z_1 \in L$ with $z_1 \leq z$. But by applying 2.1 again we infer that α fails to be regular with respect to L , which is a contradiction. \square

3.5. Lemma. *Let $\emptyset \neq \alpha \subseteq (L^+)^{\mathbb{N}}$. Assume that α is regular with respect to $G(L)$. Then α is regular with respect to $G(L)$.*

Proof. This is an immediate consequence of 2.1. \square

3.6. Theorem. *Let $\emptyset \neq \alpha \subseteq (L^+)^{\mathbb{N}}$. Assume that α is regular with respect to $G(L)$. Then $\alpha \cup A_0$ is regular with respect to $G(L)$ and with respect to L .*

Proof. In view of 3.4, α is regular with respect to $G(L')$. Because $G(L')$ is projectable, according to 3.3 we obtain that $\alpha \cup A_0$ is regular with respect to $G(L')$. Thus 3.5 yields that $\alpha \cup A_0$ is regular with respect to $G(L)$. Now it follows from 2.7 that $\alpha \cup A_0$ is regular with respect to L . \square

3.7. Corollary. *Let $\alpha \in 0\text{-Conv}_g L$. Then $\alpha \vee d^0$ does exist in $0\text{-Conv}_g L$ and in $0\text{-Conv} L$.*

3.8. Proposition. *The following conditions are equivalent:*

- (i) $0\text{-Conv} L$ has the greatest element.
- (ii) $0\text{-Conv}_g L$ has the greatest element.

Proof. We obviously have (ii) \Rightarrow (i). Let (i) hold and let β be the greatest element of $0\text{-Conv} L$. Let $\alpha \in 0\text{-Conv}_g L$. According to 3.7, the element $\alpha \vee d^0$ does exist in $0\text{-Conv} L$. Thus $\alpha \leq \alpha \vee d^0 \leq \beta$. Hence β is the greatest element of $0\text{-Conv}_g L$. \square

3.9. Corollary. *Let $0\text{-Conv} L$ have the greatest element. Then $0\text{-Conv} L$ is a complete lattice and $0\text{-Conv} L$ is a principal dual ideal of $0\text{-Conv}_g L$ generated by the element d^0 .*

Let us remark that if L_1 is a convex ℓ -subgroup of $G(L)$, then it is a linear subspace of L .

3.10. Theorem. *There exists a convex ℓ -subgroup L_1 of $G(L)$ such that the following conditions are satisfied:*

- (i) $\text{Conv} L_1$ is a complete lattice.
- (ii) If L_2 is a convex ℓ -subgroup of $G(L)$ such that $\text{Conv} L_2$ is a complete lattice, then $L_2 \leq L_1$.

Proof. This follows from 3.8 and from [10], Theorem 5.5. \square

Let L_1 be a vector lattice. If neither the operation $+$ nor the multiplication of elements of L_1 by reals is taken into account, then we obtain a lattice which will be denoted by L_1^0 .

3.11. Theorem. *Let L_i ($i = 1, 2$) be archimedean vector lattices. Assume that the lattices L_1^0 and L_2^0 are isomorphic and that $\text{Conv} L_1$ possesses a greatest element. Then $\text{Conv} L_2$ possesses a greatest element as well.*

P r o o f. According to 1.10, $0\text{-Conv } L_1$ possesses a greatest element. Then in view of 3.8, $0\text{-Conv}_g L$ has a greatest element. Since L_1^0 is isomorphic to L_2^0 , by applying [10], Theorem 3.5 we conclude that $0\text{-Conv}_g L_2$ has a greatest element as well. Now according to 3.8 and 1.10, $\text{Conv } L_2$ possesses a greatest element. \square

4. DISJOINT SEQUENCES

A sequence (x_n) in L will be said to be disjoint (or orthogonal) if $x_n \wedge x_m = 0$ whenever n and m are distinct positive integers.

The following assertion follows from the results proved in [4].

