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ON CONDENSING DISCRETE DYNAMICAL SYSTEMS

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Dedicated to the memory of M. A. Krasnosel'skij

Abstract. In the paper the fundamental properties of discrete dynamical systems generated by an α-condensing mapping (α is the Kuratowski measure of noncompactness) are studied. The results extend and deepen those obtained by M. A. Krasnosel'skij and A. V. Lusnikov in [21]. They are also applied to study a mathematical model for spreading of an infectious disease investigated by P. Takáč in [35], [36].

Keywords: condensing discrete dynamical system, stability, singular interval, continuous branch connecting two points, continuous curve

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Introduction

By his work, M. A. Krasnosel'skij has immensely influenced the developement of nonlinear functional analysis. This can be seen in his books, see e.g. [18], [19], [20]. Among others, he investigated the problem, when an operator has a continuum of fixed points. This problem has been solved by several methods. Some of them have been developed within the theory of differential equations.

The first method studied a continuum of solutions of the initial value problem for ordinary differential systems and was originated by H. Kneser in 1923 (see [9, p. 212]). There are several papers dealing with this problem, among them let us mention [12]. The general setting of this method was given by Z. Kubáček in [22], [23] and in [38].

M. A. Krasnosel'skij and A. I. Perov in [17] started another method which represents a combination of the previous one with the theory of fixed point index (see

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[17], [19] and [42, p. 564]). An extension of this method was given by B. Rudolf in [30]. M. A. Krasnosel'skij and A. V. Lusnikov proposed a modification of this method in [21] and B. Rudolf completed it in [32].

The existence of a compact convex set of solutions of a boundary value problem was investigated by B. Rudolf and Z. Kubáček in [33]. In a more general setting it was established by V. Šeda, J. J. Nieto, M. Gera in [37] and in [39].

The last method to show the existence of a continuous curve of equilibria appeared in the papers [35], [36] by P. Takáč and in [16] by P. Hess on discrete dynamical systems. The systems are generated by a mapping which is, roughly speaking, completely continuous. It is also strongly increasing.

The aim of this paper is twofold. First, to investigate fundamental properties of discrete dynamical systems generated by an α-condensing mapping (α is the Kuratowski measure of noncompactness). Secondly, to extend and to deepen the results by M. A. Krasnosel'skij and A. V. Lusnikov in [21]. Among the results attained it has been shown that in each partially ordered Banach space a compact continuous branch (the notion has been introduced by M. A. Krasnosel'skij and A. V. Lusnikov in [21]) contains a continuum (Lemma 8) and each continuum with the smallest and the greatest element contains a continuous curve connecting these two elements (Theorem 3). The results have been applied to a study of a mathematical model for spreading of an infectious disease (compare with [35], [36]).

The paper is organized as follows: In the first part the condensing discrete dynamical systems are studied in a complete metric space. In this space three important sets M_1, M_2 and M_3 are specified and the relations between them are studied. Then this study is continued in a Fréchet space where a convex set C_2 plays an important role.

In the second part the condensing dynamical systems are studied in a partially ordered Banach space. The study of these systems is based on Lemma 7 and Theorem 4. Another important result is contained in Lemma 11. Theorems 7 and 8 guarantee the existence of a continuous curve of equilibria.

Part 3 deals with an application of the previous results to a τ-periodic Kamke system. The existence of a continuum of τ-periodic solutions of that system depends on their stability.

First we recall the definition of the Kuratowski measure of noncompactness and the definition of the α-condensing mapping. (Compare with [9, pp. 41 and 69]).

Let (E, ϱ) be a complete metric space and $\mathcal{B}$ the set of all bounded subsets of E. Then $\alpha\colon \mathcal{B} \to \mathbb{R}^+$ defined by

$$\alpha[B] = \inf\{d > 0\colon\ B \text{ admits a finite cover by sets of diameter} \leqslant d\}$$

is called the *Kuratowski measure of noncompactness.*

Further, let α be the Kuratowski measure of noncompactness, $\emptyset \neq M \subset E$, let $T\colon M \to E$ be continuous and bounded, i.e. T maps bounded subsets of M into bounded sets. Then T is said to be *α-condensing* if

$$\alpha[T(B)] < \alpha[B]$$

whenever $B \subset M$ is bounded and $\alpha[B] > 0$.

By Lemma 1.6.11 [1, p. 41] and Remark 1.6.13 [1, p. 43] we get

Proposition 1. *Let (E, ϱ) be a complete metric space, $\emptyset \neq M$ a closed bounded set in E, α the Kuratowski measure of noncompactness and $T\colon M \to M$ an α-condensing mapping. Then*

$$\lim_{k\to\infty} \alpha[T^k(M)] = 0.$$

Proposition 2. ([24, pp. 6, 111]) *Let (E, ϱ) be a complete metric space and α the Kuratowski measure of noncompactness. If $\{F_k\}_{k=1}^{\infty}$ is a decreasing sequence (that is, $F_1 \supset F_2 \supset \ldots$) of nonempty, closed sets such that*

$$\lim_{k\to\infty} \alpha[F_k] = 0,$$

then $\bigcap\limits_{k=1}^{\infty} F_k$ is a nonempty and compact set. Moreover, if all F_k are nonempty, closed and connected sets, then $\bigcap\limits_{k=1}^{\infty} F_k$ is a nonempty, compact and connected set.

Our considerations will be based on the following assumption

(H1) *Let (E, ϱ) be a complete metric space, $\emptyset \neq M$ a closed, bounded and connected set in E and*

$$T\colon M \to M$$

an α-condensing mapping.

For $x \in M$ let

$$\gamma^+(x) := \{T^k(x)\colon\ k = 0, 1, 2, \ldots\},\ T^0(x) := x$$

be the *positive semiorbit of* x and

$$\omega(x) := \{w \in E\colon\ \exists\, k_l \to \infty \text{ such that } T^{k_l}(x) \to w \text{ as } l \to \infty\}$$

the *ω-limit set of x.*

If $\emptyset \neq A \subset M$, then

$$\gamma^+(A) := \bigcup_{x \in A} \gamma^+(x),\ \omega(A) := \bigcup_{x \in A} \omega(x).$$

A set $\emptyset \neq A \subset M$ is called *invariant* (*positively invariant*) if $T(A) = A$ ($T(A) \subset A$). A point $x \in M$ is *k-periodic* ($k \geqslant 2$) if $T^k(x) = x$. A set A is called a *k-cycle* if $A = \gamma^+(x)$ for some k-periodic point x. Any fixed point of T is also called *equilibrium.* The set of all equilibria (the union of all cycles) will be denoted by F_p (C).

Further, for a given sequence of sets $A_k \subset E$, $k = 1, 2, \ldots$ let

$$\mathop{\underline{\lim}}_{k \to \infty} A_k := \{x \in E\colon\ \exists a_k \in A_k \text{ such that } \lim_{k \to \infty} a_k = x\}$$

be the *lower limit of the sequence* $\{A_k\}_{k=1}^{\infty}$, and

$$\mathop{\overline{\lim}}_{k \to \infty} A_k := \{x \in E\colon\ \exists k_l \to \infty \text{ and a sequence } \{a_{k_l}\} \text{ such that } a_{k_l} \in A_{k_l} \text{ and } a_{k_l} \to x \text{ as } l \to \infty\}$$

the *upper limit of the sequence* $\{A_k\}_{k=1}^{\infty}$.

Proposition 3. ([6, p. 54]) *The following statements hold:*
(i) $\mathop{\underline{\lim}}_{k \to \infty} A_k = \mathop{\underline{\lim}}_{k \to \infty} \overline{A}_k$, $\mathop{\overline{\lim}}_{k \to \infty} A_k = \mathop{\overline{\lim}}_{k \to \infty} \overline{A}_k$;
(ii) *the sets* $\mathop{\underline{\lim}}_{k \to \infty} A_k$ *and* $\mathop{\overline{\lim}}_{k \to \infty} A_k$ *are closed;*
(iii) $\bigcap_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} A_i \subset \mathop{\underline{\lim}}_{k \to \infty} A_k \subset \mathop{\overline{\lim}}_{k \to \infty} A_k \subset \bigcap_{k=1}^{\infty} \overline{\bigcup_{i=k}^{\infty} A_i} \subset \bigcup_{k=1}^{\infty} \overline{A_k}$.

Lemma 1. *Under assumption* (H1) *the set*

$$M_1 := \bigcap_{k=1}^{\infty} \overline{T^k(M)} \tag{1}$$

has the following properties:

(i) $\emptyset \neq M_1 \subset M$ *and* M_1 *is compact and connected;*
(ii) $M_1 = \varliminf_{k\to\infty} T^k(M) = \varlimsup_{k\to\infty} T^k(M)$*;*
(iii)

$$T(M_1) \subset M_1. \tag{2}$$

P r o o f. Since $T(M) \subset M$ and M is closed, $M_1 \subset M$. As $\{\overline{T^k(M)}\}_{k=1}^{\infty}$ is a decreasing sequence of nonempty, closed and connected sets, and by Proposition 1 we have $\lim_{k\to\infty} \alpha[\overline{T^k(M)}] = \lim_{k\to\infty} \alpha[T^k(M)] = 0$, Proposition 2 implies statement (i).

By Proposition 3,

$$M_1 = \varliminf_{k\to\infty} \overline{T^k(M)} = \varlimsup_{k\to\infty} \overline{T^k(M)} = \varliminf_{k\to\infty} T^k(M) = \varlimsup_{k\to\infty} T^k(M) \tag{3}$$

and hence, (ii) is proved.

(2) follows from the inclusions

$$T\Big(\bigcap_{k=1}^{\infty} \overline{T^k(M)}\Big) \subset \bigcap_{k=1}^{\infty} T(\overline{T^k(M)}) \subset \bigcap_{k=1}^{\infty} \overline{T^{k+1}(M)}$$

where the continuity of T has been used. □

Definition 1. The point $z \in M$ will be called stable with respect to a set A, $\emptyset \neq A \subset M$, if $z \in A$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that the implication

$$\varrho(x,z) < \delta \Rightarrow \varrho(T^k(x), T^k(z)) < \varepsilon \text{ for each } x \in A \text{ and for each } k = 0,1,\ldots$$

holds.

Stability with respect to M is simply called *stability.*

Now we will deal with the properties of the ω limit sets. The following general property of these sets has been given in [5, Lemma 3, p. 71].

Proposition 4. *Let X be a compact metric space and let $T\colon X \to X$ be a continuous map of this space into itself. If $L = \omega(x)$ is a limit set and if S is a non-empty proper closed subset of L, then*

$$S \cap \overline{T(L \setminus S)} \neq \emptyset. \tag{4}$$

This proposition can be sharpened. By using a modification of its proof, the following lemma can be proved.

Lemma 2. *Let (X, d) be a metric space and $T\colon X \to X$ a continuous map. If $L = \omega(x)$ is a limit set, which is compact and invariant, and S is a non-empty proper closed subset of L, then* (4) *is true. In particular, if L is finite, then it is either a cycle or an equilibrium.*

P r o o f. Suppose that S and $\overline{T(L \setminus S)}$ are disjoint. Since both S and $\overline{T(L \setminus S)}$ are compact, there exists an $\varepsilon > 0$ such that the ε-neighbourhoods $U(S, \varepsilon)$ and $U(\overline{T(L \setminus S)}, \varepsilon)$ of the sets S and $\overline{T(L \setminus S)}$, respectively, satisfy

$$\overline{U(S,\varepsilon)} \cap \overline{U(\overline{T(L \setminus S)}, \varepsilon)} = \emptyset.$$

Put $G_2 = U(S, \varepsilon)$. Further, for each $z \in L \setminus S$ there exists $\delta(z) > 0$ such that for each $y \in X$, $d(z, y) < \delta(z) \Rightarrow d(T(z), T(y)) < \varepsilon$ and hence $T(y) \in U(\overline{T(L \setminus S)}, \varepsilon)$.

