125 (2000) MATHEMATICA BOHEMICA No. 3, 341–354

DIRECT PRODUCT DECOMPOSITIONS OF INFINITELY DISTRIBUTIVE LATTICES

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(Received August 17, 1998)

Abstract. Let α be an infinite cardinal. Let $\mathcal{T}_\alpha$ be the class of all lattices which are conditionally α-complete and infinitely distributive. We denote by $\mathcal{T}'_\sigma$ the class of all lattices X such that X is infinitely distributive, σ-complete and has the least element. In this paper we deal with direct factors of lattices belonging to $\mathcal{T}_\alpha$. As an application, we prove a result of Cantor-Bernstein type for lattices belonging to the class $\mathcal{T}'_\sigma$.

Keywords: direct product decomposition, infinite distributivity, conditional α-completeness

MSC 2000: 06B35, 06D10

1. Introduction

Let L be a partially ordered set and $s^0 \in L$. The notion of the internal direct product decomposition of L with the central element s^0 was investigated in [10] (the definition is recalled in Section 2 below).

We denote by $F(L, s^0)$ the set of all internal direct factors of L with the central element s^0; this set is partially ordered by the set-theoretical inclusion. In the present paper we suppose that L is a lattice. Then $F(L, s^0)$ is a Boolean algebra (cf. Section 3).

Let α be an infinite cardinal. We denote by $\mathcal{T}_\alpha$ the class of all lattices which are conditionally α-complete and infinitely distributive. We prove

Theorem 1. *Let $L \in \mathcal{T}_\alpha$ and $s^0 \in L$. Then the Boolean algebra $F(L, s^0)$ is α-complete.*

In the particular case when the lattice L is bounded we denote by $\operatorname{Cen} L$ the center of L. For each $s^0 \in L$, $F(L, s^0)$ is α-complete and if $\operatorname{Cen} L$ is a closed sublattice of

L, then $\operatorname{Cen} L$ is α-complete and thus $F(L, s^0)$ is α-complete as well. Some sufficient conditions under which the center of a complete lattice is closed were found in [2], [11], [12], [13], [14]; these results were generalized in [4]. For related results cf. also [3].

We denote by $\mathcal{T}'_\sigma$ the class of all lattices L belonging to $\mathcal{T}_{\aleph_0}$ which have the least element and are σ-complete.

As an application of Theorem 1 we prove the following result of Cantor-Bernstein type:

Theorem 2. *Let L_1 and L_2 be lattices belonging to $\mathcal{T}'_\sigma$. Suppose that*

(i) *L_1 is isomorphic to a direct factor of L_2;*

(ii) *L_2 is isomorphic to a direct factor of L_1.*

Then L_1 is isomorphic to L_2.

This generalizes a theorem of Sikorski [15] on σ-complete Boolean algebras (proven independently also by Tarski [17]).

Some results of Cantor-Bernstein type for lattice ordered groups and for MV-algebras were proved in [5], [6], [7], [8].

2. Internal direct factors

Assume that L and L_i $(i \in I)$ are lattices and that φ is an isomorphism of L onto the direct product of lattices L_i; then we say that

$$\varphi\colon\ L \to \prod_{i \in I} L_i \tag{1}$$

is a direct product decomposition of L; the lattices L_i are called direct factors of L.

For $x \in L$ and $i \in I$ we denote by $x(L_i, \varphi)$ the component of x in L_i, i.e.,

$$x(L_i, \varphi) = (\varphi(x))_i.$$

Let $s^0 \in L$ and $i \in I$. Put

$$L_i^{s^0} = \{y \in L\colon\ y(L_j, \varphi) = s^0(L_j, \varphi) \quad \text{for each } j \in I \setminus \{i\}\}.$$

Then for each $x \in L$ and each $i \in I$ there exists a uniquely determined element y_i in $L_i^{s^0}$ such that

$$x(L_i, \varphi) = y_i(L_i, \varphi).$$

The mapping

$$\varphi^{s^0}\colon\ L \to \prod_{i \in I} L_i^{s^0} \tag{2}$$

defined by

$$\varphi^{s^0}(x) = (\ldots, y_i, \ldots)_{i \in I}$$

is also a direct product decomposition of L. Moreover, the following conditions are valid:

(i) For each $i \in I$, $L_i^{s^0}$ is a closed convex sublattice of L and $s^0 \in L_i^{s^0}$.

