

DIGRAPHS CONTRACTIBLE ONTO $*K_3$.

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Abstract. We show that any digraph on $n \geq 3$ vertices and with not less than $3n - 3$ arcs is contractible onto $*K_3$

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INTRODUCTION

The notation of contraction is well known for non-oriented graphs (cf. [4]). In this paper, Hadwiger gave the following conjecture: If $\chi(G) = p$ then G is contractible onto the complete graph on p vertices K_p . Here $\chi(G)$ denotes the chromatic number of G . Dirac [2] showed this conjecture to be true for $p \leq 4$. Wagner [11] showed that the four color theorem implies the case $p = 5$. Robertson, Seymour and Thomas [9] proved the case $p = 6$. For a good survey on the relationship between the minor's existence in G and the generalization of the coloring notion to the digraphs we refer the reader to [5], where an oriented version of Hadwiger's conjecture is given, too. Recently, Jagger [6] has shown that if p is large enough, then any digraph on n vertices having at least $10^5 p \sqrt{\log_2 p} \cdot n$ arcs is contractible onto $*K_p$. Nevertheless, this nice asymptotical result does not give a right information about the "little" cases. In this direction, Duchet and Kaneti [3] proved that any digraph on n vertices with not less than $5n - 8$ arcs is contractible onto $*K_4$. We give a short proof of the following result discovered by Meyniel [8].

Theorem. *Any digraph on $n \geq 3$ vertices and with not less than $3n - 3$ arcs is contractible onto $*K_3$, and this bound is attained for any n .*

We consider only finite digraphs without loops and parallel arcs. An arc of a digraph $G = (V(G), A(G))$ from x to y is the couple (x, y) . We say that (x, y) is *incident* to x and y . The couple of arcs (x, y) and (y, x) is called a *symmetrical* arc and is denoted by xy . We will say *edge* instead of arc whenever the orientation is insignificant. The set of *out-neighbours* (*in-neighbours*) of x is $A^+(x) = \{y \in V(G) : (x, y) \in A(G)\}$ ($A^-(x) = \{y \in V(G) : (y, x) \in A(G)\}$). $A(x) = A^+(x) \cup A^-(x)$ is the set of neighbours of x . We denote by $d^+(x)$ ($d^-(x)$) the *in-degree* (*out-degree*) and by $d(x) = d^+(x) + d^-(x)$ the *degree* of x . By contracting one arc we mean identifying its extremities and omitting the loop(s) created. We say that the digraph G is *contractible* onto G' (or G' is a *minor* of G) and we denote $G \geq G'$ if G' can be obtained from G by a sequence (possibly empty) of contractions of arcs or removing of arcs or removing of vertices. Clearly, this relation is transitive. The digraph $*K_p$ contains p vertices and a symmetrical arc between any pair of vertices. The digraph $*K_3$ is given in Fig. 1:

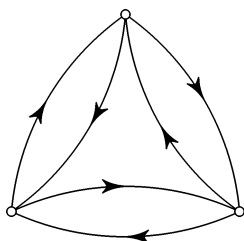


Fig. 1. The digraph $*K_3$.

PROOF AND REMARKS

Proof. Let $G = (V(G), A(G))$ be a digraph with $n = |V(G)|$ and $m = |A(G)|$. The proof is done by induction on $n + m$. The result is clearly true for $n = 3$, so we suppose that G has at least 4 vertices. If G contains a vertex x with $d(x) \leq 3$, then it is easy to see that $G' = G - x$ verifies the induction hypothesis. This means that $G' \geq *K_3$ and by the transitivity of " \geq ", we have $G \geq *K_3$. We can assume, in the following, that $d(x) \geq 4$ for every vertex x of G . If all vertices of G have a degree ≥ 6 , then G has at least $3n$ arcs and the induction hypothesis applies to the graph G' obtained from G by removing one arc. So, we can suppose that G contains at least one vertex u such that $d(u) \in \{4, 5\}$. We can also assume that the following condition is verified:

- (*) If G' is obtained from G by contraction of one arc with both its end-vertices in $A(u)$, then $|A(G')| \leq |A(G)| - 4$.

Otherwise, the induction hypothesis applies to G' .

Let us suppose that $d^+(u) \geq d^-(u)$. We can ensure this to be always the case by changing the orientation of all arcs.

Now, let $d(u) = 4$. If $|A(u)| = 2$, let x and y be the neighbours of u . G contains the symmetrical arcs ux and uy because $d(u) = 4$. The condition (*) implies the existence of the symmetrical arc xy .

If $|A(u)| = 3$, let $A(u) = \{x, y, z\}$ and suppose that there is a symmetrical arc ux . By condition (*), there must be at least one symmetrical arc either between x and y or x and z (see Fig. 2).

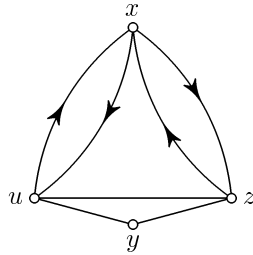


Fig. 2.