Consider the set $G_1 = \bigcup\limits_{z \in L \setminus S} U(z, \delta(z))$. Then $T(\overline{G_1}) \subset \overline{U(\overline{T(L \setminus S)}, \varepsilon)}$. Thus G_1, G_2 are open sets such that $L \setminus S \subset G_1$, $S \subset G_2$ and

$$\overline{G_2} \cap T(\overline{G_1}) = \emptyset. \tag{5}$$

All terms $T^k(x)$ with sufficiently large index k belong either to G_1 or to G_2 and there are subsequences belonging to each of them. Hence there is a subsequence $\{k_l\} \subset \mathbb{N}$ such that $T^{k_l}(x) \in G_1$ and $T^{k_l+1}(x) \in G_2$. If y is a limit point of $\{T^{k_l}(x)\}$, then $y \in \overline{G_1}$ and $T(y) \in \overline{G_2}$, which contradicts (5). $\square$

Under hypothesis (H1) the properties of $\omega(x)$ are given by

Lemma 3. *If assumption* (H1) *is fulfilled, then for each $x \in M$ the following statements are true:*

(i) *$\gamma^+(x)$ is relatively compact.*

(ii) *$\omega(x)$ is a nonempty, compact subset of M_1 and*

$$T(\omega(x)) = \omega(x). \tag{6}$$

(iii) *If S is a non-empty proper closed subset of $\omega(x)$, then* (4) *is true with $L = \omega(x)$. Especially, if $\omega(x)$ is finite, then it is either a cycle or an equilibrium.*

(iv)

$$\bigcup_{y \in \omega(x)} \omega(y) \subset \omega(x). \tag{7}$$

(v) *If $z \in \omega(x)$ and z is stable with respect to $\overline{T^{k_0}(M)}$ for some $k_0 \in \mathbb{N}$, then*

$$\omega(x) = \omega(z).$$

In particular, if $z \in \omega(x)$ is a stable equilibrium, then $\omega(x) = \{z\}$.

(vi) *If $\omega(x)$ is finite or there exists a point $z \in \omega(x)$ which is stable with respect to $\overline{T^{k_0}(M)}$ for some $k_0 \in \mathbb{N}$, then*

$$\omega(x) = \bigcup_{y \in \omega(x)} \omega(y). \tag{8}$$

P r o o f. Let $x \in M$ be arbitrary but fixed.

(i) If $\gamma^+(x)$ were not relatively compact, then we would have

$$\alpha[\gamma^+(x)] = \alpha[\{x\} \cup T(\gamma^+(x))] = \alpha[T(\gamma^+(x))] < \alpha[\gamma^+(x)],$$

which is a contradiction.

(ii) Relative compactness of $\gamma^+(x)$ implies that $\omega(x) \neq \emptyset$. By the definition of $\overline{\lim\limits_{k\to\infty}} T^k(M)$ and by (3) we get that $\omega(x) \subset M_1$. By the equivalent definition of $\omega(x)$ in [5, p. 70], $\omega(x) = \bigcap\limits_{j=0}^{\infty} (\overline{\bigcup\limits_{k=j}^{\infty} T^k(x)})$ and hence $\omega(x)$ is closed. Since $\omega(x) \subset M_1$ and M_1 is compact, $\omega(x)$ is also compact. It is clear that $T(\omega(x)) \subset \omega(x)$. To prove the inverse inclusion, we consider an arbitrary point $w = \lim\limits_{l\to\infty} T^{k_l}(x) \in \omega(x)$. Then the sequence $T^{k_l-1}(x)$ has a subsequence $T^{k_m-1}(x)$ which converges to $z \in \omega(x)$ and $w = \lim\limits_{m\to\infty} T^{k_m}(x) = \lim\limits_{m\to\infty} T(T^{k_m-1}(x)) = T(z)$. Hence $\omega(x) \subset T(\omega(x))$.

(iii) The statement follows from Lemma 2.

(iv) Statement (ii) implies (7).

(v) Clearly $\omega(z) \subset \omega(x)$. If $z \in \omega(x)$ is stable with respect to $\overline{T^{k_0}(M)}$ for a $k_0 \in \mathbb{N}$ and $y \in \omega(x)$ is an arbitrary but fixed element, then there exist two increasing sequences $\{l_k\}$ and $\{m_k\}$ of natural numbers tending to ∞ such that

$$\lim_{k\to\infty} T^{l_k}(x) = y, \quad \lim_{k\to\infty} T^{m_k}(x) = z.$$

Choosing a suitable subsequence of $\{l_k\}$ and denoting it again by $\{l_k\}$ we can assume that

$$2m_k < l_k, \qquad k = 1, 2, \ldots.$$

Let

$$n_k = l_k - m_k, \qquad k = 1, 2, \ldots.$$

Then

$$\varrho(T^{n_k}(z), y) \leqslant \varrho(T^{n_k}(z), T^{n_k}(T^{m_k}(x))) + \varrho(T^{l_k}(x), y)$$

and hence $\lim\limits_{k\to\infty} \varrho(T^{n_k}(z), y) = 0$. So $y \in \omega(z)$ and (8) is true.

(vi) If $\omega(x)$ is finite, then it is a cycle or an equilibrium. Hence, (8) is true. The rest of the proof follows from statement (v). $\square$

Lemma 4. *Under assumption* (H1) *the set*

$$M_3 := \omega(M) \tag{9}$$

is a nonempty, relatively compact subset of M_1 *such that*

$$T(M_3) = M_3. \tag{10}$$

Moreover, $\omega(M_3) \subset M_3$ *and* $\omega(M_3)$ *contains all equilibria and cycles. If each point* x *of* $M_3 \setminus (F_p \cup C)$ *is stable with respect to* $\overline{T^{k_0}(M)}$ *where* k_0 *depends on* x*, then*

$$\omega(M_3) = M_3.$$

Proof. Lemma 3 implies that M_3 is a nonempty subset of M_1 and by Lemma 1, M_3 is relatively compact. Further, (6) implies that

$$T(M_3) = \bigcup_{x \in M} T(\omega(x)) = M_3.$$

As $M_3 \subset M$, we have the inclusion $\omega(M_3) \subset M_3$. Clearly all equilibria and all cycles belong to M_3 and by (8) also to $\omega(M_3)$. Again, by Lemma 3, if each point of $M_3 \setminus (F_p \cup C)$ is stable in the sense given above, then $\omega(M_3) = M_3$. □

Remark 1. By virtue of (10), the set

$$C_T := \bigcap_{k=0}^{\infty} T^k(M)$$

called the *center of* T ([13, p. 213]) is nonempty, $M_3 \subset C_T \subset M_1$ and hence C_T is relatively compact.

Now we shall study the properties of the multifunction ω determined by the relation $x \mapsto \omega(x)$ for every $x \in M$.

Let (E, ϱ) be a metric space (not necessarily complete) and let $F\colon D \subset E \to 2^E \setminus \{\emptyset\}$ be a multifunction. We recall that $F(D_0) = \bigcup_{x \in D_0} F(x)$ for $D_0 \subset D$ and the graph of F is $G(F) = \{(x, y) \in D \times E\colon x \in D,\ y \in F(x)\}$. Further, by Definition 4′, [34, pp. 1057–1058], F is *closed at a point* $x \in D$ if and only if the following implication holds:

If $\{x_k\}$ and $\{y_k\}$ are two sequences in E such that

$$\{x_k\} \subset D,\ \lim_{k\to\infty} x_k = x,\ y_k \in F(x_k), \quad k = 1, 2, \ldots,\ \lim_{k\to\infty} y_k = y, \tag{11}$$

then

$$y \in F(x). \tag{12}$$

By [9, p. 299], F is *upper semicontinuous* (usc for short) *at a point* $x_0 \in D$ if for an arbitrary open set $V \supset F(x_0)$ there exists a neighbourhood $U(x_0)$ of the point x_0 such that V includes $F(x)$ for each $x \in U(x_0) \cap D$. F is usc in D iff F is usc at every $x_0 \in D$. See also [34] and [15].

Some properties of usc multifunctions are given by

Proposition 5. (See Proposition 24.1, [9, p. 300], Theorem 2.3 [8, p. 381], Theorem 7, [34, p. 1059] and Definition 3′, [34, p. 1056]). *The following statements are true:*

(a) *Let $F(x)$ be closed for all $x \in D$. If F is usc in D and D is closed, then the graph of F is closed. If $\overline{F(D)}$ is compact and D is closed, then F is usc in D if and only if the graph of F is closed.*

(b) *If D is compact, F is usc in D and $F(x)$ is compact for all $x \in D$, then $F(D)$ is compact.*

(c) *If D is connected, F is usc in D and $F(x)$ is connected for all $x \in D$, then $F(D)$ is connected.*

(d) *If $\overline{F(D)}$ is compact and F is closed at a point $x \in D$, then F is usc at x and the set $F(x)$ is compact.*

Remark 2. Proposition 24.1 was formulated for Banach spaces, but the proof works also in a metric space. Theorem 2.3 in [8] has been proved under an additional assumption that $F(x)$ is compact for all $x \in D$, but again this assumption is not necessary for the validity of the theorem.

Lemma 5. (Compare with Theorem 55.1 in [25, pp. 124–125]). *Suppose that assumption* (H1) *is fulfilled and D is a non-empty subset of M. Then the following statement holds:*

If $x \in D$ is stable with respect to the set D, then the multifunction ω is usc at x.

Proof. Let $x \in D$ be stable with respect to the set D. Lemma 4 implies that together with $\omega(M)$ also $\omega(D)$ is relatively compact. Thus, Proposition 5 can be applied and it suffices to show that for $F = \omega$ the implication (9) $\Rightarrow$ (10) holds. Consider two sequences $\{x_k\}$, $\{y_k\}$ such that

$$\lim_{k\to\infty} x_k = x, \lim_{k\to\infty} y_k = y, \{x_k\} \subset D,$$

and

$$y_k \in \omega(x_k), \quad k = 1, 2, \ldots. \tag{13}$$

We shall show that $y \in \omega(x)$.