(ii) For each $i \in I$, $L_i^{s^0}$ is isomorphic to L_i.

(iii) If $i \in I$ and $x \in L_i^{s^0}$, then $x(L_i^{s^0}, \varphi^{s^0}) = x$.

(iv) If $i \in I$, $j \in I \setminus \{i\}$ and $x \in L_j^{s^0}$, then $x(L_i^{s^0}, \varphi^{s^0}) = s^0$.

We say that (2) is an internal direct product decomposition of L with the central element s^0; the sublattices $L_i^{s^0}$ are called internal direct factors of L with the central element s^0.

The condition (ii) yields that if we are interested only in considerations "up to isomorphisms", then we need not distinguish between (1) and (2).

We denote by $F(L, s^0)$ the collection of all internal direct factors of L with the central element s^0. Then in view of (i), $F(L, s^0)$ is a set. On the other hand, it is obvious that the collection of all direct factors of L is a proper class.

3. Auxiliary results

Assume that the relation (2) is valid. Let I_1 and I_2 be nonempty subsets of I such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$. Denote

$$L(I_1) = \{x \in L\colon\ x(L_i^{s^0}, \varphi^{s^0}) = s^0 \quad \text{for each } i \in I_2\},$$

$$L(I_2) = \{x \in L\colon\ x(L_i^{s^0}, \varphi^{s^0}) = s^0 \quad \text{for each } i \in I_1\}.$$

Consider the mapping

$$\psi\colon\ L \to L(I_1) \times L(I_2) \tag{3}$$

defined by $\psi(x) = (x^1, x^2)$, where

$$x^1 = (\ldots, x(L_i^{s^0}, \varphi^{s^0}), \ldots)_{i \in I_1}, \quad x^2 = (\ldots, x(L_i^{s^0}, \varphi^{s^0}), \ldots)_{i \in I_2}.$$

Then (3) is also an internal direct product decomposition of L with the central element s^0.

Further suppose that we have another internal direct product decomposition of L with the central element s^0,

$$\psi^{s^0}\colon\ L \to \prod_{j \in J} P_j^{s^0}. \tag{4}$$

3.1. Proposition. *Let* (2) *and* (4) *be valid. Suppose that there are* $i(1) \in I$ *and* $j(1) \in J$ *such that* $L^{s^0}_{i(1)} = P^{s^0}_{j(1)}$. *Then for each* $x \in L$ *the components of* x *in* $L^{s^0}_{i(1)}$ *and* $P^{s^0}_{j(1)}$ *are equal, i.e.,*

$$x(L^{s^0}_{i(1)}, \varphi^{s^0}) = x(P^{s^0}_{j(1)}, \psi^{s^0}).$$

Proof. This is a consequence of Theorem (A) in [10]. $\square$

We denote by $\operatorname{Con} L$ the set of all congruence relations on L; this set is partially ordered in the usual way. $R_{\min}$ and $R_{\max}$ denote the least element of $\operatorname{Con} L$ or the greatest element of $\operatorname{Con} L$, respectively. For $x \in L$ and $R \in \operatorname{Con} L$ we put $x_R = \{y \in L\colon\ yRx\}$.

