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A REMARK ON SPACES OVER A SPECIAL LOCAL RING

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Abstract. This paper deals with $\mathbf{A}$-spaces in the sense of McDonald over linear algebras $\mathbf{A}$ of a certain type. Necessary and sufficient conditions for a submodule to be an $\mathbf{A}$-space are derived.

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According to [1] we define:

1. Definition. Let $\mathbf{A}$ be a local ring. Let $\mathbf{M}$ be an $\mathbf{A}$-module. Then $\mathbf{M}$ is called an $\mathbf{A}$-*space* if there exist $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbf{M}$ with

(a) $\mathbf{M} = \mathbf{A}\mathbf{e}_1 \oplus \ldots \oplus \mathbf{A}\mathbf{e}_n$,

(b) the map $\mathbf{A} \to \mathbf{A}\mathbf{e}_i$ defined by $\xi \mapsto \xi\mathbf{e}_i$ is an $\mathbf{A}$-isomorphism for $1 \leqslant i \leqslant n$.
The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called an $\mathbf{A}$-*basis of* $\mathbf{M}$.

2. Remarks.

A module $\mathbf{M}$ over a local ring $\mathbf{A}$ is an $\mathbf{A}$-space if and only if it is a free finitely dimensional module.

If $\mathbf{A}$ is a local ring and $\mathbf{M}$ is an $\mathbf{A}$-space then all bases of $\mathbf{M}$ have the same number n of elements and we say $\mathbf{M}$ has $\mathbf{A}$-*dimension* n. (See [1].)

Every direct summand of an $\mathbf{A}$-space is an $\mathbf{A}$-space. (See [1].)

3. Definition. A direct summand K of an $\mathbf{A}$-space $\mathbf{M}$ is called an $\mathbf{A}$-*subspace of* $\mathbf{M}$.

4. Definition. Let $\mathbf{T}$ be a commutative field. The *plural* $\mathbf{T}$-*algebra of order* m is every linear algebra $\mathbf{A}$ on $\mathbf{T}$ having as a vector space over $\mathbf{T}$ a basis

$$\{1, \eta, \eta^2, \ldots, \eta^{m-1}\} \text{ with } \eta^m = 0.$$

5. Notation. In what follows we denote by $\mathbf{A}$ the plural $\mathbf{T}$-algebra of order m introduced by Definition 4.

Propositions 6, 7 and Lemma 9 are proved in [3]. Thus the proofs of them will be omitted.

6. Proposition. *$\mathbf{A}$ is a local ring with the maximal ideal $\eta\mathbf{A}$. All ideals of $\mathbf{A}$ are just $\eta^j\mathbf{A}$, $1 \leqslant j \leqslant m$.*

7. Proposition. *The ring $\mathbf{A}$ is isomorphic to the factor ring of polynomials $\mathbb{R}[x]/(x^m)$.*

8. Theorem. *Let K be a submodule of an $\mathbf{A}$-space $\mathbf{M}$. Then K is an $\mathbf{A}$-subspace of $\mathbf{M}$ if and only if K is an $\mathbf{A}$-space.*

Proof. It follows from Definition 3 and from Theorem 7 in [4]. □

9. Lemma. *Let K be an $\mathbf{A}$-space and let $\{\mathbf{e}_1, \dots, \mathbf{e}_s\}$ be some $\mathbf{A}$-basis of K. Then K is a vector-space over $\mathbf{T}$ having dimension (called $\mathbf{T}$-dimension) sm and the set $\{\mathbf{e}_1, \dots, \mathbf{e}_s, \eta\mathbf{e}_1, \dots, \eta\mathbf{e}_s, \dots, \eta^{m-1}\mathbf{e}_1, \dots, \eta^{m-1}\mathbf{e}_s\}$ forms a basis of K over $\mathbf{T}$ ($\mathbf{T}$-basis).*

Let us define a linear operator η on an $\mathbf{A}$-space $\mathbf{M}$ by the relation:

$$\forall \mathbf{x} \in \mathbf{M}\colon\ \eta(\mathbf{x}) = \eta.\mathbf{x}.$$

10. Theorem. *Let K be a submodule of the $\mathbf{A}$-space $\mathbf{M}$ and let $\vartheta = \eta|K$. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ is an $\mathbf{A}$-basis of K if and only if $\{\eta^{m-k}\mathbf{u}_1, \dots, \eta^{m-k}\mathbf{u}_s\}$ is a $\mathbf{T}$-basis of $\operatorname{Ker}\vartheta^k$ relatively* to $\operatorname{Ker}\vartheta^{k-1}$ for every $k = 1, \dots, m$.*

Proof. The operator η is a nilpotent linear operator on the vector space $\mathbf{M}$. Using well-known properties of nilpotent linear operators on vector spaces (see [2]) we get the following properties of kernels of powers of η and of factor modules $K/\operatorname{Ker}\vartheta^{m-1}, \dots, \operatorname{Ker}\vartheta^2/\operatorname{Ker}\vartheta, \operatorname{Ker}\vartheta$.