(13) means that for each natural k there exists a sequence $\{n_{k,l}\}$ of natural numbers such that $\lim\limits_{l\to\infty} n_{k,l} = \infty$ and

$$\lim_{l\to\infty} T^{n_{k,l}}(x_k) = y_k, \quad k = 1, 2, \ldots.$$

Hence there exists n_k such that

$$\varrho(T^{n_k}(x_k), y_k) < \tfrac{1}{k}, \quad k = 1, 2, \ldots. \tag{14}$$

Without loss of generality we can assume that the sequence $\{n_k\}$ is increasing and $\lim\limits_{k\to\infty} n_k = \infty$. Now our aim is to prove that

$$\lim_{k\to\infty} T^{n_k}(x) = y. \tag{15}$$

By virtue of the inequality

$$\varrho(T^{n_k}(x), y) \leqslant \varrho(T^{n_k}(x), T^{n_k}(x_k)) + \varrho(T^{n_k}(x_k), y_k) + \varrho(y_k, y),$$

the stability of x with respect to D, (14) and $\lim\limits_{k\to\infty} y_k = y$, we have (15) and the proof is complete. □

Proposition 6. (Theorem 5, [11, p. 244]). *If all spaces X_σ of an inverse system $S = \{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ are continua, then the limit $X = \varprojlim S$ of that system is also a continuum (a connected and compact space).*

Lemmas 1, 3 and 4 will be completed by

Theorem 1. *If assumption* (H1) *is satisfied, M_1 and M_3 are determined by* (1) *and* (9)*, respectively, then there exists a set M_2 with the following properties:*

(i)

$$M_3 \subset M_2 \subset M_1,\ M_2 \text{ is compact and connected and } T(M_2) = M_2. \tag{16}$$

(ii) *The set M_2 with properties* (16) *is minimal, that is, if M_4 has the same properties and $M_4 \subset M_2$, then $M_4 = M_2$.*

(iii) *If each $x \in M_2$ is stable with respect to M_2, then $\omega(M_2)$ is compact. Moreover, if also each $x \in M_3 \setminus (F_p \cup C)$ is stable with respect to $\overline{T^{k_0}(M)}$ for some k_0 depending on x, then $\omega(M_2) = M_3$ and M_3 is compact.*

(iv) *If each $x \in M_2$ is stable with respect to M_2 and for each $x \in M_2$, $\omega(x)$ is connected, then $\omega(M_2)$ is compact and connected.*

P r o o f. (i), (ii). Let

$$S_1 = \{F \in 2^M \colon\ M_3 \subset F \subset M_1,\ F \text{ is compact and connected and } T(F) \subset F\}.$$

By Lemmas 1 and 4, $M_1 \in S_1$. S_1 can be partially ordered by the relation

$$F_1 \leqslant F_2 \quad \text{if and only if} \quad F_2 \subset F_1. \tag{17}$$

Let U be a totally ordered subset of S_1. Let $V = \bigcap\limits_{F \in U} F$. Then by (10), $M_3 = T(M_3) \subset T(V) \subset V \subset M_1$ and V is compact. We will show that V is connected, too, and thus, $V \in S_1$ is an upper bound of U. By the Kuratowski-Zorn lemma, this will mean that S_1 has a maximal element M_2. M_2 as well as $T(M_2)$ belong to S_1. Therefore $T(M_2) = M_2$ and the proof of (i), (ii) will be complete.

Clearly the family U is directed by the relation $\leqslant$ defined by (17). Let us define $\Pi_{F_2}^{F_1} \colon F_1 \to F_2$ for $F_2 \leqslant F_1$ to be the embedding of F_1 in F_2. Then the system $S = \{F, \Pi_{F_2}^{F_1}, U\}$ where the space assigned to the element $F \in U$ is F itself, is an inverse system of topological spaces. (For definition of such a system, see [11, pp. 87–88]). An element $\{x_F\}$ of the Cartesian product $\prod\limits_{F \in U} F$ belongs to the limit of the inverse system S if and only if $x_F = x$ for every $F \in U$ and $x \in V$. Therefore $\varprojlim S$ is homeomorphic to V (see Example 2 in [11, p. 88]) and by Proposition 6, V is also connected.

(iii), (iv) If each $x \in M_2$ is stable with respect to M_2, then by Lemma 5 the multifunction ω is usc in M_2. Proposition 5 implies that $\omega(M_2)$ is compact and if $\omega(x)$ is connected for every $x \in M_2$, then $\omega(M_2)$ is also connected. If each $x \in M_3 \setminus (F_p \cup C)$ is stable with respect to $\overline{T^{k_0}(M)}$ for some k_0 depending on x, then by Lemma 4, $M_3 = \omega(M_3)$ and thus $M_3 \subset \omega(M_2) \subset \omega(M) = M_3$, which implies $\omega(M_2) = M_3$. $\square$

Now we will work in a Fréchet space $(E, \{p_m\})$ where the seminorms p_m define a topology and a metric in the usual way. We will use the following assumption

(H2) *Let $(E, \{p_m\})$ be a Fréchet space, $\emptyset \neq M$ a closed, bounded and convex set in E, and*

$$T \colon M \to M$$

an α-condensing mapping.

Clearly (H2) implies (H1).

Let M_3 have the same meaning as in Theorem 1. Let a be the cardinal number of the set

$$S = \{P \in 2^M : \ M_3 \subset P, P \ \text{ is a closed and convex set}, T(P) \subset P\}. \tag{18}$$

By the Cantor theorem, [14, p. 16], the cardinal number $2^a > a$. Let b be the initial ordinal number of the power 2^a. Then we define a transfinite sequence $\{P_\gamma\}$ of the type b with values in S in the following way (compare with the proof of Theorem 1.5.11 in [1, p. 33]):

$$\begin{aligned} &P_0 = M, \ \text{and for } \gamma > 0 \\ &P_\gamma = \begin{cases} \overline{\text{co}}\, T(P_{\gamma-1}), & \text{if } \gamma - 1 \text{ exists} \\ \bigcap\limits_{\beta<\gamma} P_\beta, & \text{in the other case } (\gamma \text{ is a limit number}). \end{cases} \end{aligned} \tag{19}$$

Here $\overline{\text{co}}\, A$ means the closed convex hull of the set A. The sequence $\{P_\gamma\}$ is decreasing with respect to the set inclusion and there exists an ordinal number $\delta < b$ such that $P_\delta = P_{\delta+1}$ which, on the basis of (19), means

$$P_\delta = \overline{\text{co}}\, T(P_\delta). \tag{20}$$

Since the Kuratowski measure of noncompactness $\alpha[\overline{\text{co}}\, T(P_\delta)] = \alpha[T(P_\delta)]$ and T is α-condensing, the set P_δ is compact and convex. If (20) were not true for any $\delta < b$, the sequence $\{P_\gamma\}$ would be injective and the cardinal number of S would be greater or equal to 2^a, which on the basis of the Cantor theorem is a contradiction with the properties of cardinal numbers.

Denote

$$C_1 := P_\delta. \tag{21}$$

By virtue of (10), (18), (20) the following lemma holds.

Lemma 6. *If assumption* (H2) *is satisfied, then the set* C_1 *determined by* (21) *is nonempty, convex, compact and satisfies*

$$M_3 \subset T(C_1) \subset \overline{\text{co}}\, T(C_1) = C_1 \subset M.$$

Consider now the set

$$C_2 := \bigcap_{P \in S} P.$$

Then $C_2 \subset C_1$ and hence C_2 is compact and convex. Further, $M_3 \subset T(C_2) \subset \overline{\mathrm{co}}\, T(C_2) \subset C_2$ and C_2 is the least set in M with these properties. Hence $\overline{\mathrm{co}}\, T(C_2) = C_2$, otherwise $C_3 := \overline{\mathrm{co}}\, T(C_2)$ would be a proper subset of C_2 with the same properties. We can proceed further in the same way as in the proof of statements (iii), (iv) of Theorem 1. Thus the following theorem is true.

Theorem 2. *If assumption* (H2) *is satisfied and M_3 is determined by* (9), *then there exists a set C_2 having the following properties:*

(i)

$$M_3 \subset T(C_2) \subset \overline{\mathrm{co}}\, T(C_2) = C_2 \subset M, \quad C_2 \text{ is compact and convex.} \tag{22}$$

(ii) *The set C_2 is the smallest set with the properties* (22).

(iii) *If each $x \in C_2$ is stable with respect to C_2, then $\omega(C_2)$ is compact. Moreover, if also each $x \in M_3 \setminus (F_p \cup C)$ is stable with respect to $\overline{T^{k_0}(M)}$ for some k_0 depending on x, then $\omega(C_2) = M_3$ and M_3 is compact.*

(iv) *If each $x \in C_2$ is stable with respect to C_2 and for each $x \in C_2$, $\omega(x)$ is connected, then $\omega(C_2)$ is compact and connected.*

Example. Let $T\colon [0,1] \to [0,1]$ be the continuous piecewise linear map defined by

$$T(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant \frac{1}{2}, \\ (-2)(x-1), & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

Then each T^k, $k = 1, 2, \ldots$, has the same properties and, by mathematical induction we get that its graph consists of 2^k segments. More precisely,

$$T^k(x) = \begin{cases} 2^k\Big(x - \dfrac{2l}{2^k}\Big), & \dfrac{2l}{2^k} \leqslant x \leqslant \dfrac{2l+1}{2^k}, \\ (-2^k)\Big(x - \dfrac{2l+2}{2^k}\Big), & \dfrac{2l+1}{2^k} \leqslant x \leqslant \dfrac{2l+2}{2^k}, \quad l = 0, 1, \ldots, 2^{k-1} - 1. \end{cases}$$

Clearly $M_1 = [0,1]$ and since M_2 is an invariant compact interval, we also have $M_2 = [0,1]$. T^k has 2^k equilibria satisfying

$$x_{2l} = \frac{2l}{2^k - 1} \in \Big[\frac{2l}{2^k}, \frac{2l+1}{2^k}\Big] \quad \text{and} \quad x_{2l+1} = \frac{2l+2}{2^k+1} \in \Big[\frac{2l+1}{2^k}, \frac{2l+2}{2^k}\Big],$$
$$l = 0, 1, \ldots, 2^{k-1} - 1.$$

Each fixed point of T^k either is a fixed point of T or belongs to an l-cycle where l is a divisor of k. In both these cases x_{2l} as well as x_{2l+1} belong to the set $\omega([0,1])$ and hence this set is dense in $[0,1]$. By Corollary 12, [5, p. 76], $\omega([0,1])$ is a closed set and hence $M_3 = [0,1]$.

Now we will work in an ordered Banach space. We will start with assumption

(H3) *Let* $(E, \leqslant)$ *be a real Banach space,* $P \subset E$ *a normal cone and* $\leqslant$ *the partial ordering in* E *defined by* P*. Let* $[a,b] := \{x \in E\colon\ a \leqslant x \leqslant b\}$ *be a cone interval* $(a < b)$ *and let*

$$T\colon\ [a,b] \to [a,b]$$

be an α*-condensing mapping.*

Clearly (H3) implies (H2).

For $x, y \in E$ we write $x < y$ if $x \leqslant y$ and $x \neq y$. If P has a nonempty interior $\operatorname{int}(P)$, we also write

$$x \ll y \quad \text{if} \quad y - x \in \operatorname{int}(P).$$

According to [16, pp. 8–9], we say that T is *order-preserving (order-reversing)* if $x \leqslant y \Rightarrow T(x) \leqslant T(y)$ $(x \leqslant y \Rightarrow T(x) \geqslant T(y))$, *strictly order-preserving (strictly order-reversing)* if $x < y \Rightarrow T(x) < T(y)$ $(x < y \Rightarrow T(x) > T(y))$ and *strongly order-preserving (strongly order-reversing)* if $x < y \Rightarrow T(x) \ll T(y)$ $(x < y \Rightarrow T(x) \gg T(y))$ for $x, y \in [a,b]$.

An element $x \in [a,b]$ is called *subequilibrium (superequilibrium)* provided $x \leqslant T(x)$ $(x \geqslant T(x))$. The subequilibrium x is a *strict subequilibrium (strong subequilibrium)* if $x < T(x)$ $(x \ll T(x))$. The *strict* and the *strong superequilibrium* are defined accordingly.