From the well-known theorem concerning direct products and congruence relations of universal algebras and from the definition of the internal direct product decomposition of a lattice we immediately obtain:

3.2. Proposition. *Let* $R(1)$ *and* $R(2)$ *be elements of* $\operatorname{Con} L$ *such that they are permutable,* $R(1) \wedge R(2) = R_{\min}$, $R(1) \vee R(2) = R_{\max}$. *Then the mapping*

$$\varphi\colon\ L \to s^0_{R(1)} \times s^0_{R(2)}$$

defined by

$$\varphi(x) = (x^1, x^2), \quad \text{where } \ \{x^1\} = x_{R(2)} \cap s^0_{R(1)}, \{x^2\} = x_{R(1)} \cap s^0_{R(2)}$$

is an internal direct product decomposition of L *with the central element* s^0.

3.3. Definition. Congruence relations $R(1)$ and $R(2)$ on L are called *interval permutable* if, whenever $[a, b]$ is an interval in L, then there are $x_1, x_2 \in [a, b]$ such that $aR(1)x_1R(2)b$ and $aR(2)x_2R(1)b$.

The following assertion is easy to verify (cf. also [1], p. 15, Exercise 13).

3.4. Lemma. *Let* $R(1)$ *and* $R(2)$ *be interval permutable congruence relations on* L. *Then*

(i) $R(1) \vee R(2) = R_{\max}$;

(ii) $R(1)$ *and* $R(2)$ *are permutable.*

If the relation (2) from Section 2 above is valid, then in view of 2.1, it suffices to express this fact by writing

$$L = (s^0) \prod_{i \in I} L_i, \tag{5}$$

where L_i has the same meaning as $L^{s^0}_i$ in (2) of Section 2.

Also, if $x \in L$, then instead of $x(L_i^{s^0}, \varphi^{s^0})$ we write simply $x(L_i)$.

If A, B are elements of $F(L, s^0)$ and $x \in L$, then the symbol $x(A)(B)$ means $(x(A))(B)$.

Let the system (F, L, s^0) be partially ordered by the set-theoretical inclusion.

3.5. Lemma. *$F(L, s^0)$ is a Boolean algebra.*

Proof. This is a consequence of Proposition 3.14 in [9]. □

It is obvious that if L is bounded, then $F(L, s^0)$ is isomorphic to the center of L.

Further, it is easy to verify that if $A, B \in F(L, s^0)$ and $L = (s^0)A \times B$, then B is the complement of A in the Boolean algebra $F(L, s^0)$; we denote $B = A'$.

4. α-completeness and infinite distributivity

Let α be an infinite cardinal. In this section we suppose that L is a lattice belonging to $\mathcal{T}_\alpha$ and that s^0 is an element of L.

Let I be a set with $\operatorname{card} I = \alpha$ and for each $i \in I$ let L_i be an element of $F(L, s^0)$. Thus for each $i \in I$ we have

$$L = (s^0)L_i \times L_i'. \tag{1}$$

For each $x \in L$ and each $i \in L$ we denote

$$x_i = x(L_i), \quad x_i' = x(L_i').$$

Let $x, y \in L$ and $i \in I$. We put xR_iy if $x_i' = y_i'$, similarly we set $xR_i'y$ if $x_i = y_i$. Then R_i and R_i' belong to $\operatorname{Con} L$, $R_i \wedge R_i' = R_{\min}$ and $R_i \vee R_i' = R_{\max}$. Moreover, R_i and R_i' are permutable.

4.1. Lemma. *Let $a, b \in L$, $a \leqslant b$. There exist elements x, y, x^i $(i \in I)$ in $[a, b]$ such that*

(i) *x^iR_ia for each $i \in I$;*

(ii) *$yR_i'a$ for each $i \in I$;*

(iii) *$x = \bigvee_{i \in I} x^i$, $x \wedge y = a$ and $x \vee y = b$.*

Proof. Let $i \in I$. There exist uniquely determined elements x^i and y^i in L such that

$$x^i \in a_{R_i} \cap b_{R_i'}, \quad y^i \in a_{R_i'} \cap b_{R_i}.$$

Hence

$$(x^i)'_i = a'_i, \quad (x^i)_i = b_i,$$
$$(y^i)'_i = b'_i, \quad (y^i)_i = a_i.$$