The kernels form the chain of inclusions

$$\{\mathbf{o}\} = \operatorname{Ker}\eta^0 \subset \operatorname{Ker}\eta \subset \ldots \subset \operatorname{Ker}\eta^{r-1} \subset \operatorname{Ker}\eta^r \subset \ldots \subset \operatorname{Ker}\eta^{m-1} \subset \operatorname{Ker}\eta^m = \mathbf{M}.$$

For every subset $\{\mathbf{o}\} \subset K \subseteq \mathbf{M}$ we obtain an integer r, $1 \leqslant r \leqslant m$, such that $K \subseteq \operatorname{Ker}\eta^r \wedge K \not\subseteq \operatorname{Ker}\eta^{r-1}$. Since K is an *η-invariant* submodule we get the following chain for the operator $\vartheta = \eta \mid K$ on K:

$$\{\mathbf{o}\} = \operatorname{Ker}\vartheta^0 \subset \operatorname{Ker}\vartheta \subset \ldots \subset \operatorname{Ker}\vartheta^{r-1} \subset \operatorname{Ker}\vartheta^r = K.$$

* = modulo

These submodules as well as factor modules

$$K/\operatorname{Ker}\vartheta^{r-1}, \operatorname{Ker}\vartheta^{r-1}/\operatorname{Ker}\vartheta^{r-2}, \dots, \operatorname{Ker}\vartheta/\operatorname{Ker}\vartheta^{0}$$

may be considered vector spaces over $\mathbf{T}$.

Let $\mathbf{u}_1, \dots, \mathbf{u}_{s_0}$ be a $\mathbf{T}$-basis of K relatively to $\operatorname{Ker}\vartheta^{r-1}$. Then there exist elements of K

$$\mathbf{u}_{s_0+1}, \dots, \mathbf{u}_{s_1}, \mathbf{u}_{s_1+1}, \dots, \mathbf{u}_{s_2}, \dots, \mathbf{u}_{s_{r-2}+1}, \dots, \mathbf{u}_{s_{r-1}}$$

such that

$$\eta\mathbf{u}_1, \dots, \eta\mathbf{u}_{s_0}, \mathbf{u}_{s_0+1}, \dots, \mathbf{u}_{s_1}$$

is a $\mathbf{T}$-basis of $\operatorname{Ker}\vartheta^{r-1}$ relatively to $\operatorname{Ker}\vartheta^{r-2}$,

$$\eta^{r-k}\mathbf{u}_1, \dots, \eta^{r-k}\mathbf{u}_{s_0}, \eta^{r-k-1}\mathbf{u}_{s_0+1}, \dots, \eta^{r-k-1}\mathbf{u}_{s_1}, \dots, \mathbf{u}_{s_{r-k-1}+1}, \dots, \mathbf{u}_{s_{r-k}}$$

is a $\mathbf{T}$-basis of $\operatorname{Ker}\vartheta^{k}$ relatively to $\operatorname{Ker}\vartheta^{k-1}$, $1 < k < r-1$,

$$\eta^{r-1}\mathbf{u}_1, \dots, \eta^{r-1}\mathbf{u}_{s_0}, \eta^{r-2}\mathbf{u}_{s_0+1}, \dots, \eta^{r-2}\mathbf{u}_{s_1}, \dots, \mathbf{u}_{s_{r-2}+1}, \dots, \mathbf{u}_{s_{r-1}},$$

is a $\mathbf{T}$-basis of $\operatorname{Ker}\vartheta$.

Viewing K as a vector space we get that the union of the above set (including the basis of K relatively to $\operatorname{Ker}\vartheta^{r-1}$) forms a $\mathbf{T}$-basis of K.

I. Let $\eta^{m-k}\mathbf{u}_1, \dots, \eta^{m-k}\mathbf{u}_s$ be a $\mathbf{T}$-basis of $\operatorname{Ker}\vartheta^{k}$ relatively to $\operatorname{Ker}\vartheta^{k-1}$ for every $k = m, \dots, 1$. Then the union $\bigcup_{k=1}^{m} \{\eta^{m-k}\mathbf{u}_1, \dots, \eta^{m-k}\mathbf{u}_s\}$ is a $\mathbf{T}$-basis of K as a vector space. It follows that every $\mathbf{x} \in K$ may be written in the form

$$\mathbf{x} = \sum_{i=1}^{s}\left(\sum_{j=0}^{m-1} x_{ij}\eta^{j}\right)\mathbf{u}_i, \quad x_{ij} \in \mathbf{T}.$$

It means that $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ forms the set of generators over $\mathbf{A}$ of the submodule K.

Supposing $\sum_{i=1}^{s} \xi_i\mathbf{u}_i = \mathbf{o}$ and $\xi_i = \sum_{j=0}^{m-1} x_{ij}\eta^{j}, x_{ij} \in \mathbf{T}$, we have $\mathbf{o} = \sum_{i=1}^{s}\sum_{j=0}^{m-1} \mathbf{x}_{ij}(\eta^{j}\mathbf{u}_i)$. This yields that (for all indices) $x_{ij} = 0$ which implies $\xi_1 = \ldots = \xi_s = 0$.