Two points $x, y \in E$ are said to be *related* if $x \leqslant y$ or $y \leqslant x$. A set $A \subset E$ is said to be *unordered* if it does not contain two related points.

The following definitions are taken from [21, pp. 303–304].

Definition 2. Let $z_1 < z_2$ be two points from $[a,b]$. The interval $[z_1, z_2]$ will be called *singular* (for the mapping T) if $T([z_1,z_2]) \subset [z_1,z_2]$, $T(z_1) = z_1$, $T(z_2) = z_2$ and for each $x \in [z_1, z_2]$ the inequality $T(x) \leqslant x$ or $T(x) \geqslant x$ implies $T(x) = x$.

Definition 3. A set $F \subset E$ will be said to form a *continuous branch connecting points* $z_1, z_2 \in E$ if for each bounded open set $B \subset E$ such that either $z_1 \in B$, $z_2 \in E \setminus \overline{B}$ or $z_1 \in E \setminus \overline{B}$, $z_2 \in B$, the intersection $\delta B \cap F$ is nonempty.

Here δB means the boundary of the set B.

Proposition 7. ([6, pp. 63–64]) *Let* $A \subset P$*, let* (P, ϱ) *be a metric space and let* $\delta A := \overline{A} \cap \overline{(P \setminus A)}$ *be the boundary of the set* A*. Then the following statements hold:*

(i) $\delta\overline{A} \subset \delta A$.
(ii) *Let* $Q \subset P$, $A \subset P$. *Then* $\delta_Q(Q \cap A) \subset Q \cap \delta_P(A)$.
(iii) $\delta(A \cap B) \subset \delta A \cup \delta B$.

In the following theorems we will keep the notation from Theorems 1 and 2. The basic set M will be the interval $[z_1, z_2]$. Hence M_3, C_2 will be defined by means of $[z_1, z_2]$ and hence $M_3 \subset C_2 \subset [z_1, z_2]$.

Lemma 7. *Let assumption* (H3) *be fulfilled, let* $[z_1, z_2] \subset [a, b]$ *be a positively invariant interval for the operator* T*, that is,* $T([z_1, z_2]) \subset [z_1, z_2]$*, and let* $z_1, z_2 \in C_2$*. Then the set* F *of all subequilibria and all superequilibria lying in* C_2 *forms a continuous branch connecting the points* z_1, z_2.

Proof. Let $B \subset E$ be an open bounded set such that $z_1 \in B$, $z_2 \in E \setminus \overline{B}$. The case $z_1 \in E \setminus \overline{B}$, $z_2 \in B$ can be dealt with similarly. By Theorem 2, $\overline{\text{co}}\ T(C_2) = C_2$ and hence C_2 is a retract of E [9, p. 45].

Consider open subsets $U_1 := C_2 \cap B$, $U_2 := C_2 \cap (E \setminus \overline{B})$ of C_2. In the rest of the proof the topological notions as open, closed and boundary which are referred to the relative topology of C_2 as a subspace of E will be denoted by a subscript C_2. By Proposition 7

$$\delta_{C_2}(C_2 \cap B) \subset C_2 \cap \delta B, \tag{23}$$

$$\delta_{C_2}(C_2 \cap (E \setminus \overline{B})) \subset C_2 \cap \delta(E \setminus \overline{B}) = C_2 \cap \delta\overline{B} \subset C_2 \cap \delta B. \tag{24}$$

Consider two homotopies

$$T_\lambda(x) := \lambda T(x) + (1 - \lambda) z_1, \tag{25}$$

$$\widetilde{T}_\lambda(x) := \lambda T(x) + (1 - \lambda) z_2, \quad 0 \leqslant \lambda \leqslant 1,\ x \in C_2. \tag{26}$$

Since $[z_1, z_2]$ is a positively invariant interval for T, we have that

$$x = T_\lambda(x) \quad (x = \widetilde{T}_\lambda(x)) \text{ implies that } x \leqslant T(x) \quad (x \geqslant T(x)). \tag{27}$$

Indeed, if $x = T_\lambda(x)$ and $\lambda = 0$, then (27) is a consequence of $x = z_1 \leqslant T(z_1)$ and for $0 < \lambda \leqslant 1$ this follows from $z_1 \leqslant x$. Similarly we can proceed in the case $x = \widetilde{T}_\lambda(x)$.

Suppose that $\delta B \cap F = \emptyset$. Then, in view of (23), (24) we have $\delta_{C_2} U_1 \cap F = \emptyset$ and $\delta_{C_2} U_2 \cap F = \emptyset$. Hence $T_\lambda(x) \neq x$ for each $x \in \delta_{C_2} U_1$ and $\widetilde{T}_\lambda(x) \neq x$ for each $x \in \delta_{C_2} U_2$, $0 \leqslant \lambda \leqslant 1$. By the homotopy invariance and the normalization property of the fixed point index $i(T, U_1, C_2)$ of T over U_1 with respect to C_2 given in Theorem 11.1 ([2, pp. 657–658]) we obtain

$$i(T, U_1, C_2) = i(T_1, U_1, C_2) = i(T_0, U_1, C_2) = 1 \tag{28}$$

and similarly

$$i(T, U_2, C_2) = i(\widetilde{T}_1, U_2, C_2) = i(\widetilde{T}_0, U_2, C_2) = 1. \tag{29}$$

On the other hand, if $R\colon E \to C_2$ is a retraction of E onto C_2, then using the definition of the index we get that

$$i(T, C_2, C_2) := d_{LS}(I - TR, R^{-1}(C_2), 0) = d_{LS}(I - TR, E, 0) := d_{LS}(I - TR, V, 0)$$

where d_{LS} is the Leray-Schauder degree, I is the identity in E and $V \subset E$ is a sufficiently large ball containing $(I - TR)^{-1}(0) \subset C_2$ and all λC_2 for $0 \leqslant \lambda \leqslant 1$. Then

$$d_{LS}(I - TR, V, 0) = d_{LS}(I - \lambda TR, V, 0) = d_{LS}(I, V, 0) = 1$$

and thus

$$i(T, C_2, C_2) = 1. \tag{30}$$

If we denote $U_3 := C_2 \cap \delta B$, then U_1, U_2, U_3 are pairwise disjoint, $U_1 \cup U_2 \cup U_3 = C_2$ and hence, $C_2 \setminus (U_1 \cup U_2) = C_2 \cap \delta B$. This enables us to apply the additivity of the fixed point index. (28) and (29) then imply that

$$i(T, C_2, C_2) = i(T, U_1, C_2) + i(T, U_2, C_2),$$

which contradicts (30). Therefore $\delta B \cap F$ is nonempty. □

In the proof of Lemma in [32] the following proposition has been proved.

Proposition 8. ([32]) *Let K be a compact subset of a Banach space E. Then there exists a closed separable subspace E_1 of E such that*

$$K \subset E_1.$$

The following proposition is a corollary to Michael's selection theorem.

Proposition 9. ([4, p. 83]) *Let G be a lower semi-continuous map from a paracompact space X to a Banach space Y. Let $H\colon X \to Y$ be a set valued map with open graph. If $G(x) \cap H(x) \neq \emptyset$ for all $x \in X$, then there exists a continuous selection of $G \cap H$.*

A simple criterion for upper semicontinuity of a map is given in the following proposition.

Proposition 10. ([4, p. 42]) *Let G be a set-valued map from a Hausdorff topological space X to a compact topological space Y whose graph is closed. Then G is upper semicontinuous.*

The next proposition deals with a property of a compact metric space. In its formulation we need the following definitions (see [6, pp. 140, 135]).

Let (P, ϱ) be a metric space.

Let $\varepsilon > 0$, $a \in P$, $b \in P$. An *ε-chain from the point a to the point b in the space P* is any finite sequence $\{a_i\}_{i=1}^m$ of points in P such that (i) $a_1 = a$; (ii) $a_m = b$; (iii) $\varrho(a_i, a_{i+1}) < \varepsilon$ for $1 \leqslant i \leqslant m - 1$.

Let $a \in P$, $b \in P$. P is *connected between the points a and b* if for each decomposition $P = A \cup B$ with separated A and B the points a, b either both belong to A or both belong to B.

A set $Q \subset P$ is called a *quasicomponent of the space P* if (i) $Q \neq \emptyset$; (ii) P is connected between any two points $a \in Q$, $b \in Q$; (iii) P is not connected between a, b whenever $a \in Q$, $b \in P \setminus Q$.

By [6, Theorem 19.1.3, p. 140, Theorem 18.3.5, p. 136 and Theorem 19.1.5, p. 141] the following proposition holds.

Proposition 11. *Let (P, ϱ) be a metric space. Then the following statements hold:*

(i) *If P is a compact space, $a \in P$, $b \in P$ and for each $\varepsilon > 0$ there exists an ε-chain from the point a to the point b in P, then P is connected between the points a and b.*

(ii) *The points $a \in P$, $b \in P$ belong to the same quasicomponent of the space P if and only if P is connected between a and b.*

(iii) *In a compact space P the quasicomponents coincide with components.*

Hence,

(iv) *if P is a compact space, $a \in P$, $b \in P$ and for each $\varepsilon > 0$ there exists an ε-chain from the point a to the point b in P, then the points a, b belong to the same component of P.*

Now we are able to prove the following lemma which describes a property of a continuous branch.

Lemma 8. *Let assumption* (H3) *be satisfied and let $[z_1, z_2] \subset [a, b]$. If a set $S \subset [z_1, z_2]$ is compact and forms a continuous branch connecting the points z_1, z_2, then S contains a continuum S_1 such that z_1, $z_2 \in S_1$.*

P r o o f. Since S is a continuous branch, there exist points x_n, $y_n \in S$ such that $\|x_n - z_1\| = \|y_n - z_2\| < \frac{1}{n}$ and hence z_1, $z_2 \in S$. In view of Proposition 11, statement (iv), we will show that for each $\varepsilon > 0$ there exists an ε-chain from the point z_1 to the point z_2 in S and this will complete the proof of the lemma.

Hence, let $\varepsilon > 0$ be given. Denote the $\frac{\varepsilon}{2}$-neighbourhood of $x \in S$ by $U(x, \frac{\varepsilon}{2})$. Then $\bigcup\limits_{x\in S} U(x, \frac{\varepsilon}{2})$ is an open cover of the compact set S and hence there exists a finite subcover $\bigcup\limits_{k=1}^{s} U(x_k, \frac{\varepsilon}{2})$ where $x_k \in S$, $k = 1, \dots, s$. We will deal with the case that z_1, $z_2 \notin \{x_1, \dots, x_s\}$. The other cases can be dealt with in a similar way. By rearranging the indices if necessary, we can suppose that $z_1 \in U(x_k, \frac{\varepsilon}{2})$, $k = 1, \dots, l$, $z_2 \in U(x_k, \frac{\varepsilon}{2})$, $k = r, \dots, s$. If $l \geqslant r$, then the searched ε-chain from z_1 to z_2 in S is $\{z_1, x_l, z_2\}$. Suppose now that $l < r$.