Then clearly

$$x^i \wedge y^i = a, \tag{2}$$
$$x^i \vee y^i = b. \tag{3}$$

Denote

$$x = \bigvee_{i \in I} x^i, \quad y = \bigwedge_{i \in I} y^i;$$

these elements exist in L since L is α-complete. By applying the infinite distributivity of L we get

$$y \wedge x = y \wedge \left(\bigvee_{i \in I} x^i \right) = \bigvee_{i \in I} (y \wedge x^i) = \bigvee_{i \in I} \bigwedge_{j \in I} (y^j \wedge x^i).$$

For $j = i$ we have $y^j \wedge x^i = a$ (cf. (2)). Hence for each $i \in I$ the relation

$$\bigwedge_{j \in I} (y^j \wedge x^i) = a$$

is valid. Thus

$$y \wedge x = a. \tag{4}$$

Further we obtain

$$x \vee y = x \vee \left(\bigwedge_{i \in I} y^i \right) = \bigwedge_{i \in I} (x \vee y^i) = \bigwedge_{i \in I} \bigvee_{j \in I} (x^j \vee y^i).$$

For $j = i$ we have $x^j \vee y^i = b$ (cf. (3)). Hence

$$\bigvee_{j \in I} (x^j \vee y^i) = b$$

for each $i \in I$. Therefore

$$x \vee y = b. \tag{5}$$

The definition of x and the relations (4), (5) yield that (iii) is valid. Now, in view of the definition of x^i, the condition (i) is satisfied. Let $i \in I$; then $y^i R'_i a$. Since $y \in [a, y^i]$, we obtain $y R'_i a$. Thus (ii) holds. $\square$

By an argument dual to that applied in the proof of 4.1 we obtain:

4.2. Lemma. *Let $a, b \in L$, $a \leqslant b$. There exist elements z, t, z^i $(i \in I)$ in $[a, b]$ such that*

(i) $z^i R_i b$ *for each* $i \in I$;

(ii) $t R'_i b$ *for each* $i \in I$;

(iii) $z = \bigwedge_{i \in I} z^i$, $z \vee t = b$ *and* $z \wedge t = a$.

4.3. Lemma. *Let a, b, x and x^i $(i \in I)$ be as in 4.1. Suppose that $u, v \in [a, x]$, $u \leqslant v$ and $u R'_i v$ for each $i \in I$. Then $u = v$.*

Proof. By way of contradiction, assume that $u < v$. From the definition of x we conclude that

$$v = u \vee (v \wedge x) = u \vee \left(v \wedge \bigvee_{i \in I} x^i \right) = \bigvee_{i \in I} (u \vee (v \wedge x^i)).$$

Hence there exists $i \in I$ such that $u < u \vee (v \wedge x^i)$. From $a R_i x^i$ we obtain

$$u \vee (v \wedge a) R_i (u \vee (v \wedge x^i)),$$

whence $u R_i (u \vee (v \wedge x^i)$. At the same time, since $u \vee (v \wedge x^i)$ belongs to the interval $[u, v]$ and $u R'_i v$, we get $r R'_i (u \vee (v \wedge x^i))$. Therefore $u = u \vee (v \wedge x^i)$, which is a contradiction. □

Analogously, by applying 4.2 we obtain

4.4. Lemma. *Let a, b and z be as in 4.2. Suppose that $u, v \in [z, b]$, $u \leqslant v$ and $u R'_i v$ for each $i \in I$. Then $u = v$.*

4.5. Lemma. *Let a, b, x, y, z and t be as in 4.1 and 4.2. Then $t = x$ and $z = y$.*

Proof. a) We have

$$t = t \wedge b = t \wedge (x \vee y) = (t \wedge x) \vee (t \wedge y).$$

The interval $[t \wedge x, x]$ is projectable to the interval $[t, t \vee x]$ and $[t, t \vee x] \subseteq [t, b]$. Hence in view of 4.2, $(t \wedge x) R'_i x$ for each $i \in I$. Thus according to 4.3, $t \wedge x = x$ and therefore $t \geqslant x$.

