A NOTE ON CONGRUENCE KERNELS IN ORTHOLATTICES

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Abstract. We characterize ideals of ortholattices which are congruence kernels. We show that every congruence class determines a kernel.

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The problem whether an ideal of a lattice $\mathcal{L}$ is a kernel of a congruence θ on $\mathcal{L}$ was solved by J. Hashimoto in the 50-ties, [2]. By his result, every ideal of $\mathcal{L}$ is a kernel of some $\theta \in \operatorname{Con} \mathcal{L}$ if and only if $\mathcal{L}$ is distributive. However, ortholattices and orthomodular lattices are distributive if and only if they are Boolean algebras. Hence, for determining whether an ideal I of an ortholattice $\mathcal{L}$ is a congruence kernel we cannot adopt Hashimoto's result. We are going to characterize such ideals by means of closedness with respect to suitable terms.

In accordance with [1], [3], by an *ortholattice* we mean an algebra

$$\mathcal{L} = (L, \vee, \wedge, ^{\perp}, 0, 1)$$

such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and $^{\perp}$ is the unary operation of *orthocomplementation*, i.e. $^{\perp}$ is order-reversing with respect to the lattice order and satisfying the following identities:

$$(x^{\perp})^{\perp} = x,$$
$$x \wedge x^{\perp} = 0 \quad \text{and} \quad x \vee x^{\perp} = 1,$$
$$(x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp} \quad \text{and} \quad (x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp},$$
$$0^{\perp} = 1 \quad \text{and} \quad 1^{\perp} = 0.$$

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Roughly speaking, ortholattices satisfy all axioms of Boolean algebras except distributivity.

By an *ideal* I of an ortholattice $\mathcal{L}$ we mean the lattice ideal of $(L, \vee, \wedge)$, i.e. $\emptyset \neq I \subseteq L$ with

$$a, b \in I \;\Rightarrow\; a \vee b \in I$$
$$a \in I,\; x \in L \;\Rightarrow\; a \wedge x \in I.$$

An example of an ortholattice which is neither distributive nor modular is shown in Fig. 1:

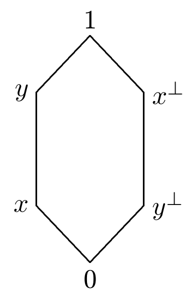

Fig. 1.

Let θ be a congruence on an ortholattice $\mathcal{L}$. By a *kernel* of θ we mean the set

$$\operatorname{Ker}\theta = \{a \in L;\; \langle a, 0\rangle \in \theta\}.$$

Remarks. (1) An ideal of an ortholattice $\mathcal{L}$ need not be a kernel of any congruence on $\mathcal{L}$. For example, $I(x) = \{x, 0\}$ is an ideal of the ortholattice in Fig. 1 but it is not a kernel of any $\theta \in \operatorname{Con}\mathcal{L}$; if $\langle x, 0\rangle \in \theta$ for $\theta \in \operatorname{Con}\mathcal{L}$ then also $\langle y, 0\rangle \in \theta$ but $y \notin I(x)$.

(2) If an ideal I of an ortholattice $\mathcal{L}$ is a kernel of some $\theta \in \operatorname{Con}\mathcal{L}$ then θ need not be unique. For example, $\{0\}$ is an ideal of $\mathcal{L}$ in Fig. 1 but it is the kernel of the identity congruence on $\mathcal{L}$ as well as of the congruence given by the partition $\{0\},\ \{x, y\},\ \{x^\perp, y^\perp\},\ \{1\}$.

For characterizing the ideals which are congruence kernels in ortholattices we recall the well-known result of A. I. Mal'cev [4]:

Proposition. *Let $\mathcal{A} = (A, F)$ be an algebra, $\emptyset \neq B \subseteq A$. B is a class of some congruence on $\mathcal{A}$ if and only if for every $c, d \in B$ and each unary polynomial $\tau(x)$ over $\mathcal{A}$, $\tau(c) \in B \;\Rightarrow\; \tau(d) \in B$.*

Recall that by a unary polynomial $\tau(x)$ over $\mathcal{A} = (A, F)$ we mean a unary function $\tau\colon A \to A$ such that there exists an $(n+1)$-ary term function $t(y, x_1, \dots, x_n)$ of type F and elements $a_1, \dots, a_n \in A$ such that $\tau(x) = t(x, a_1, \dots, a_n)$.