If $U(x_i, \frac{\varepsilon}{2}) \cap U(x_j, \frac{\varepsilon}{2}) \neq \emptyset$ for $1 \leqslant i, j \leqslant s$, $i \neq j$, then $\|x_i - x_j\| < \varepsilon$ ($\|.\|$ is the norm in E) and we call $U(x_i, \frac{\varepsilon}{2})$, $U(x_j, \frac{\varepsilon}{2})$ adjacent. Now we consider all subsequences $\{U(x_{k_m}, \frac{\varepsilon}{2})\}_{m=1}^{p}$ (the so called *admissible subsequences*) such that $k_1 \in \{1, \dots, l\}$, the sequence $\{k_m\}_{m=1}^{p}$ is injective, $U(x_{k_i}, \frac{\varepsilon}{2})$, $U(x_{k_{i+1}}, \frac{\varepsilon}{2})$ are adjacent and $1 \leqslant p \leqslant s$. If there is an admissible subsequence which contains the term with the index $k_p \in \{r, \dots, s\}$, then the searched ε-chain is $\{z_1, x_{k_1}, \dots, x_{k_p}, z_2\}$. Otherwise we would have two disjoint open bounded sets $O_1 = \bigcup\{U(x_{k_m}, \frac{\varepsilon}{2})\}_{m=1}^{p}$ where the union is taken over all admissible subsequences, and $O_2 = \bigcup\limits_{k=1}^{s} U(x_k, \frac{\varepsilon}{2})\backslash O_1$, $\bigcup\limits_{k=r}^{s} U(x_k, \frac{\varepsilon}{2}) \subset O_2 \neq \emptyset$. Thus $z_2 \in O_2$, $z_1 \in E\backslash\overline{O_2}$ and since S is a continuous branch, we have $\delta O_2 \cap S \neq \emptyset$, which contradicts the fact that $S \subset O_1 \cup O_2$ and $O_1 \cap \overline{O_2} = \emptyset$ (open disjoint sets are separated, see [11, p. 242]). $\square$

The last result can be stregthened by the following continuous selection theorem which asserts that each continuum S_1 with the smallest z_1 and the greatest element z_2 in a partially ordered Banach space contains a continuous curve connecting z_1, z_2.

Theorem 3. *Let $(E, \leqslant)$ be a partially ordered Banach space with a normal cone P and let $S_1 \subset E$ be a continuum with the smallest element z_1 and the greatest element z_2. Then there exists an interval $[\alpha_1, \alpha_2] \subset \mathbb{R}$ and a continuous function $s\colon [\alpha_1, \alpha_2] \to S_1$ such that*

$$s(\alpha_1) = z_1, \qquad s(\alpha_2) = z_2. \tag{31}$$

Proof. By Proposition 8, there exists a closed separable subspace E_1 of the Banach space E such that $S_1 \subset E_1$. When the norm and ordering in E_1 are induced by the norm and ordering, respectively, from E, then E_1 is a partially ordered Banach space with the normal cone $P_1 = P \cap E_1$. By Proposition 19.3 in [9, p. 222], in the separable Banach space E_1 there exists a strictly positive linear continuous functional $x^\star$ from the dual cone $P_1^\star$.

Denote $x^\star(z_i) = \alpha_i$, $i = 1, 2$. Then $\alpha_1 < \alpha_2$, $\alpha_1 < x^\star(x) < \alpha_2$ for each $x \in S_1 \setminus \{z_1, z_2\}$ and $x^\star(S_1) = [\alpha_1, \alpha_2]$. Consider the multifunction ${x^\star}^{-1}$ (the inverse of the functional $x^\star$). By Example 24.1 in [9, p. 301], ${x^\star}^{-1}$ is lower-semicontinuous. Further, ${x^\star}^{-1}(\alpha) \cap S_1 \neq \emptyset$. We shall show that the multifunction

$$S_1(\alpha) = {x^\star}^{-1}(\alpha) \cap S_1, \qquad \alpha_1 \leqslant \alpha \leqslant \alpha_2, \tag{32}$$

has a continuous selection.

Let $V_k = \{x \in E_1 \colon \|x\| < \frac{1}{2^k}\}$, $k = 1, 2, \ldots$. Consider the multifunction

$$\overline{S}_1(\alpha) = {x^\star}^{-1}(\alpha) \cap (S_1 + V_1), \qquad \alpha_1 \leqslant \alpha \leqslant \alpha_2. \tag{33}$$

Since $H_1(\alpha) = S_1 + V_1$, $\alpha_1 \leqslant \alpha \leqslant \alpha_2$, has an open graph, by Proposition 9 there exists a continuous selection s_1 of $\overline{S}_1$. Now we consider the multifunction

$$\overline{S}_2(\alpha) = {x^\star}^{-1}(\alpha) \cap (S_1 + V_1) \cap (s_1(\alpha) + V_2), \qquad \alpha_1 \leqslant \alpha \leqslant \alpha_2.$$

Again the multifunction $H_2(\alpha) = (S_1 + V_1) \cap (s_1(\alpha) + V_2)$, $\alpha_1 \leqslant \alpha \leqslant \alpha_2$, has an open graph and $H_2(\alpha) \neq \emptyset$. Therefore, by Proposition 9, there exists a continuous selection s_2 of $\overline{S}_2$ on $[\alpha_1, \alpha_2]$.

Suppose that we already have continuous functions $s_1, \ldots, s_j$ with the property

$$s_k(\alpha) \in {x^\star}^{-1}(\alpha) \cap (S_1 + V_{k-1}) \cap (s_{k-1}(\alpha) + V_k), \quad \alpha_1 \leqslant \alpha \leqslant \alpha_2, \quad k = 2, \ldots, j. \tag{34}$$

Then there exists a continuous function s_{j+1} on $[\alpha_1, \alpha_2]$ such that

$$s_{j+1}(\alpha) \in {x^\star}^{-1}(\alpha) \cap (S_1 + V_j) \cap (s_j(\alpha) + V_{j+1}).$$

By mathematical induction there exists a sequence $\{s_k\}_{k=1}^\infty$ of continuous functions with property (34). Since $s_{k+1}(\alpha) \in (s_k(\alpha) + V_{k+1})$, $\{s_k\}$ is a Cauchy sequence which converges uniformly on $[\alpha_1, \alpha_2]$ to a continuous function s. As $s(\alpha) \in {x^\star}^{-1}(\alpha) \cap \bigcap_{k=1}^{\infty} (S_1 + \overline{V}_k)$, we have that $s(\alpha) \in {x^\star}^{-1}(\alpha) \cap S_1$, $\alpha_1 \leqslant \alpha \leqslant \alpha_2$. Thus s is a continuous curve lying in S_1 and connecting the points z_1, z_2. □

Remark 3. The multifunction S_1 defined by (32) has a closed graph. Indeed, if $\alpha_n \to \alpha$ and $x_n \to x$, $x_n \in S_1(\alpha_n)$, then the points x_n as well as x belong to S_1 and $x^\star(x_n) = \alpha_n \to x^\star(\alpha)$. Thus $x \in S_1(\alpha)$ and (α, x) belongs to the graph of S_1. By Proposition 10, S_1 is upper semicontinuous. Nevertheless, S_1 contains a continuous selection.

Now let us go back to Lemma 7. Keeping the notation from that lemma, the set $F \subset C_2$ is closed and since C_2 is compact, F is also compact. By Lemma 7, F forms a continuous branch connecting the points z_1, z_2. Then Lemma 8 implies that F contains a continuum F_1 such that $z_1, z_2 \in F_1$. By Theorem 3 we get the following theorem.

Theorem 4. *Let assumption* (H3) *be fulfilled, let* $[z_1, z_2] \subset [a, b]$ *be a positively invariant interval for the operator* T *and let* $z_1, z_2 \in C_2$. *Then the set* F *of all subequilibria and all superequilibria lying in* C_2 *forms a continuous branch connecting the points* z_1, z_2 *and contains a continuous curve* s *connecting* z_1, z_2.

Remark 4. By Theorem 2, each equilibrium belongs to C_2. Further, if $z = s(\alpha)$ is a subequilibrium (superequilibrium) and there is a sequence $\alpha_k \to \alpha$ such that $z_k = s(\alpha_k)$ are superequilibria (subequilibria), then $z_k \to z$ and z is an equilibrium. We also have that the set of all equilibria lying on the curve s is closed and thus, the set of all sub- and superequilibria on that curve is open (with respect to that curve). By the continuity of s, the corresponding values of the parameter α form a closed and an open subset of $[\alpha_1, \alpha_2]$, respectively.

On the basis of Remark 4, Theorem 4 implies the following theorem and lemma.

Theorem 5. *If assumption* (H3) *is satisfied and* $[z_1, z_2] \subset [a, b]$ *is a singular interval for the mapping* T, *then the set* F_p *of all equilibria lying in* $[z_1, z_2]$ *forms a continuous branch connecting the points* z_1, z_2 *and contains a continuous curve* s *connecting* z_1, z_2.

Lemma 9. *Let assumption* (H3) *be fulfilled, let* $[z_1, z_2] \subset [a, b]$ *be a positively invariant interval for* T *and let* z_1, z_2 *be two equilibria. Then the following alternative holds: Either*

(a) *there exists a further equilibrium in* $[z_1, z_2]$,

or

(b) *there exists a continuous curve* s *in* $[z_1, z_2]$ *connecting* z_1, z_2 *such that all points of the curve except* z_1, z_2 *are strict subequilibria,*

or

(c) *there exists a continuous curve* s *in* $[z_1, z_2]$ *connecting* z_1, z_2 *such that all points of the curve except* z_1, z_2 *are strict superequilibria.*

The following lemma is a little modification of Lemma 1.1 in [16, p. 9].

Lemma 10. *Let assumption* (H3) *be satisfied. Let* $[z_1, z_2] \subset [a, b]$ *and let* $T\colon [z_1, z_2] \to [z_1, z_2]$ *be an order-preserving mapping. Let* $x \in [z_1, z_2]$ *be a subequilibrium* ($y \in [z_1, z_2]$ *a superequilibrium). Then the following statements hold:*

1. *The sequence*

$$x_{k+1} := T(x_k) \text{ for each } k \in \mathbb{N},\ x_0 = x \tag{35}$$

is an increasing sequence converging to the least equilibrium v in $[x, z_2]$, while the sequence

$$y_{k+1} := T(y_k) \text{ for each } k \in \mathbb{N},\ y_0 = y$$

is a decreasing sequence converging to the greatest equilibrium u in $[z_1, y]$. Hence $\omega(x) = \{v\}$, $\omega(y) = \{u\}$.

2. *The elements x_k and y_k are again sub- and superequilibria, respectively. If T is strictly order-preserving and x is a strict subequilibrium (y is a strict superequilibrium), then also x_k (y_k) is a strict subequilibrium (a strict superequilibrium).*

Proof. We only prove the convergence of the sequence $\{x_k\}_{k\in\mathbb{N}}$. The other statements can be easily proved. By Lemma 3, $\omega(x) \neq \emptyset$. Assume that there exist two subsequences $\{x_{k_l}\}_{l\in\mathbb{N}}$ and $\{x_{k_m}\}_{m\in\mathbb{N}}$ of the sequence (35) such that

$$\lim_{l\to\infty} x_{k_l} = w, \qquad \lim_{m\to\infty} x_{k_m} = z.$$

Then we proceed as in the proof of Lemma 1.1 in [16, p. 9] and obtain that $w = z$. $\square$

In the sequel we will use the following definition. (Compare with [16, p. 10]).