b) Analogously,

$$y = y \vee a = y \vee (t \wedge z) = (y \vee t) \wedge (y \vee z).$$

The interval $[y \wedge z, y]$ is projectable to the interval $[z, z \vee y]$ and $y \wedge z, y] \subseteq [a, y]$. Hence in view of 4.1, $zR_i'(z \vee y)$ for each $i \in I$. Then by applying 4.4 we get $y = z \vee y$, whence $z \geqslant y$.

c) Since L is distributive, if either $t > x$ or $z > y$ then $t \wedge z > a$, which is impossible in view of 4.2 (iii). Thus $t = x$ and $z = y$. $\square$

5. The relations R and R'

We apply the same assumptions and the same notation as in the previous section. If $a, b \in L$, $a \leqslant b$ and if x, y are as in 4.1, then we write

$$x = x(a, b), \quad y = y(a, b).$$

Let $p, q \in L$. We put pRq if

$$x(p \wedge q, p \vee q) = p \vee q.$$

Further we put $pR'q$ if

$$y(p \wedge q, p \vee q) = p \vee q.$$

Thus $pR'q$ if and only if $pR_i'q$ for each $i \in I$. Hence we have

5.1. Lemma. *R' is a congruence relation on L.*

In view of the definition, the relation R is reflexive and symmetric.

5.2. Lemma. *Let $p, q \in L$. Then the following conditions are equivalent:*

(i) *pRq.*

(ii) *There exists no interval $[u, v] \subseteq L$ such that $[u, v] \subseteq [p, \wedge q, p \vee q]$, $u < v$ and $uR_i'v$ for each $i \in I$.*

P r o o f. Denote $p \wedge q = a$, $p \vee q = b$. Let (i) be valid. Then in view of 4.2, the condition (ii) is satisfied. Conversely, assume that (ii) holds. Put $x(a, b) = x$, $y(a, b) = y$. If $y > a$, then by putting $[u, v] = [a, y]$ we arrive at a contradiction with the condition (ii). Hence $y = a$. Then 4.1 yields that $x = b$, whence (i) is valid. $\square$

5.2.1. Corollary. *Let $a_1, a_2, b_1, b_2 \in L$, $a_1 \leqslant b_1 \leqslant b_2 \leqslant a_2$, a_1Ra_2. Then b_1Rb_2.*

5.3. Lemma. *Let $a_1, a_2, a_3 \in L$, $a_1 \leqslant a_2 \leqslant a_3$, a_1Ra_2, a_2Ra_3. Then a_1Ra_3.*

P r o o f. Suppose that $[u,v] \subseteq [a_1,a_3]$ and $uR'v$. Denote

$$u_1 = u \wedge a_2, \quad v_1 = v \wedge a_2, \quad u_2 = u \vee a_2, \quad v_2 = v \vee a_2,$$
$$s = v_1 \vee u.$$

Thus $u \leqslant s \leqslant v$. Hence if $u < v$, then either $u < s$ or $s < v$.

It is easy to verify that $[u,s]$ is projectable to a subinterval of $[a_1,a_2]$ (namely, to the interval $[v_1 \wedge u, v_1]$). Hence $(v_1 \wedge u)R'v_i$ and thus $v_1 \wedge u = v_1$. Therefore $u = s$. Analogously we obtain the relation $s = v$. Thus $u = v$. According to 5.2, a_1Ra_2. $\square$

5.4. Lemma. *Let $a_1,a_2 \in L$, $s \in L$, a_1Ra_2. Then $(a_1 \vee s)R(a_2 \vee s)$ and $(a_1 \wedge s)R(a_2 \wedge s)$.*

