UNIFORMITY OF CONGRUENCES IN COHERENT VARIETIES

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Abstract. An algebra $\mathcal{A}$ is uniform if for each $\theta \in \operatorname{Con} \mathcal{A}$, every two classes of θ have the same cardinality. It was shown by W. Taylor that coherent varieties need not be uniform (and vice versa). We show that every coherent variety having transferable congruences is uniform.

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Following [8], an algebra $\mathcal{A}$ is *uniform* if for each $\theta \in \operatorname{Con} \mathcal{A}$, every two classes of θ have the same cardinality. A variety $\mathcal{V}$ is *uniform* if each $\mathcal{A} \in \mathcal{V}$ has this property.

The well-known examples of uniform algebras are groups, rings, quasigroups and Boolean algebras. It was proved by W. Taylor [8] that the class of uniform varieties is not definable by a Mal'cev condition.

Recall the following concepts: A variety $\mathcal{V}$ is

—*regular* if every two congruences on each $\mathcal{A} \in \mathcal{V}$ coincide whenever they have a class in common,

—*permutable* if $\theta \cdot \varphi = \varphi \cdot \theta$ for every two congruences θ, φ on each $\mathcal{A} \in \mathcal{V}$,

—*coherent* if for each $\mathcal{A} \in \mathcal{V}$, any subalgebra of $\mathcal{A}$ containing a congruence class of some $\theta \in \operatorname{Con} \mathcal{A}$ is a union of classes of θ.

The relationship between regularity, permutability and coherence was investigated in [2]. For uniformity, the following results are known:

Proposition 1. (see e.g. [8]) *Every uniform variety is regular.*

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An easy proof is based on the fact (see [9]) that a variety $\mathcal{V}$ is regular if and only if for each $\mathcal{A} = (A, F) \in \mathcal{V}$, if $\theta \in \operatorname{Con} \mathcal{A}$ has a one element class then θ is the identity relation on A. The following result was proved by D. M. Clark and P. H. Krauss, see [6] or [7]:

Proposition 2. *If $\mathcal{V}$ is a locally finite variety all of whose finite algebras are uniform then $\mathcal{V}$ is permutable.*

The aforementioned results motivated us to search for uniform varieties among those which are regular and permutable. The best candidate are coherent varieties:

Proposition 3. (D. Geiger [5]) *Every coherent variety is regular and permutable.*

The converse of Proposition 3 does not hold, see [2]. The following condition must be added to regularity and permutability to obtain coherency (see [2]):

An algebra $\mathcal{A} = (A, F)$ has *subalgebras closed under translation of principal congruences* (briefly $\mathcal{A}$ is CUT) if for every subalgebra $\mathcal{B}$ of $\mathcal{A}$ all $x \in A$, $y, z \in B$ and every polynomial φ over $\mathcal{A}$

$$[z]_\theta \subseteq B \quad \text{and} \quad \varphi(z, \ldots, z) = y \quad \text{imply} \quad \varphi([z]_\theta) \subseteq B; \quad \theta = \theta(x, y)$$

(for the sake of brevity, $\varphi(C)$ denotes the set $\{\varphi(c_1, \ldots, c_n); c_i \in C\}$). A variety $\mathcal{V}$ is CUT if each $\mathcal{A} \in \mathcal{V}$ has this property. Recall the main result of [2]:

Proposition 4. *For a variety $\mathcal{V}$ the following conditions are equivalent:*

(1) *$\mathcal{V}$ is coherent;*

(2) *$\mathcal{V}$ is regular, permutable and CUT;*

(3) *there exists an $(n+1)$-ary term h and ternary terms $t_1, \ldots, t_n$ such that $t_i(x, x, z) = z$ for $i = 1, \ldots, n$ and*

$$h(y, t_1(x, y, z), \ldots, t_n(x, y, z)) = x$$

Remarks. (a) The equivalence (1) $\Leftrightarrow$ (3) of Proposition 4 was proved by D. Geiger [5].