Definition 4. A sequence $\{x_k\}_{k\in\mathbb{Z}}$ in $S \subset [a, b]$ with

$$x_{k+1} = T(x_k), \qquad k \in \mathbb{Z}$$

will be called an *entire orbit of the discrete dynamical system $\{T^k\}_{k\in\mathbb{N}}$ in S* (shortly an entire orbit in S). The entire orbit $\{x_k\}_{k\in\mathbb{Z}}$ in S is *connecting points* $z_1 \in \overline{S}$, $z_2 \in \overline{S}$ (in this order) if

$$\lim_{k\to-\infty} x_k = z_1 \qquad \text{and} \qquad \lim_{k\to\infty} x_k = z_2.$$

The entire orbit $\{x_k\}_{k\in\mathbb{Z}}$ connecting points z_1, z_2 is *positively finite* if there exists an integer l such that

$$x_k = z_2 \qquad \text{for all } k \geqslant l.$$

The next lemma gives another sufficient condition for the curve s from Theorem 4 to contain only equilibria.

Lemma 11. *Let assumption* (H3) *be satisfied, let z_1, z_2 be two equilibria such that $a \leqslant z_1 < z_2 \leqslant b$ and let T be order-preserving in $[z_1, z_2]$. Further, let all* *equilibria in* $[z_1, z_2]$ *be stable. Then there is a continuous curve of equilibria in* $[z_1, z_2]$ *connecting* z_1, z_2.

P r o o f. Clearly $[z_1, z_2]$ is a positively invariant interval for T. If there were a strict subequilibrium on the curve s, then by Remark 4 there would exist an interval (α_3, α_4) such that $s(\alpha_3)$, $s(\alpha_4)$ are equilibria and $s(\alpha)$ are strict subequilibria for all $\alpha \in (\alpha_3, \alpha_4)$. On the basis of Lemma 10, this contradicts the stability of $s(\alpha_3)$. $\square$

The following theorem extends the statement of Proposition 2.1 in [16, p. 10] to order-preserving condensing mappings. Its proof is similar to that of the proposition mentioned. For the sake of completeness it is given here.

Theorem 6. *Let assumption* (H3) *be fulfilled, let* $z_1 < z_2$, z_1, $z_2 \in [a, b]$ *be two equilibria and let* T *be order-preserving in* $[z_1, z_2]$. *Then the following statement holds: Either*

(a) *there exists another equilibrium in* $[z_1, z_2]$,

or

(b) *there exists an entire orbit* $\{x_k\}_{k\in\mathbb{Z}}$ *in* C_2 *connecting the points* z_1 *and* z_2 *such that either all terms of the orbit are strict subequilibria or this orbit is positively finite and the terms of the orbit different from* z_2 *are strict subequilibria,*

or

(c) *there exists an entire orbit* $\{x_k\}_{k\in\mathbb{Z}}$ *in* C_2 *connecting the points* z_2 *and* z_1 *such that either all terms of the orbit are strict superequilibria or this orbit is positively finite and the terms of the orbit different from* z_1 *are strict superequilibria.*

P r o o f. Clearly $[z_1, z_2]$ is a positively invariant interval for T and hence Lemma 9 can be applied. Let $B(a, \varepsilon)$ denote the open ball in C_2 with center $a \in C_2$ and radius $\varepsilon > 0$. The subscript C_2 will have the same meaning as in the proof of Lemma 7.

Suppose that there is no further equilibrium in $[z_1, z_2]$. Then, by Lemma 9, we have two cases:

(i) There exist strict subequilibria in C_2 as close to z_1 as we wish.

(ii) In each neighbourhood of z_2 there exists a strict superequilibrium in C_2.

In the first case we will derive alternative (b). Dealing with the case (ii) we would come to statement (c).

Let $\delta_0 > 0$ be such that $z_2 \notin \overline{B}_{C_2}(z_1, \delta_0)$. By continuity of T at z_1 there exists δ_1, $0 < \delta_1 < \delta_0$ such that $\|T(z) - z_1\| \leqslant \delta_0$ for each $z \in \overline{B}_{C_2}(z_1, \delta_1)$ and there is a strict subequilibrium $v_1\colon\ v_1 \in \partial_{C_2} B(z_1, \delta_1)$, $z_1 < v_1 < T(v_1)$.

Further, there exists $\delta_2\colon\ 0 < \delta_2 < \delta_1 < \delta_0$ such that $\|T(z) - z_1\| \leqslant \delta_1$ for each $z \in \overline{B}_{C_2}(z_1, \delta_2)$ and there exists a strict subequilibrium v_2, $v_2 \in \partial_{C_2} B(z_1, \delta_2)$. Hence $z_1 < v_2 < T(v_2) < T^2(v_2) < \ldots$ and, by Lemma 9, $\lim\limits_{k\to\infty} T^k(v_2) = z_2$, since there is no further equilibrium in $[z_1, z_2]$. Then there exists an index $n(2)$ such that $\delta_1 \leqslant \|T^{n(2)}(v_2) - z_1\| \leqslant \delta_0$, whereby $n(2) \geqslant 1$.

In this way we get a sequence $\{T^{n(k)}(v_k)\}_{k=1}^{\infty}$ of strict subequilibria such that $\delta_1 \leqslant \|T^{n(k)}(v_k) - z_1\| \leqslant \delta_0$ and $n(k) \geqslant k-1$.

Since $T(C_2)$ is compact, there exists a subsequence $\{T^{n(k')}(v_{k'})\}$ converging in C_2 to some x_0. Clearly $\delta_1 \leqslant \|x_0 - z_1\| \leqslant \delta_0$. Then the sequence $\{T^{n(k')-1}(v_{k'})\}$ contains a subsequence (index k'') converging to some x_{-1}. Since $T^{n(k'')-1}(v_{k''}) < T^{n(k'')}(v_{k''})$, we have $\lim\limits_{k''\to\infty} T^{n(k'')-1}(v_{k''}) = x_{-1} \leqslant x_0 = \lim\limits_{k''\to\infty} T^{n(k'')}(v_{k''})$ and $T(x_{-1}) = x_0$. But $x_{-1} \neq z_1$, since $\|x_0 - z_1\| \geqslant \delta_1$. As z_1, z_2 are the only equilibria in $[z_1, z_2]$, we have $x_{-1} < x_0$ and x_{-1} is a strict subequilibrium.

By induction we get a negative semiorbit $\{x_{-p}\}_{p\in\mathbb{N}}$ of strict subequilibria. As $x_{-p} \in C_2$ for each $p \in \mathbb{N}$, the decreasing semiorbit $\{x_{-p}\}_{p\in\mathbb{N}}$ converges to some $x \in C_2$ with $T(x) = x < x_0 < z_2$. Since z_1 is the only equilibrium in $[z_1, z_2]$ smaller than z_2, we have $x = z_1$. By Lemma 9, $x_{k+1} := T(x_k)$, $k \in \mathbb{N}$, are subequilibria and either all of them are strict subequilibria or there is the smallest integer l such that $x_l = T(x_l)$ and hence the entire orbit $\{x_k\}_{k\in\mathbb{Z}}$ is positively finite, x_l is an equilibrium greater than z_1 and hence $x_l = z_2$. All terms x_k, $k < l$, of the orbit are strict subequilibria. $\square$

R e m a r k 5. If the entire orbit is positively finite, $x_{l-1} < x_l$ and $x_k = z_2$ for all $k \geqslant l$, then $T(x) = z_2$ for all $x \in [x_{l-1}, z_2]$. Hence the following corollary holds.

Corollary 1. *If all assumptions of Theorem* 6 *are satisfied and T is not constant on any of subintervals $[z_1, z_3]$ and $[z_4, z_2]$ of $[z_1, z_2]$ where $z_1 < z_3 < z_4 < z_2$ (in particular if T is strictly order-preserving in $[z_1, z_2]$), then in alternative* (b) *(alternative* (c)*) all terms of the entire orbit connecting the points z_1 and z_2 (the points z_2 and z_1) are strict subequilibria (strict superequilibria).*

If T is order-preserving, then Theorem 5 can be strengthened.

Theorem 7. *If assumption* (H3) *is satisfied, $z_1 < z_2$ are two equilibria in $[a, b]$, T is order-preserving in $[z_1, z_2]$, and either*

(i) *$[z_1, z_2]$ is a singular interval for the mapping T,*

or

(ii) *each equilibrium in $[z_1, z_2]$ is stable,*

then the set F_p of all equilibria in $[z_1, z_2]$ has the following two properties:

(a) *If z_3 is an equilibrium satisfying $z_1 < z_3 < z_2$, then the set F_p contains a continuous curve*

$$G = \{x \in C_2 \colon x = \varphi(t), \quad 0 \leqslant t \leqslant 1,\ \varphi(0) = z_1,\ \varphi(1) = z_2\} \tag{36}$$

such that $z_3 \in G$ *and* G *is strictly increasing in the following sense: If* $0 \leqslant t_1 < t_2 \leqslant 1$, *then* $\varphi(t_1) < \varphi(t_2)$.

(b) F_p *is a continuum.*

P r o o f. Case (i). Since C_2 is compact, similarly as in the proof of Theorem 3 we get the existence of a separable partially ordered Banach space $(E_1, \leqslant)$ and of a strictly positive linear continuous functional $x^\star$ such that the norm and ordering in E_1 are induced from E, $C_2 \subset E_1$ and $x^\star$ is from the dual cone $P_1^\star$ where $P_1 = P \cap E_1$.

F_p is a subset of C_2 and is closed, hence it is compact. Further, F_p is a partially ordered set by the ordering induced from E_1.

Let $z_3 \in F_p$ be an arbitrary but fixed element such that $z_1 < z_3 < z_2$. Denote $x^\star(z_i) = \alpha_i$, $i = 1, 2, 3$. Then $\alpha_1 < \alpha_3 < \alpha_2$. z_1, z_2, z_3 form a chain in F_p. By the Hausdorff maximal-chain theorem [14, p. 65], there exists a maximal chain $U \subset F_p$ containing z_1, z_2, z_3. We shall show that the set U is closed. If $x_k \in U$, $x_k \to x$ as $k \to \infty$ and $y \in U$ is an arbitrary element, then in case that there exists a subsequence $\{x_{k_l}\}$ of $\{x_k\}$ such that $x_{k_l} \leqslant y$ $(x_{k_l} \geqslant y)$ we have $x \leqslant y$ $(x \geqslant y)$ and thus, $x \in F_p$ is comparable with each element $y \in U$. Maximality of U implies that $x \in U$. Therefore U is a closed subset of F_p and hence compact. Then $x^\star(U) = A \subset [\alpha_1, \alpha_2]$ is compact, $\alpha_1, \alpha_2 \in A$ and hence $[\alpha_1, \alpha_2] \setminus A$ is an open subset of $\mathbb{R}$.

Suppose that $[\alpha_1, \alpha_2] \setminus A \neq \emptyset$ and let the open interval (α_4, α_5) be a component of $[\alpha_1, \alpha_2] \setminus A$. Then there exist two points $z_4 < z_5$ of U such that $x^\star(z_4) = \alpha_4$, $x^\star(z_5) = \alpha_5$. Again, by Theorem 5, there exists another point $z_6 \in F_p$ such that $z_4 < z_6 < z_5$. Then, in view of maximality of U, $z_6 \in U$ and $\alpha_4 < x^\star(z_6) < \alpha_5$, which contradicts the fact that (α_4, α_5) contains no points from A. Therefore $A = [\alpha_1, \alpha_2]$ and $x^\star\colon U \to [\alpha_1, \alpha_2]$ is continuous and bijective. Then its inverse mapping $\varphi\colon [\alpha_1, \alpha_2] \to U$ is continuous, too. By using a strictly increasing homeomorphic mapping of $[0, 1]$ onto $[\alpha_1, \alpha_2]$ we may assume that φ is defined on $[0, 1]$ and $\varphi(0) = z_1$, $\varphi(1) = z_2$. Clearly φ is strictly increasing and there is an $\bar{\alpha} \in (0, 1)$ such that $\varphi(\bar{\alpha}) = z_3$.