P r o o f. If $[u,v]$ is a subinterval of $[a_1 \vee s, a_2 \vee s]$, then $[u,v]$ is projectable to the interval $[a_2 \wedge u, a_2 \wedge v]$ and this is a subinterval of $[a_1,a_2]$. Hence in view of 5.2, if $uR'v$, then $u = v$. Therefore $(a_1 \vee s)R(a_2 \vee s)$. Similarly we verify that $(a_1 \wedge s)R(a_2 \wedge s)$. $\square$

5.5. Lemma. *The relation R is transitive.*

P r o o f. Let $p_1,p_2,p_3 \in L$, p_1Rp_2, p_2Rp_3. Denote

$$p_1 \wedge p_2 = u_1, \quad p_2 \wedge p_3 = u_2, \quad u_1 \wedge u_2 = u_3,$$
$$p_1 \vee p_2 = v_1, \quad p_2 \vee p_3 = v_2, \quad v_1 \vee v_2 = v_3.$$

In view of 5.4 we have $p_1Rp_1 \wedge p_2$, thus p_1Ru_1. Analogously we obtain p_2Ru_2. The interval $[u_3,u_1]$ is projectable to some subinterval of $[u_2,p_2]$, hence u_3Ru_1. Similarly we verify that p_1Rv_1 and v_3Rv_1. Thus u_3Rv_3 by 5.2.1. Since $[p_1 \wedge p_3, p_1 \vee p_3] \subseteq [u_3,v_3]$, 5.2 yields that p_1Rp_3. $\square$

From 5.4 and 5.5 we infer

5.6. Lemma. *R is a congruence relation on L.*

5.7. Lemma. *$R \wedge R' = R_{\min}$, $R \vee R' = R_{\max}$ and R,R' are permutable.*

P r o o f. In view of 5.2 we have $R \wedge R' = R_{\min}$. Let $a,b \in L$, $a \leqslant b$. Let x and y be as in 4.1. Then we have

$$aRx, \quad aR'y. \tag{1}$$

Further, $x \wedge y = a$ and $x \vee y = b$. Thus in view of the projectability we obtain

$$xR'b, \quad yRb. \tag{2}$$

Hence $a(R \vee R')b$. From this we easily obtain $R \vee R' = R_{\max}$. Further, from (1), (2) and 3.4 we conclude that R and R' are permutable. $\square$

P r o o f o f T h e o r e m 1. Let $L \in \mathcal{T}_\alpha$ and $s^0 \in L$. Let $\{L_i\}_{i\in I}$ be a subset of $F(L, s^0)$ such that $\operatorname{card} I \leqslant \alpha$. First we verify that $\bigvee_{i\in I} L_i$ exists in the Boolean algebra $F(L, s^0)$. Let us apply the notation as above.

Consider the lattices s^0_R and $s^0_{R'}$. According to 5.1, 5.6, 5.7 and 3.2 we have

$$L = (s^0) s^0_R \times s^0_{R'}. \tag{3}$$

According to the definition of R' we obviously have

$$s^0_{R'} = \bigcap_{i\in I} L'_i. \tag{4}$$

Then (3) and (4) yield

$$s^0_{R'} = \bigwedge_{i\in I} L'_i. \tag{5}$$

Further, in view of the definition of R, $L_i \subseteq s^0_R$ for each $i \in I$. Let $X \in F(L, s^0)$ and suppose that $L_i \subseteq X$ for each $i \in I$. Put $Y = X \cap s^0_R$. Then $Y \in F(L, s^0)$ and $L_i \subseteq Y$ for each $i \in I$. Moreover, Y is a closed sublattice of L.