We prove that $\mathbf{u}_1, \dots, \mathbf{u}_s$ is an $\mathbf{A}$-basis of K.

II. Let us suppose that $\mathbf{u}_1, \dots, \mathbf{u}_s$ form an $\mathbf{A}$-basis of K. According to Lemma 9, K is a vector space over $\mathbf{T}$ having a $\mathbf{T}$-basis

$$B = \{\mathbf{u}_1, \dots, \mathbf{u}_s, \eta\mathbf{u}_1, \dots, \eta\mathbf{u}_s, \dots, \eta^{m-1}\mathbf{u}_1, \dots, \eta^{m-1}\mathbf{u}_s\}.$$

We prove that $\{\eta^{m-k}\mathbf{u}_1, \dots, \eta^{m-k}\mathbf{u}_s\}$ is a basis of $\operatorname{Ker}\vartheta^{k}$ relatively to $\operatorname{Ker}\vartheta^{k-1}$, $k = 1, \dots, m$.

i) the linear independence (over $\mathbf{T}$) relatively to $\operatorname{Ker}\vartheta^{k-1}$: Let $\sum_{i=1}^{s} c_i\eta^{m-k}\mathbf{u}_i \in \operatorname{Ker}\vartheta^{k-1}$. Thus

$$\mathbf{o} = \eta^{k-1}\sum_{i=1}^{s} c_i\eta^{m-k}\mathbf{u}_i = \sum_{i=1}^{s} c_i(\eta^{m-1}\mathbf{u}_i).$$

As $\{\eta^{m-1}\mathbf{u}_1,\ldots,\eta^{m-1}\mathbf{u}_s\} \subseteq B$ is linearly independent over $\mathbf{T}$ we get $c_1 = \ldots = c_s = 0$.

ii) Let $\mathbf{x}\in \operatorname{Ker}\vartheta^k$, $\mathbf{x}=\sum_{i=1}^{s}\sum_{j=0}^{m-1}\mathbf{x}_{ij}\eta^j\mathbf{u}_i$. Then

$$\mathbf{o} = \eta^k\mathbf{x} = \sum_{i=1}^{s}\sum_{j=0}^{m-k-1}\mathbf{x}_{ij}(\eta^{j+k}\mathbf{u}_i).$$

Since B is linearly independent over $\mathbf{T}$ we obtain

$$x_{10} = \ldots = x_{1,m-k-1} = x_{20}\ldots = x_{2,m-k-1} = \ldots = x_{s0}\ldots = x_{s,m-k-1} = 0,$$

which implies

$$\mathbf{x} = \sum_{i=1}^{s} x_{i,m-k}\eta^{m-k}\mathbf{u}_i + \sum_{i=1}^{s}\sum_{j=m-k+1}^{m-1} x_{ij}\eta^j\mathbf{u}_i$$

where the second summand belongs to $\operatorname{Ker}\vartheta^{k-1}$. It means $\{\eta^{m-k}\mathbf{u}_1,\ldots,\eta^{m-k}\mathbf{u}_s\}$ forms a set of generators of $\operatorname{Ker}\vartheta^k$ relatively to $\operatorname{Ker}\vartheta^{k-1}$. □

11. Theorem. *Let K be a submodule of the $\mathbf{A}$-space $\mathbf{M}$ and let $\vartheta = \eta \mid K$. Then K is an $\mathbf{A}$-subspace of $\mathbf{M}$ if and only if there exists an integer s such that $s = \dim\operatorname{Ker}\vartheta^k$ relatively to $\operatorname{Ker}\vartheta^{k-1}$ for every $k = 1,\ldots,m$.*

In this case s is the $\mathbf{A}$-dimension of K.

P r o o f. Let K be an $\mathbf{A}$-subspace. Then according to the previous theorem the bases of all factor modules considered have the same number of elements and it is equal to the $\mathbf{A}$-dimension of K.

Let K be a submodule such that the factor modules considered have the same dimension. Let $\{\mathbf{u}_1,\ldots,\mathbf{u}_s\}$ be a $\mathbf{T}$-basis of K relatively to $\operatorname{Ker}\vartheta^{m-1}$. Constructing bases of factor modules $\operatorname{Ker}\vartheta^k$ relatively to $\operatorname{Ker}\vartheta^{k-1}$, $k = m-1,\ldots,1$, by the introductory part of the proof of the previous theorem we obtain that the set $\{\eta^{m-k}\mathbf{u}_1,\ldots,\eta^{m-k}\mathbf{u}_s\}$ forms a $\mathbf{T}$-basis of $\operatorname{Ker}\vartheta^k$ relatively to $\operatorname{Ker}\vartheta^{k-1}$ for all k, $1 \leqslant k \leqslant m-1$. Using the previous theorem we get that K is an s-dimensional $\mathbf{A}$-subspace of $\mathbf{M}$. □

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