Case (ii). We proceed in the same way as before. The only difference is that instead of Theorem 5 we apply Lemma 11. □

Theorem 8. *Let assumption* (H3) *be fulfilled, let the cone* P *have a nonempty interior* $\mathrm{int}(P)$, *let* $z_1 < z_2$ *be two equilibria in* $[a, b]$, *let* T *be strongly order-preserving in* $[z_1, z_2]$ *and let either*

(i) $[z_1, z_2]$ *be a singular interval for* T,

or

(ii) *each equilibrium in* $[z_1, z_2]$ *be stable.*

Then the set F_p *of all equilibria is a continuous curve* G *given by* (36).

P r o o f. Case (i). We will show that F_p is totally ordered. Consider any nontrivial positive linear continuous functional $x_1{}^\star \in P^\star$ (not necessarily strictly positive). Since T is strongly order-preserving, if $x < y$ are two equilibria, then $x \ll y$ and, by Proposition 19.3 in [9, p. 222], $x_1{}^\star(x) < x_1{}^\star(y)$. If $x_1{}^\star(x) = \alpha \in \mathbb{R}$, then x will be denoted briefly by x_α. Hence $x_1{}^\star(x_\alpha) = \alpha$.

Assume that u and $\bar{u}$ are not order-related elements of F_p. Since $F_p \subset M_3 \subset C_2$, F_p is compact. Let v_2 be a minimal equilibrium above u, $\bar{u}$. Its existence can be proved by the Kuratowski-Zorn lemma. Indeed, denote $F_{u,\bar{u}} = \{x \in F_p \colon\ x \geqslant u,\ x \geqslant \bar{u}\}$. Clearly $F_{u,\bar{u}} \neq \emptyset$. Let G_2 be a totally ordered subset of $F_{u,\bar{u}}$. Let the sequence $\alpha_k \in x_1{}^\star(G_2)$ be such that $\alpha_k \searrow \inf x_1{}^\star(G_2)$ as $k \to \infty$. As F_p is compact and $\{x_{\alpha_k}\} \subset G_2$ is a decreasing sequence, similarly as in Lemma 10 we get that there exists $v \in F_p$ such that $\lim\limits_{k\to\infty} x_{\alpha_k} = v$. Clearly $v \in F_{u,\bar{u}}$ and v is a lower bound of G_2. Then, by the Kuratowski-Zorn lemma, $F_{u,\bar{u}}$ has a minimal element $v_2 > u$, $v_2 > \bar{u}$. By the strong monotonicity of T, $u \ll v_2$, $\bar{u} \ll v_2$. This implies that v_2 is an element of F_p which is isolated from below. Otherwise, there would exist a sequence $\{u_k\} \subset F_p$ such that $u_k < v_2$ and $\lim\limits_{k\to\infty} u_k = v_2$. Then $u < u_k < v_2$, $\bar{u} < u_k < v_2$ for large k, contradicting the minimality of v_2.

Let v_1 be a maximal fixed point of T below v_2 which exists again by the Kuratowski-Zorn lemma. To prove this, denote $F_{v_2} = \{x \in F_p \colon\ x < v_2\}$. Then $z_1 \in F_{v_2}$. Let G_1 be a totally ordered subset of F_{v_2} and let the sequence α_k be such that $\alpha_k \nearrow \sup x_1{}^\star(G_1)$ as $k \to \infty$. Then there exists $v \in F_p$ such that $\lim\limits_{k\to\infty} x_{\alpha_k} = v$ and $v \in F_{v_2}$ due to the fact that v_2 is isolated from below. Thus, v is an upper bound of G_1 and, by the Kuratowski-Zorn lemma, there exists a maximal point $v_1 \in F_{v_2}$ below v_2. This contradicts Theorem 5 with $z_1 = v_1$, $z_2 = v_2$. Hence the set F_p of all equilibria in $[z_1, z_2]$ is totally ordered.

Similarly as in the proof of Theorem 7 we get that for each $\alpha \in (\alpha_1, \alpha_2)$ there exists an $x_\alpha \in F_p$. F_p is compact. Thus $x_1{}^\star\colon\ F_p \to [\alpha_1, \alpha_2]$ is an increasing homeomorphism of F_p onto $[\alpha_1, \alpha_2]$. Therefore F_p is a continuous curve which can be written in the form (36).

Case (ii) differs from the previous one only by using Lemma 11 instead of Theorem 5. $\square$

R e m a r k 6. Theorems 5, 7, 8 represent an extension of Theorem 5 in [21, p. 304] to α-condensing operators. Theorems 7 and 8 contain a new sufficient condition for the existence of a continuous curve of equilibria. They also complete Theorem 1.5 in [36, p. 229]. Similarly Theorem 4 in [32] is extended and sharpened by the theorems mentioned. Theorem 8 is similar to Theorem 3.3 in [16, p. 12].

Let $n \geqslant 2$ be a natural number, $0 < p_i < \infty$, $i = 1, \dots, n$ arbitrary real numbers. Denote the order interval $[0, p] := [0, p_1] \times \ldots \times [0, p_n]$ in a partially ordered space $\mathbb{R}^n$ where $a = (a_1, \dots, a_n) \leqslant b = (b_1, \dots, b_n)$ iff $a_i \leqslant b_i$, $i = 1, \dots, n$ for any two $a, b \in \mathbb{R}^n$. The euclidean norm in $\mathbb{R}^n$ will be denoted by $|.|$. Consider the differential system

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(t, x_1, \dots, x_n), \quad i = 1, \dots, n \tag{37}$$

where $f_i \in C([0, \infty) \times [0, p], \mathbb{R})$, $i = 1, \dots, n$, is such that each initial value problem for (37) with initial values in $[0, \infty) \times [0, p]$ is uniquely locally solvable. We also assume that the following assumption is fulfilled:

(H4)

(a) *There exists a* $\tau > 0$ *such that* $f_i(t + \tau, x_1, \dots, x_n) = f_i(t, x_1, \dots, x_n)$, $0 \leqslant t < \infty$, $x = (x_1, \dots, x_n) \in [0, p]$;

(b) $f_i(t, 0, \dots, 0) \equiv 0$, $0 \leqslant t < \infty$, $i = 1, \dots, n$;

(c) $f_i(t, x_1, \dots, x_n) < 0$, $0 \leqslant t < \infty$, $x_i = p_i$, $i = 1, \dots, n$, $0 \leqslant x_j \leqslant p_j$, $j \neq i$, $j = 1, \dots, n$;

(d) *for each* $t \in [0, \infty)$ *the function* $f(t, ., \dots, .)$ *is of type* K *in* $[0, p]$, *that is, for each subscript* $i = 1, \dots, n$ *we have* $f_i(t, a) \leqslant f_i(t, b)$ *for any two points* $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ *in* $[0, p]$ *with* $a_i = b_i$ *and* $a_k \leqslant b_k$, $k = 1, \dots, n$, $k \neq i$.

Remark 7. In view of (a) and (d), system (37) will be called a τ*-periodic Kamke system* (see [26, p. 9], [7, p. 27], [40, p. 42]). Further, (b) and (d) imply the inequalities

(e) $f_i(t, b_1, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_n) \geqslant 0$ for each $0 \leqslant t < \infty$, $0 \leqslant b_k \leqslant p_k$, $k = 1, \dots, n$, $k \neq i$ and hence, the system represents a mathematical model for spreading of an infectious disease (see [35, Ex. 3.2, p. 40], [36, Ex. 4.2, p. 241]). In this model we have n disjoint population classes, p_i is the number of individuals in class i and x_i is the number of infected ones in class i, $i = 1, \dots, n$. In view of (a), we also have that

(f) f is uniformly continuous and bounded (by a constant $M > 0$) on $[0, \infty) \times [0, p]$.

Denote $x(t, t_0, x_0)$, $x_0 = (x_{01}, \dots, x_{0n})$, the noncontinuable to the right solution of system (37) satisfying the initial condition $x_i(t_0) = x_{0i}$, $i = 1, \dots, n$.

On the basis of Theorem 10 in [7, p. 29], (b), (c) and (d) imply the following statement:

(i) $0 \leqslant x(t, t_0, c) \leqslant p$, t being from the maximal to the right interval of existence, $0 \leqslant t_0 < \infty$, $c \in [0, p]$, and hence, $x(t, t_0, c)$ is defined in $[t_0, \infty)$.

Further,

(ii) $x(t, t_0, c) \leqslant x(t, t_0, d)$ for $t_0 \leqslant t < \infty$, $0 \leqslant t_0 < \infty$ and for any c, $d \in \mathbb{R}^n$, $0 \leqslant c \leqslant d \leqslant p$.

By (a), we have

(iii) $x(t + k\tau, t_0, c) = x(t, t_0, x(t_0 + k\tau, t_0, c))$ for $t_0 \leqslant t < \infty$, $k \in \mathbb{N}$, $0 \leqslant t_0 < \infty$, $c \in [0, p]$.

In particular, if $x(t_0 + \tau, t_0, c) = x(t_0, t_0, c)$, then $x(t + \tau, t_0, c) = x(t, t_0, c)$, $t_0 \leqslant t < \infty$, $0 \leqslant t_0 < \infty$, $0 \leqslant c \leqslant p$.

Statement (i) allows to define the period mapping $T_{t_0}\colon [0, p] \to [0, p]$ for each $0 \leqslant t_0 < \infty$ by

$$T_{t_0}(c) = x(t_0 + \tau, t_0, c).$$

By virtue of the uniqueness of the initial value problems for (37), T_{t_0} is continuous and hence, a compact mapping. Further, by (ii) and (iii),

(iv) T_{t_0} is order-preserving and $T_{t_0}^k(c) = x(t_0 + k\tau, t_0, c)$ for $k \in \mathbb{N}$, $c \in [0, p]$ and $0 \leqslant t_0 < \infty$.

(v) $T_{t_0}(c) = c$ iff $x(t, t_0, c)$ is a τ-periodic function (in $[t_0, \infty)$) for each admissible t_0.

Since $x(t, t_0, c) = x(t + k\tau, t_0 + k\tau, c)$, the following equality holds:

(vi) $T_{t_0 + k\tau}(c) = T_{t_0}(c)$ for each $0 \leqslant t_0 < \infty$, $k \in \mathbb{N}$, $c \in [0, p]$.

Further,

(vii) $T_{t_0}(c) = c$ iff $T_{t_1}(c_1) = c_1$ for $c_1 = x(t_1, t_0, c)$, $t_0 \leqslant t_1 \leqslant t_0 + \tau$, $0 \leqslant t_0 < \infty$, $c \in [0, p]$.

(iv) implies the first part of the statement

(viii) If the solution $x(t, t_0, c)$ is Lyapunov stable ($0 \leqslant t_0 < \infty$), then the point $c \in [0, p]$ is stable with respect to $[0, p]$ (and the mapping T_{t_0}). Conversely, if $T_{t_0}(c) = c$ is stable (with respect to $[0, p]$ and the mapping T_{t_0}), then the periodic solution $x(t, t_0, c)$ is Lyapunov stable. Hence the Lyapunov stability of τ-periodic solutions of (37) is equivalent to the stability of equilibria (of T_{t_0}).