- (A) Assume that L possesses a disjoint sequence all members of which are strictly positive. Then there exist infinitely many elements α_i of $0\text{-Conv}_g L$ such that each α_i is generated by a disjoint sequence.

4.1. Lemma. (Cf. [4].) *Let (x_n) be a disjoint sequence in L . Then the set (x_n) is regular with respect to $G(L)$.*

4.2. Lemma. *Let (x_n) be a disjoint sequence in L . Then the set $\{(x_n)\} \cup A_0$ is regular with respect to $G(L)$ and with respect to L .*

P r o o f. This is a consequence of 4.1 and 3.6. \square

If $(x_n) \in (L^+)^{\mathbb{N}}$ and the set $\{(x_n)\}$ is regular in $G(L)$ then the least element α of $0\text{-Conv}_g L$ satisfying the relation $\{(x_n)\} \cup A_0 \subseteq \alpha$ will be denoted by $\alpha(x_n)$.

Let (x_n) be a disjoint sequence in L such that $x_n > 0$ for each $n \in \mathbb{N}$. Then $(x_n) \notin d_0$. On the other hand, (x_n) can belong to d^0 (cf. Proposition 4.6 below).

4.3. Lemma. *Let (x_n) and (y_n) be disjoint sequences in L such that $x_n \wedge y_m = 0$ for each $m, n \in \mathbb{N}$. Let $y_n > 0$ for each $n \in \mathbb{N}$ and $(y_n) \notin d^0$. Then $(y_n) \notin \alpha(x_n)$.*

P r o o f. By way of contradiction, assume that $y_n \in \alpha(x_n)$. Then in view of [10], Lemma 2.3 there are $m, k \in \mathbb{N}$ and $(z_n^1), \dots, (z_n^k) \in (L^+)^{\mathbb{N}}$ such that each (z_n^i) ($i = 1, 2, \dots, k$) is a subsequence of a sequence belonging to $\{(x_n)\} \cup A_0$ and

$$0 < y_n \leq m(z_n^1 \vee \dots \vee z_n^k) \quad \text{for each } n \in \mathbb{N}.$$

Since $(y_n) \notin A_0$, without loss of generality we can assume that $(z_n^1), \dots, (z_n^{k-1})$ are subsequences of (x_n) and that (z_n^k) is a subsequence of $(\frac{1}{n}x)$ for some $0 < x \in L$. Thus

$$0 < y_n \leq (mz_n^1 \vee \dots \vee mz_n^{k-1}) \vee \frac{1}{n}x' \quad \text{for each } n \in \mathbb{N},$$

where $x' = mx$. But $y_n \wedge (mz_n^1 \vee \dots \vee mz_n^{k-1}) = 0$, whence $y_n \leq \frac{1}{n}x'$ for each $n \in \mathbb{N}$. Since $(y_n) \notin d^0$, we have arrived at a contradiction. \square

4.4. Theorem. *Assume that L possesses an infinite orthogonal subset. Next, suppose that no disjoint sequence (x_n) in L with $x_n > 0$ for each $n \in \mathbb{N}$ belongs to d^0 . Then $0\text{-Conv } L$ is infinite.*

Proof. In view of the assumption there are disjoint sequences (x_n^i) ($i \in \mathbb{N}$) in L such that $x_n^i > 0$ for each $n, i \in \mathbb{N}$, and $x_n^i \wedge x_m^j = 0$ whenever $m, n, i, j \in \mathbb{N}$ and $i \neq j$. In view of 4.2 we have $\alpha(x_n^i) \in 0\text{-Conv}_g L$ for each $i \in \mathbb{N}$. Let i, j be distinct elements of \mathbb{N} . According to 4.3, $\alpha(x_n^i) \neq \alpha(x_n^j)$. \square

For a relevant result concerning convergences in a lattice ordered group cf. [4].