Let $p \in s^0_R$. Put $a = p \wedge s^0$ and $b = p \vee s^0$. Thus $a, b \in s^0_R$. Hence $s^0 R b$. In view of the definition of R there exist $x^i \in [s^0, b]$ $(i \in I)$ such that $x^i \in L_i$ and $\bigvee_{i\in I} x^i = b$. Then all x^i belong to Y; since Y is closed, we get $b \in Y$. By a dual argument (using Lemma 4.2) we obtain the relation $a \in Y$. Hence, by the convexity of Y, the element p belongs to Y. Therefore, $s^0_R \subseteq Y$. Thus

$$s^0_R = \bigvee_{i\in I} L_i. \tag{6}$$

Further, we have to verify that each subset of $F(L, s^0)$ having the cardinality $\leqslant \alpha$ possesses the infimum. But this is a consequence of the just proved result concerning the existence of suprema and of the fact that each Boolean algebra is self-dual. □

5.8. Corollary. *Under the assumptions as in* Theorem 1 *and under the notation as above we have*

$$L = (s^0)\Big(\bigvee_{i\in I} L_i\Big) \times \Big(\bigwedge_{i\in I} L'_i\Big).$$

P r o o f. This is a consequence of (3)–(6). □

Let L be a lattice which belongs to the class $\mathcal{T}'_\sigma$. The least element of L will be denoted by s^0. Suppose that for each $n \in \mathbb{N}$ we have

$$L = (s^0)L_n \times L'_n. \tag{1}$$

We apply analogous notation as in Sections 4 and 5 above (with the distinction that we now have $\mathbb{N}$ instead of I).

6.1. Lemma. *Suppose that $L_{n(1)} \cap L_{n(2)} = \{s^0\}$ whenever $n(1)$ and $n(2)$ are distinct positive integers. For each $n \in \mathbb{N}$ let $x^n \in L_n$ and let $x = \bigvee\limits_{n\in\mathbb{N}} x^n$. Then $x(L_n) = x^n$.*

Proof. We have $x^n(L_n) = x^n$. Since $x^n \leqslant x$, we obtain $x^n \leqslant x(L_n)$. Further, from $x = x(L_n) \vee x(L'_n)$ we get $x(L_n) \leqslant x$.

Clearly, $x(L_n) \wedge x^m = s^0$ for each $m \in \mathbb{N} \setminus \{n\}$, thus

$$x(L_n) = x(L_n) \wedge x = x(L_n) \wedge \left(\bigvee_{m\in\mathbb{N}} x^m\right) = \bigvee_{m\in\mathbb{N}} (x(L_n) \wedge x^m) = x(L_n) \wedge x^n.$$

Summarizing, we obtain $x(L_n) = x^n$. □

6.2. Lemma. *s^0_R is the set of all elements $x \in L$ which can be expressed in the form $x = \bigvee\limits_{n\in\mathbb{N}} x^n$, where $x^n \in L_n$ for each $n \in \mathbb{N}$.*

Proof. This is a consequence of 5.6 and of the fact that s^0 is the least element of s^0_R. □

6.3. Lemma. *Let $(L_n)_{n\in\mathbb{N}}$ be as in 6.1. Then the mapping*

$$\varphi^0 \colon s^0_R \to \prod_{n\in\mathbb{N}} L_n,$$

where $\varphi^0(x) = (\ldots, x(L_n), \ldots)_{n\in\mathbb{N}}$, is an internal direct product decomposition of s^0_R with the central element s^0.

Proof. In view of the definition, φ^0 is a homomorphism. According to 6.1 and 6.2, φ^0 is surjective and injective. □

If A and B are elements of $F(L,s^0)$ such that $A \subseteq B$, then there exists $C \in F(L,s^0)$ with $B = (s^0)A \times C$; moreover, C is uniquely determined. We denote

$$C = B - A.$$

In fact, C is the relative complement of the element A in the interval $[\{s^0\}, B]$ of the Boolean algebra $F(L,s^0)$.

We will use Theorem 1 and apply the method which is analogous to the well-known argument of the proof of Cantor-Bernstein Theorem of set theory.