P r o o f of the second part of the statement. Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that

$$|x(t, t_0, c) - x(t, t_0, c_1)| < \varepsilon \quad \text{for } t_0 \leqslant t \leqslant t_0 + \tau,\ |c - c_1| < \delta. \tag{38}$$

Further, by (iv), there is $\delta_1 > 0$ implying $|c - x(t_0 + k\tau, t_0, c_1)| < \delta$ for $|c - c_1| < \delta_1$, $k \in \mathbb{N}$. Let $|c - c_1| < \delta_1$, let $k \in \mathbb{N}$ be arbitrary but fixed. Then, by (iii) and in view of (38),

$$|x(t+k\tau,t_0,c)-x(t+k\tau,t_0,c_1)|=|x(t,t_0,c)-x(t,t_0,x(t_0+k\tau,t_0,c_1))|<\varepsilon$$

for $t_0+k\tau\leqslant t+k\tau\leqslant t_0+(k+1)\tau$. □

Further, by an *ω-limit point of the solution* $x(t,t_0,c)$ we understand a point $q\in[0,p]$ such that there exists a sequence $t_0\leqslant t_1<t_2<\ldots\to\infty$ and

$$\lim_{k\to\infty}x(t_k,t_0,c)=q.$$

The set of all ω-limit points of $x(t,t_0,c)$ will be denoted by $\omega(x(t,t_0,c))$. Denote by $\omega_{t_0}(c)$ the ω-limit set of c under T_{t_0}. Then the following statement holds.

(ix) $\omega_{t_0}(c)\subset\omega(x(t,t_0,c))$ and for each $q\in\omega(x(t,t_0,c))$ there exists $t'\in[0,\tau]$ and $d\in\omega_{t_0}(c)$ such that $x(t_0+t',t_0,d)=q$.

Proof. In view of (iv), the first part of the statement is clear. Let $q\in\omega(x(t,t_0,c))$ and let $x(t_k,t_0,c)\to q$ as $k\to\infty$. Then $t_k=t_0+l_k\tau+t'_k$ where $\{l_k\}$ is a nondecreasing subsequence of $\mathbb{N}$ tending to ∞ and $0\leqslant t'_k<\tau$ is uniquely determined. Choosing a subsequence if necessary, we may assume that $\lim\limits_{k\to\infty}t'_k=t'\in[0,\tau]$. By (f),

$$\begin{aligned}|x(t_0+l_k\tau+t'_k,t_0,c)-x(t_0+l_k\tau+t',t_o,c)|&=\Big|\int_{t_0+l_k\tau+t'}^{t_0+l_k\tau+t'_k}f[t,x(t,t_0,c)]\,\mathrm{d}t\Big|\\&\leqslant M|t'_k-t'|\to 0\text{ as }k\to\infty\end{aligned}$$

and hence, (iii) and (iv) yield

$$\begin{aligned}q=&\lim_{k\to\infty}x(t_0+l_k\tau+t'_k,t_0,c)=\lim_{k\to\infty}x(t_0+l_k\tau+t',t_0,c)\\&+\lim_{k\to\infty}[x(t_0+l_k\tau+t'_k,t_0,c)-x(t_0+l_k\tau+t',t_0,c)]\\=&\lim_{k\to\infty}x(t_0+t',t_0,x(t_0+l_k\tau,t_0,c))\\=&\lim_{k\to\infty}x(t_0+t',t_0,T_{t_0}^{l_k}(c)).\end{aligned}\tag{39}$$

From the sequence $\{T_{t_0}^{l_k}(c)\}$ we can extract a convergent subsequence. Denoting it again by $\{T_{t_0}^{l_k}(c)\}$ we get that there exists $d\in\omega_{t_0}(c)$ such that $T_{t_0}^{l_k}(c)\to d$ as $k\to\infty$. The relation $\lim\limits_{k\to\infty}x(t_0+t',t_0,T^{l_k}(c))=x(t_0+t',t_0,d)$ together with (39) implies the second part of the statement. □

Two cases for the τ-periodic Kamke system (37) may occur. Either it has only one τ-periodic solution, namely the trivial one, or it has also a nontrivial τ-periodic solution. In the former case the following theorem is true.

Theorem 9. *Let assumption* (H4) *be fulfilled. Suppose that* $x(t) \equiv 0$, $0 \leqslant t < \infty$, *is the only* τ*-periodic solution of* (37). *Then this solution is stable and*

$$\lim_{t\to\infty} x(t, t_0, c) = 0 \text{ for each } 0 \leqslant t_0 < \infty \text{ and each } c \in [0, p]. \tag{40}$$

P r o o f. Let $0 \leqslant t_0 < \infty$ and let $c \in [0, p]$ be arbitrary but fixed. By (v), 0 is the only equilibrium and p is a superequilibrium (under T_{t_0}). By Lemma 10, $\{T_{t_0}^k(p)\}$ is decreasing and $\lim\limits_{k\to\infty} T_{t_0}^k(p) = 0$. Since $0 \leqslant T_{t_0}^k(c) \leqslant T_{t_0}^k(p)$ for each $k \in \mathbb{N}$, we have $\lim\limits_{k\to\infty} T_{t_0}^k(c) = 0$, and hence $\omega_{t_0}(c) = \{0\}$. Let $q \in \omega(x(t, t_0, c))$. By (ix), there exists $t' \in [0, \tau]$ such that $q = x(t_0 + t', t_0, 0) = 0$. Hence (40) is true.

On the basis of Theorem 7.5 ([40, pp. 50–51]) and Lemma 7.1 ([40, p. 49]), stability of the trivial solution will be proved if we show that to any $\varepsilon > 0$ and $t_0 \geqslant 0$ there exists a $\delta(t_0) > 0$ and a $T(t_0, \varepsilon) \geqslant 0$ such that $|c| < \delta(t_0)$ implies $|x(t, t_0, c)| < \varepsilon$ for all $t \geqslant t_0 + T(t_0, \varepsilon)$. Hence, let $\varepsilon > 0$, $t_0 \geqslant 0$ be arbitrary but fixed. By (ii) we have $x(t, t_0, c) \leqslant x(t, t_0, p)$, $t_0 \leqslant t < \infty$ for each $c \in [0, p]$ and since $\lim\limits_{t\to\infty} x(t, t_0, p) = 0$, there exists a $T(t_0, \varepsilon) \geqslant 0$ such that

$$|x(t, t_0, c)| \leqslant |x(t, t_0, p)| < \varepsilon \quad \text{for all } t \geqslant t_0 + T(t_0, \varepsilon).$$

The proof of the theorem is complete. □

R e m a r k 8. In view of Definition 7.1 in [26, p. 93], under the assumptions of Theorem 9 system (37) has the property of convergence.

Further, by Definition 9.1, p. 77, and Theorem 9.3, p. 78 in [40], we get

Corollary 2. *Under the assumptions of Theorem* 9 *the zero solution of* (37) *is uniformly asymptotically stable in the large.*

R e m a r k 9. Corollary 2 can be partially reversed. By Remark 23.2 in [3, p. 343], any asymptotically stable τ-periodic solution $x(t)$ of (37) in $[t_0, \infty)$ is isolated, which means that there exists an $\varepsilon > 0$ such that for any other τ-periodic solution $y(t)$ of (37) in $[t_0, \infty)$ we have $|y(t) - x(t)| \geqslant \varepsilon$ for all $t \geqslant t_0$. This implies that a nonconstant τ-periodic solution of the autonomous equation

$$x' = f(x), \tag{41}$$

where f is continuous in $[0, p]$, cannot be asymptotically stable ([3, p. 345]).

Suppose, now, that there exists a nontrivial τ-periodic solution $x(t,t_1,c_1)$ of (37). Denote by S_{t_1} the set of all τ-periodic solutions $x(t,t_1,c)$. Then S_{t_1} is a nonempty subset of the Banach space $X = C([t_1,t_1+\tau],\mathbb{R}^n)$ equipped with the norm

$$\|y\| = \sup_{t_1\leqslant t\leqslant t_1+\tau} |y(t)| \text{ for each } y \in X.$$

X can be partially ordered by the natural ordering $x \leqslant y$ (in X) iff $x(t) \leqslant y(t)$ (in $\mathbb{R}^n$) for all $t_1 \leqslant t \leqslant t_1+\tau$. By this definition the cone $K = \{x \in X\colon\ x \geqslant 0\}$ in X is normal.

Theorem 10. *Let assumption* (H4) *be satisfied and let there exist a nontrivial τ-periodic solution $x(t,t_1,c_1)$. Then S_{t_1} contains the greatest τ-periodic solution $x(t,t_1,c_2)$ and either S_{t_1} contains an unstable solution or S_{t_1} is a continuum in the space X. In the latter case, if $x(t,t_1,c_3)$ is an arbitrary τ-periodic solution such that $0 < c_3 < c_2$, then the set S_{t_1} contains a continuous curve*

$$H = \{x \in S_{t_1}\colon\ x = \psi(t), \quad 0 \leqslant t \leqslant 1,\ \psi(0) = x(t,t_1,0) \equiv 0,\ \psi(1) = x(t,t_1,c_2)\}$$

such that $x(t,t_1,c_3) \in H$ and H is strictly increasing in the following sense: If $0 \leqslant t_1 < t_2 \leqslant 1$, then $\psi(t_1) < \psi(t_2)$ (in X).

Proof. Consider the mapping T_{t_1}. By (iv) and (v), $T_{t_1}\colon [0,p] \to [0,p]$ is order-preserving and c is an equilibrium (in notation $c \in F_p$) iff $x(t,t_1,c) \in S_{t_1}$. Let $S\colon F_p \to S_{t_1}$ be defined by $S(c) = x(t,t_1,c)$, $c \in F_p$. Then S is continuous and, by (ii), order-preserving. By Lemma 10, there exists the greatest equilibrium c_2 in $[0,p]$ which is defined by $c_2 = \lim_{k\to\infty} T_{t_1}^k(p)$, and the solution $S(c_2) = x(t,t_1,c_2)$ is the greatest τ-periodic solution in X. When all τ-periodic solutions in S_{t_1} are Lyapunov stable, then (viii) implies that all $c \in F_p$ are stable, and by Theorem 6, there exists another equilibrium in $[0,c_2]$. Theorem 7 gives that F_p is a continuum and hence, S_{t_1} is also a continuum. Further, for any $c_3 \in F_p$ there exists a continuous curve G in F_p which contains c_3 and is strictly increasing. Then the image of this curve under S is the curve H with the properties mentioned in Theorem 10. □

On the basis of Remark 9, we get from the last theorem the following corollary.

Corollary 3. *Let both the assumptions of Theorem* 10 *be satisfied. Then the following implication holds:*

If there exists an asymptotically stable τ-periodic solution of (37)*, or more generally, an isolated τ-periodic solution of* (37) *(in S_{t_1}), then there is another τ-periodic solution of the equation (in S_{t_1}) which is unstable.*

R e m a r k 10. A criterion for the stability of a nonconstant τ-periodic solution of an autonomous differential system is given by the Andronov-Witt theorem ([10, p. 312]). Further results on the stability of a τ-periodic solution can be found in [28], [29].

R e m a r k 11. Theorem 10 and its corollary completes the statement from [27] dealing with the strongly cooperative τ-periodic differential system.

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