If this is not the case, then it is easy to see that for any orientation of the arcs between vertices of $A(u)$, the contraction of any arc incident to u decreases the number of arcs by at most 3. So, let xz be a symmetrical arc. By the same argument, we can see that there is at least one arc between y and z (drawn as a segment because we don't know its orientation). The graph obtained by the contraction of the edge uz must verify (*).

This implies that either the two arcs incident to y go to y , or they come out from y . But then, for any orientation of uz , by contracting either the edge uy or the edge yz , we obtain the desired $*K_3$.

Let $|A(u)| = 4$. The condition (*) implies that there is at least one arc between any pair of vertices of $A(u)$. If $A(u) = A^+(u)$, then it is clear by (*) that the graph induced by $A(u)$ is $*K_4$. If $A^+(u) = \{x, y\}$ and $A^-(u) = \{z, v\}$ then we obtain a $*K_3$ by identifying x with z and y with v . If $A^+(u) = \{x, y, z\}$ and $A^-(u) = \{v\}$, then the graph induced by $A^+(u)$ is a $*K_3$ and this completes the case $d(u) = 4$.

Suppose now that u has the degree $d(u) = 5$. Let $|A(u)| = 3$ and let x, y and z be the neighbours of u . Suppose (u, x) is not a symmetrical arc. The condition (*) implies that there is at least one symmetrical arc either between x and y or between x and z , for example between x and y , and at least one of the edges xz or xy , say xz . Then we obtain a $*K_3$ by contracting xz .

If $|A(u)| = 4$ and $A^+(u) = \{x, y, z, v\}$ and $A^-(u) = \{x\}$ then the graph G_1 induced by $u \cup A(u)$ contains at least 14 arcs and if $n = 5$ we have $14 \geq 3n - 3$. So, we can remove one of these arcs and apply the induction hypothesis to G_1 .

Suppose now that $A^+(u) = \{x, y, z\}$ and $A^-(u) = \{x, v\}$. If there is at least one symmetrical arc between x and one of y, z or v , say between x and y , then (*) implies the existence of at least one of edges vy or vz and at least one edge yz . We obtain a $*K_3$ by contracting all these edges. If there is only one edge between x and all the other vertices of $A(u)$, then there is (by (*)) a symmetrical arc yz and the edge zv . We obtain a $*K_3$ by contracting xy and zv .

If $|A(u)| = 5$ and $|A^+(u)| = 4$ or 5 then the graph G_1 induced by $u \cup A(u)$ contains at least 16 arcs and for $n = 6$ we have $16 > 3n - 3$, so the induction hypothesis applies to G_1 . Suppose that $A^+(u) = \{x, y, z\}$ and $A^-(u) = \{v, w\}$. If there is one arc (a, b) for any $a \in A^-(u)$ and for any $b \in A^+(u)$ then, by contracting (w, x) and (v, y) , we obtain a $*K_3$. On the contrary, if (w, x) is not an arc of G then G contains (by (*)) the arcs $(x, y), (x, z), (y, x), (w, y), (w, z)$ and (v, w) . We obtain a $*K_3$ by identifying the vertices of the sets $\{y, w\}$ and $\{x, z, v\}$ and this completes the proof of the inequality in the theorem. The graph G drawn in Fig. 3 is an example showing that the bound of the theorem is attained for any n .

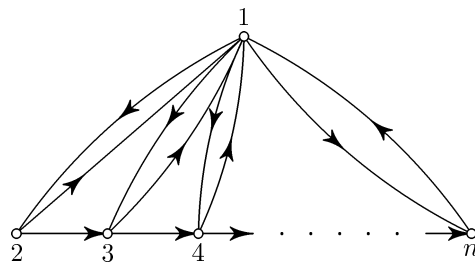


Fig. 3. The bound is attained for any n .

In this graph there is a symmetrical arc between 1 and i for $i = 2, \dots, n$ and an arc $(i, i + 1)$ for $i = 2, \dots, n - 1$. Thus, G has $3n - 4$ arcs and it is not contractible onto $*K_3$. This completes the proof of the theorem. \square

We conclude this paper with some remarks. First, by our Theorem and the result of Duchet and Kaneti [3] the following intuitive conjecture is suggested:

C o n j e c t u r e. If a digraph G on p vertices has at least $(2h - 3)p - h(h - 2)$ arcs, then G is contractible onto $*K_h$ for any integer $h \geq 3$.

We remark that this conjecture is not true for “great” valued of p . This fact is a consequence of a result from Bollobas, Catlin and Erdős [1], by taking a “great” a non-oriented graph and by replacing any edge by a symmetrical arc.

Concerning the contraction of non-oriented graphs onto cliques, Kostochka and then Thomason [10] have shown that if there is a constant c_p that any graph on n vertices having $c_p n$ edges is contractible onto K_p then $c_p = o(p\sqrt{\log p})$. Unfortunately, this bound does not apply to the digraphs.

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