4.5. Theorem. *Assume that L possesses no infinite orthogonal subset. Then $0\text{-Conv } L$ is a one-element set.*

Proof. The case $L = \{0\}$ is trivial; let $L \neq \{0\}$. The system $0\text{-Conv}_g L$ was described in [4], Section 6. According to [4], if $\alpha \in 0\text{-Conv}_g L$ and $(\frac{1}{n}x) \in \alpha$ for each $0 < x \in L$, then α is the greatest element of $0\text{-Conv}_g L$; hence only this greatest element of $0\text{-Conv}_g L$ can belong to $0\text{-Conv } L$. \square

4.6. Proposition. *Assume that L is orthogonally complete. Then each disjoint sequence in L belongs to d^0 .*

Proof. Let (x_n) be a disjoint sequence in L . Then (nx_n) is disjoint as well. Since L is orthogonally complete, there exists $x = \bigvee_{n \in \mathbb{N}} nx_n$ in L . For each $n \in \mathbb{N}$ we have $0 \leq x_n \leq \frac{1}{n}x$, whence $(x_n) \in d^0$. \square

4.7. Corollary. *The assertion (A) does not hold in general if $0\text{-Conv}_g L$ is replaced by $0\text{-Conv } L$.*

4.8. Proposition. *Assume that $L \neq \{0\}$ has a strong unit and that (x_n) is a disjoint sequence in L such that $x_n > 0$ for each $n \in \mathbb{N}$. Then there is a sequence (a_n) with $a_n \in \mathbb{N}$ for each $n \in \mathbb{N}$ having the property that $(a_n x_n) \notin d^0$.*

Proof. Let e be a strong unit in L . Since L is archimedean, for each $n \in \mathbb{N}$ there is $a_n \in \mathbb{N}$ such that

$$(1) \quad a_n x_n \not\leq e.$$

By way of contradiction, assume that $(a_n x_n) \in d^0$. Hence in view of 2.8 there is a subsequence $(b_n y_n)$ of $(a_n x_n)$ such that $(b_n y_n) \in A_1$. Thus there are $m \in \mathbb{N}$ and $0 < x \in L$ such that $b_n y_n \leq \frac{1}{n}x$ for each $n \geq m$. Next, since e is a strong unit in L , there is $k \in \mathbb{N}$ with $x \leq ke$. Thus

$$b_n y_n \leq \frac{k}{n}e \quad \text{for each } n \geq m.$$

Hence for $n > \max\{m, k\}$ we have $b_n y_n \leq e$. But in view of (1) the relation $b_n y_n \not\leq e$ is valid for each $n \in \mathbb{N}$, which is a contradiction. \square

4.9. Proposition. *Assume that L has a strong unit. Then (A) is valid with $\text{Conv}_g L$ replaced by $\text{Conv } L$.*

Proof. This is a consequence of 4.3 and 4.8. \square

5. DIRECT SUMS OF LINEARLY ORDERED VECTOR LATTICES

Let us denote by \mathcal{S} the class of all archimedean vector lattices which can be expressed as the direct sum of linearly ordered vector lattices. Next, let \mathcal{L} be the class of all linearly ordered vector lattices.

In this section it will be shown that if $L \in \mathcal{S}$, then $0\text{-Conv } L$ is a complete lattice which has no atom.

The case $L = \{0\}$ being trivial, we assume in the present section that L is a nonzero archimedean vector lattice which can be represented as

$$(1) \quad L = \sum_{i \in I} L_i, \quad \text{where } L_i \in \mathcal{L} \text{ for each } i \in I.$$

Also, without loss of generality we can suppose that $L_i \neq \{0\}$ for each $i \in I$.