We are ready to formulate our first result:

Theorem 1. *An ideal I of an ortholattice $\mathcal{L}$ is a kernel of some $\theta \in \operatorname{Con}\mathcal{L}$ if and only if for each $(n+1)$-ary term t, for every $a_1, \dots, a_n \in L$ and every $i_1, i_2, i_3 \in I$ we have $(i_1^{\perp} \wedge t(i_2, a_1, \dots, a_n))^{\perp} \wedge t(i_3, a_1, \dots, a_n) \in I$.*

Proof. Let I be a kernel of some $\theta \in \operatorname{Con}\mathcal{L}$, let t be an $(n+1)$-ary term of $\mathcal{L}$ and $a_1, \dots, a_n \in L$, $i_1, i_2, i_3 \in I$. Since $0 \in I$ we have $\langle i_1, 0\rangle \in \theta$, $\langle i_2, 0\rangle \in \theta$, $\langle i_3, 0\rangle \in \theta$. Moreover,

$$(0^{\perp} \wedge t(0, a_1, \dots, a_n))^{\perp} \wedge t(0, a_1, \dots, a_n) = 0,$$

whence, by the substitution property of θ, also

$$\begin{aligned}\langle (i_1^{\perp} \wedge t(i_2, a_1, \ \dots, a_n))^{\perp} \wedge t(i_3, a_1, \dots, a_n), 0\rangle = \\ \langle (i_1^{\perp} \wedge t(i_2, a_1, \dots, a_n))^{\perp} \wedge t(i_3, a_1, \dots, a_n), \\ (0^{\perp} \wedge t(0, a_1, \dots, a_n))^{\perp} \wedge t(0, a_2, \dots, a_n)\rangle \in \theta\end{aligned}$$

i.e. $(i_1^{\perp} \wedge t(i_2, a_1, , \dots, a_n))^{\perp} \wedge t(i_3, a_1, \dots, a_n) \in \operatorname{Ker}\theta = I$.

Conversely, let I be an ideal of an ortholattice $\mathcal{L}$ which satisfies the condition of Theorem 1. Suppose $i, j \in I$ and $\tau(i) \in I$ for a unary polynomial $\tau(x)$ over $\mathcal{L}$. Hence, $\tau(x) = t(x, a_1, \dots, a_n)$ for some $(n+1)$-ary term t and some elements $a_1, \dots, a_n \in L$. Applying our condition for $i_1 = \tau(i)$, $i_2 = i$, $i_3 = j$, we obtain

$$\tau(j) = (\tau(i)^{\perp} \wedge \tau(i))^{\perp} \wedge \tau(j) \in I.$$

By the Proposition, we are done since I is a 0-class of some $\theta \in \operatorname{Con}\mathcal{L}$, i.e. $I = \operatorname{Ker}\theta$. □

Theorem 2. *Let $\mathcal{L}$ be an ortholattice. Then for each $\theta \in \operatorname{Con}\mathcal{L}$, the kernel $\operatorname{Ker}\theta$ is determined by every class of θ.*

Proof. Let $\theta \in \operatorname{Con}\mathcal{L}$ and let C be an arbitrary class of θ. Define a subset I of $\mathcal{L}$ as follows:

$(*)$ $a \in I$ iff there exists $c \in C$ such that $a \wedge c = 0$ and $a \vee c \in C$. We prove that $I = \operatorname{Ker}\theta$.

(i) $0 \in I$ since $c \wedge 0 = 0$ and $c \vee 0 = c \in C$ for each $c \in C$.

(ii) Let $a \in I$. Denote $d = a \vee c$. Then $c, d \in C$ imply $\langle c, d\rangle \in \theta$ and $d \wedge a = (a \vee c) \wedge a = a$ whence $\langle a, 0\rangle = \langle d \wedge a, c \wedge a\rangle \in \theta$, i.e. $a \in \operatorname{Ker}\theta$.

(iii) Suppose $a \in \operatorname{Ker}\theta$. Then $\langle a, 0\rangle \in \theta$, thus also $\langle a^\perp, 1\rangle = \langle a^\perp, 0^\perp\rangle \in \theta$. Hence, for each $c_0 \in C$ we have $\langle c_0, a^\perp \wedge c_0\rangle = \langle 1 \wedge c_0, a^\perp \wedge c_0\rangle \in \theta$, i.e. also $a^\perp \wedge c_0 \in C$. Further,

$$\langle a^\perp \wedge c_0, (a^\perp \wedge c_0) \vee a\rangle = \langle (a^\perp \wedge c_0) \vee 0, (a^\perp \wedge c_0) \vee a\rangle \in \theta,$$

i.e. also $(a^\perp \wedge c_0) \vee a \in C$. We can set $c = a^\perp \wedge c_0$. Then $c \in C$, $c \wedge a = a^\perp \wedge c_0 \wedge a = 0$ and $c \vee a = (a^\perp \wedge c_0) \vee a \in C$. By $(*)$ we have $a \in I$. Together, $I = \operatorname{Ker}\theta$, which proves the assertion. □

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