6.4. Lemma. *Let A, B be elements of $F(L,s^0)$ such that $A \supseteq B$. Assume that B is isomorphic to L. Then A is isomorphic to L as well.*

Proof. There exists an isomorphism f of L onto B. Put $A_1 = L$, $A_2 = A$. Inductively we define

$$A_{n+2} = f(A_n)$$

for each $n \in \mathbb{N}$. Hence

$$A_{n+2} \simeq A_n \quad \text{for each } n \in \mathbb{N}, \tag{2}$$

where $\simeq$ is the relation of isomorphism between lattices.

By induction we can verify that $A_n \in F(L,s^0)$ and

$$A_n \supseteq A_{n+1} \quad \text{for each } n \in \mathbb{N}. \tag{3}$$

For $n \in \mathbb{N}$ we denote

$$L_n = A_n - A_{n+1}. \tag{4}$$

Then (2) yields

$$L_{n+2} \simeq L_n \quad \text{for each } n \in \mathbb{N}. \tag{5}$$

If $n(1)$ and $n(2)$ are distinct positive integers, then

$$L_{n(1)} \cap L_{n(2)} = \{s^0\}. \tag{6}$$

Put

$$C = \bigcap_{n=1}^{\infty} A_n.$$

According to Theorem 1, C belongs to $F(L, s^0)$. Consider the complement C' of C in $F(L, s^0)$.

From (4) we obtain

$$A_{n+1} \subseteq L'_n \quad \text{for each } m \in \mathbb{N}.$$

Hence according to (3),

$$C \subseteq \bigcap_{n \in \mathbb{N}} L'_n.$$

Let $x \in \bigcap_{n \in \mathbb{N}} L'_n$. Then $x \in A_1$. Suppose that $x \in A_n$ for some $n \in \mathbb{N}$. Since $x \in L'_n$ we get $x(L_n) = s^0$ and thus, in view of (4), $x \in A_{n+1}$. Therefore we obtain by induction that x belongs to C. Summarizing we have

$$C = \bigcap_{n \in \mathbb{N}} L'_n.$$

Hence according to Theorem 1 we obtain

$$L = (s^0)s^0_R \times C.$$

Moreover, in view of 6.3, we get an internal direct product decomposition with the central element s^0

(7) $$\varphi^1\colon\ L \to \left(\prod_{n \in \mathbb{N}}^{\infty} L_n\right) \times C.$$

Since $A = A_1$, we obtain analogously an internal direct product decomposition with the central element s^0,

(8) $$\varphi^2\colon\ A \to \left(\prod_{n=2}^{\infty} L_n\right) \times C.$$

Now, (7), (8) and (5) yield that L and A are isomorphic. $\square$

P r o o f o f T h e o r e m 2. Let the assumptions of Theorem 2 be valid. The least element of L_1 and L_2 will be denoted by s^0 or by t^0, respectively. In view of the assumption there exist $A_1 \in F(L_1, s^0)$, $B_1 \in F(L_2, t^0)$, an isomorphism f of L_1 onto B_1 and an isomorphism g of L_2 onto A_1. Put $A_2 = g(B_1)$. Then $A_2 \in F(A_1, s^0)$, whence $A_2 \in F(L_1, s^0)$ and A_2 is isomorphic to B_1. Thus A_2 is isomorphic to L_1. Then in view of 6.4, A_1 is isomorphic to L_1. Hence L_1 is isomorphic to L_2. $\square$

If L is a Boolean algebra, then each interval of L is isomorphic to a direct factor of L. Further, each Boolean algebra is infinitely distributive and contains the least element. Hence Theorem 2 yields as a corollary the following result:

6.5. Theorem. (Sikorski [13]; cf. also Sikorski [14] and Tarski [15].) *Let L_1 and L_2 be σ-complete Boolean algebras. Suppose that*

(i) *there exists $a_2 \in L_2$ such that L_1 is isomorphic to the interval $[0, a_2]$ of L_2;*

(ii) *there exists $a_1 \in L_1$ such that L_2 is isomorphic to the interval $[0, a_1]$ of L_1.*

Then L_1 and L_2 are isomorphic.

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