5.1. Proposition. *$0\text{-Conv } L$ is a complete lattice.*

Proof. From (1) it follows that L is completely distributive. Hence in view of 2.10, $0\text{-Conv } L$ possesses a greatest element. Thus $0\text{-Conv } L$ is a complete lattice. \square

5.2. Lemma. *Let (x_n) be a disjoint sequence in L such that $x_n > 0$ for each $n \in \mathbb{N}$. Then (x_n) is not upper-bounded in L .*

Proof. This is an immediate consequence of (1). \square

In view of 5.2 and 2.8 we obtain

5.3. Corollary. *Let (x_n) be as in 5.2. Then (x_n) does not belong to d^0 .*

5.4. Proposition. *Let I be finite. Then $0\text{-Conv } L$ is a one-element set.*

Proof. From (1) we infer that L has no infinite orthogonal subset. Hence in view of 4.5, $0\text{-Conv } L$ is a one-element set. \square

5.5. Proposition. *Let I be infinite. Then $0\text{-Conv } L$ is infinite.*

Proof. According to (1), L possesses an infinite orthogonal subset. Then 4.4 and 5.3 yield that $0\text{-Conv } L$ is infinite. \square

5.6. Lemma. *Let $\alpha \in 0\text{-Conv } L$. Assume that $(x_n) \in \alpha$, $x_n > 0$ for each $n \in \mathbb{N}$, and that the sequence (x_n) is disjoint. Then α fails to be an atom of $0\text{-Conv } L$.*

Proof. Consider the sequences (x_{2n}) and (x_{2n+1}) . In view of 5.3, $(x_{2n}) \notin d^0$ and $(x_{2n+1}) \notin d^0$. Hence by applying the notation from Section 4 we have

$$d^0 < \alpha(x_{2n}) \leq \alpha, \quad d^0 < \alpha(x_{2n+1}) \leq \alpha.$$

Next, according to 4.3, $\alpha(x_{2n}) \neq \alpha(x_{2n+1})$. Hence α cannot be an atom of $0\text{-Conv } L$. \square

For $x \in L$ and $i \in I$, let $x(i)$ be the component of x in L_i . We put $\text{Sup } x = \{i \in I : x(i) \neq 0\}$. If (x_n) is a sequence in L , then we denote

$$\text{Sup}(x_n) = \bigcup_{n \in \mathbb{N}} \text{Sup } x_n.$$

5.7. Lemma. *Let $(x_n) \in (L^+)^{\mathbb{N}}$ be such that $\{(x_n)\}$ is regular and suppose that $\text{Sup}(x_n)$ is finite. Then $\alpha(x_n) = d^0$.*

Proof. In view of the assumption there is a finite subset $I(1)$ of I such that $x_n \in L(1) = \sum_{i \in I(1)} L_i$ for each $n \in \mathbb{N}$. Then according to 4.5, (x_n) belongs to the least element of $0\text{-Conv } L(1)$. Next, in view of 2.8, (x_n) belongs to d^0 . Hence $\alpha(x_n) = d^0$. \square

5.8. Lemma. *Let $(x_n) \in (L^+)^{\mathbb{N}}$ be such that $\{(x_n)\}$ is regular and suppose that $\text{Sup}(x_n)$ is infinite. Then $\alpha(x_n)$ contains a disjoint sequence with strictly positive elements.*

Proof. Since $\text{Sup}(x_n)$ is infinite and (1) holds, there is a subsequence (y_n) of (x_n) such that for each $n \in \mathbb{N}$, $\text{Sup } y_n$ is not a subset of the set

$$\text{Sup } y_1 \cup \dots \cup \text{Sup } y_{n-1}.$$

Therefore the sequence (y_n) is disjoint and belongs to $\alpha(x_n)$. \square

5.9. Theorem. *Let $L \in \mathcal{S}$. Then $0\text{-Conv } L$ has no atom.*

Proof. By way of contradiction, assume that α is an atom of $0\text{-Conv } L$. Then there is $(x_n) \in (L^+)^{\mathbb{N}}$ such that $\alpha = \alpha(x_n)$. If $\text{Sup}(x_n)$ is finite, then 5.7 yields a contradiction. If $\text{Sup}(x_n)$ is infinite, then by means of 5.8 and 5.6 we arrive at a contradiction. \square

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