(b) It was shown in [2] that CUT, regularity and permutability are independent conditions. As was pointed out by W. Taylor (Theorem 4.2 in [8]), there are coherent varieties which are not uniform and there are uniform varieties which are not coherent. We are going to add one more condition to coherency to obtain uniformity. To this end, let us recall from [1]:

An algebra $\mathcal{A} = (A, F)$ has *transferable congruences* if for every $a, b, c \in A$ there exists $d \in A$ such that $\theta(a, b) = \theta(c, d)$. A variety $\mathcal{V}$ has *transferable congruences* if each $\mathcal{A} \in \mathcal{V}$ has this property.

Theorem 1. *For a variety $\mathcal{V}$ the following conditions are equivalent:*
(1) *$\mathcal{V}$ is coherent and has transferable congruences;*
(2) *$\mathcal{V}$ is permutable, CUT and has transferable congruences;*
(3) *there exist ternary terms t, h such that*

$$t(x, x, z) = z \quad \text{and} \quad h(y, z, t(x, y, z)) = x$$

Proof. (1) $\Rightarrow$ (2) by Proposition 4. Prove (2) $\Rightarrow$ (3). Let $\mathcal{A} = F_v(x, y, z)$ be the free algebra of $\mathcal{V}$ with three free generators x, y, z. Let $\theta(x, y) \in \operatorname{Con} \mathcal{A}$. Since $\mathcal{V}$ has transferable congruences, there exists an element d of $\mathcal{A}$ with $\theta(x, y) = \theta(z, d)$. Since $\mathcal{A} = F_v(x, y, z)$, we have $d = t(x, y, z)$ for some ternary term t. However, $\langle z, t(x, y, z)\rangle \in \theta(x, y)$ and $\mathcal{A}/\theta(x, y) \cong F_v(x, z)$ is the free algebra of $\mathcal{V}$ again, thus $t(x, x, z) = z$ in $\mathcal{V}$.

Denote by $R(a, b)$ the least reflexive and compatible relation on $\mathcal{A}$ containing the pair $\langle a, b\rangle$. According to [10], permutability of $\mathcal{V}$ yields $\theta(a, b) = R(a, b)$. Hence,

$$\theta(x, y) = \theta(z, t(x, y, z)) \quad \text{implies} \quad \langle y, x\rangle \in R(z, t(x, y, z)),$$

i.e. there exists a unary polynomial φ over $\mathcal{A}$ such that $x = \varphi(t(x, y, z))$ and $y = \varphi(z)$. Consider a subalgebra $\mathcal{B}$ of $\mathcal{A}$ generated by the set $\{y, z, t(x, y, z)\}$. Since $\mathcal{V}$ is CUT and $y, z \in \mathcal{B}$, $\varphi(z) = y$ thus also $x = \varphi(t(x, y, z)) \in \mathcal{B}$. Hence, there exists a ternary term h such that

$$x = h(y, z, t(x, y, z)).$$

(3) $\Rightarrow$ (1): Let $\mathcal{A} = (A, F) \in \mathcal{V}$ and $a, b, c \in A$. Then

$$\begin{aligned}
\langle c, t(a, b, c)\rangle &= \langle t(a, a, c), t(a, b, c)\rangle \in \theta(a, b),\\
\langle a, b\rangle &= \langle h(b, c, t(a, b, c)), h(b, c, t(b, b, c))\rangle\\
&= \langle h(b, c, t(a, b, c)), h(b, c, c)\rangle \in \theta(c, t(a, b, c))
\end{aligned}$$

i.e. $\theta(a, b) = \theta(c, t(a, b, c))$ and $\mathcal{V}$ has transferable congruences.

Further, set $t_1(x, y, z) = z$ and $t_2(x, y, z) = t(x, y, z)$. Then $t_i(x, x, z) = z$ for $i = 1, 2$ and

$$h(y, t_1(x, y, z), t_2(x, y, z)) = h(y, z, t(x, y, z)) = x,$$

thus, by Proposition 4, $\mathcal{V}$ is coherent. $\square$

E x a m p l e s. (1) For groups, we can set $t(x,y,z) = y^{-1} \cdot x \cdot z$ and $h(x,y,z) = x \cdot z \cdot y^{-1}$. Then, of course,

$$t(x,x,z) = x^{-1} \cdot x \cdot z = z \quad \text{and} \quad h(y,z,t(x,y,z)) = y(y^{-1}xz)z^{-1} = x.$$

(2) Analogously, for rings we can take $t(x,y,z) = x - y + z = h(x,y,z)$. Since every Boolean algebra is term equivalent to a Boolean ring, we can use these terms also. Of course, the operation $+$ is the symmetrical difference, i.e. $a + b = a - b = (a \wedge b') \vee (a' \wedge b)$.

(3) Analogously, we can choose the terms t and h for quasigroups.

Theorem 2. *If $\mathcal{V}$ is a coherent variety with transferable congruences then $\mathcal{V}$ is uniform.*

P r o o f. Let $\mathcal{A} = (A, F) \in \mathcal{V}$ and $\theta \in \operatorname{Con} \mathcal{A}$. Consider two classes B and C of θ. Let $y \in B$ and $z \in C$. We apply the terms t and h of Theorem 1 to construct mappings $\tau(x) = t(x,y,z)$ and $\varphi(v) = h(y,z,v)$. Of course, for $x \in B$ we have $\langle x,y \rangle \in \theta$, thus also $\langle t(x,y,z), z \rangle = \langle t(x,y,z), t(y,y,z) \rangle \in \theta$, i.e. $\tau\colon B \to C$. By (3) of Theorem 1, $x = \varphi(\tau(x))$ for each x of $\mathcal{A}$, thus τ is an injective mapping of B into C. Hence $| B | \leqslant | C |$. The converse can be shown analogously, i.e. $| B | = | C |$. $\square$

We proceed to generalize the foregoing result to the local case. We say that $\mathcal{A}$ is an *algebra with* 0 if 0 is either a nullary operation of $\mathcal{A}$ or an equationally defined constant, i.e. a nullary term operation of $\mathcal{A}$. Recall the following concept from [3]:

An algebra $\mathcal{A}$ with 0 is 0-*locally uniform* if $\operatorname{card} [a]_\theta \leqslant \operatorname{card} [0]_\theta$ for each $\theta \in \operatorname{Con} \mathcal{A}$ and every element a of $\mathcal{A}$. A variety $\mathcal{V}$ with 0 is 0-*locally uniform* if each $\mathcal{A} \in \mathcal{V}$ has this property. For investigation of 0-local uniformity, we use the following concepts adopted from [1], [4]:

An algebra $\mathcal{A}$ with 0 is *weakly coherent* if for each $\theta \in \operatorname{Con} \mathcal{A}$ and every subalgebra $\mathcal{B}$ of $\mathcal{A}$,

$$[0]_\theta \subseteq B \quad \text{implies} \quad [b]_\theta \subseteq B \quad \text{for each} \quad b \in B.$$

$\mathcal{A}$ has 0-*transferable congruences* if for each a, b of $\mathcal{A}$ there is c of $\mathcal{A}$ such that $\theta(a,b) = \theta(0,c)$. A variety $\mathcal{V}$ with 0 is weakly coherent or has 0-transferable congruences if each $\mathcal{A} \in \mathcal{V}$ has this property.

One can find Mal'cev conditions characterizing the aforementioned congruence conditions in [1] and [4]. Especially, every weakly coherent variety is permutable and weakly regular.

The proof of the following theorem is word for word analogous to that of Theorem 1 and hence omitted:

Theorem 3. *Let $\mathcal{V}$ be a variety with* 0. *$\mathcal{V}$ is weakly coherent and has* 0-*transferable congruences if and only if there exist binary terms d, b such that $b(x,x) = 0$ and $d(y, b(x,y)) = x$.*

Setting $z = 0$ in the proof of Theorem 2 and applying the mappings $\mu(x) = b(x,y)$, $\psi(v) = d(y,v)$ instead of τ and φ, we obtain the proof of

Theorem 4. *If $\mathcal{V}$ is a variety with* 0 *which is weakly coherent and has* 0-*transferable congruences then $\mathcal{V}$ is* 0-*locally uniform.*

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