ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SOME LINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we investigate the asymptotic properties of all solutions of the delay differential equation

$$y'(x) = a(x)y(\tau(x)) + b(x)y(x), \qquad x \in I = [x_0, \infty).$$

We set up conditions under which every solution of this equation can be represented in terms of a solution of the differential equation

$$z'(x) = b(x)z(x), \qquad x \in I$$

and a solution of the functional equation

$$|a(x)|\varphi(\tau(x)) = |b(x)|\varphi(x), \qquad x \in I.$$

Keywords: asymptotic behaviour, differential equation, delayed argument, functional equation

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1. Introduction

We consider the linear differential equation with the delayed argument in the form

$$y'(x) = a(x)y(\tau(x)) + b(x)y(x), \qquad x \in I = [x_0, \infty). \tag{1.1}$$

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The asymptotic behaviour of solutions of equation (1.1) has been studied in many papers (for results and references see, e.g., [7]). Among the works related to our present results we can mention papers [2] by N. G. de Bruijn, [9] by T. Kato and J. B. McLeod, [8] by M. L. Heard, [11] by F. Neuman, [6] by I. Győri and M. Pituk, [5] by J. Diblík and [3], [4].

The idea that we wish to generalize first appeared in [9]. The authors derived asymptotic formulas for solutions of the equation

$$y'(x) = a\,y(\lambda x) + b\,y(x), \qquad x \in [0,\infty)$$

in terms of functions $\varphi(x) = |\psi(x)|$, where $\psi(x) = x^\beta$, $\beta = \frac{\log \frac{a}{-b}}{\log \lambda^{-1}}$. Note that the function $\psi(x)$ defines a solution of the functional (nondifferential) equation

$$a\,\psi(\lambda x) + b\,\psi(x) = 0, \qquad x \in [0,\infty)$$

and the function $\varphi(x) = |\psi(x)|$ fulfils

$$|a|\,\varphi(\lambda x) = |b|\,\varphi(x), \qquad x \in [0,\infty).$$

M. L. Heard [8] considered a more general equation

$$y'(x) = a\,y(\tau(x)) + b\,y(x), \qquad x \in I \tag{1.2}$$

under the hypothesis $a \neq 0$, $b < 0$, $\tau \in C^2(I)$, τ' being decreasing on I. The asymptotic behaviour of all solutions of this equation was related to the behaviour of a solution of the equation

$$a\,\psi(\tau(x)) + b\,\psi(x) = 0, \qquad x \in I.$$

The generalization of this asymptotic result to equation (1.2) with variable coefficients has been carried out in [3]. Similarly as in [8], the assumption $b(x) < 0$ was necessary to preserve the validity of the corresponding estimates.

Our aim is to discuss the relationship between the asymptotic behaviour of solutions of equation (1.1) and the functional equation

$$|a(x)|\varphi(\tau(x)) = |b(x)|\varphi(x), \qquad x \in I \tag{1.3}$$

in the case $b(x) > 0$. We show, under additional assumptions, that every solution $y(x)$ of (1.1) is asymptotic to a solution $z(x)$ of the equation

$$z'(x) = b(x)z(x), \qquad x \in I$$

and, moreover, the difference of any two solutions $y_1(x)$, $y_2(x)$ of (1.1) such that $y_1(x)$ is asymptotic to $y_2(x)$, approaches a solution $\varphi(x)$ of (1.3).

Throughout this paper we denote $I = [x_0, \infty)$ and $I^* = [\tau(x_0), \infty)$. By a solution of (1.1) we understand a function $y(x) \in C^0(I^*) \cap C^1(I)$ fulfilling (1.1) for every $x \in I$. Further, by the symbol $\tau^n(x)$ we denote the n-th iterate of $\tau(x)$ (for positive integers n) or the $-n$-th iterate of the inverse function $\tau^{-1}(x)$ (for negative integers n) and put $\tau^0(x) = x$.

2. Results

We start with the study of equation (1.3) under the assumption $|a(x)| = K|b(x)|$ for every $x \in I$ and a suitable $K > 0$. The following statement yields the form of a solution $\varphi(x)$ of (1.3) in terms of a solution $\alpha(x)$ of the Abel equation

$$\alpha(\tau(x)) = \alpha(x) - 1, \qquad x \in I. \tag{2.1}$$

Proposition. *Let $b(x)$, $\tau(x) \in C^0(I)$, $b(x) \neq 0$, $|a(x)| = K|b(x)|$ for every $x \in I$ and a suitable $K > 0$, $\tau(x) < x$ and $\tau(x)$ being increasing on I. Then there exists an increasing solution $\alpha(x) \in C^0(I^*)$ of equation* (2.1) *and the function*

$$\varphi(x) = K^{\alpha(x)}, \qquad x \in I^* \tag{2.2}$$

defines a continuous positive and monotonic solution of (1.3).

P r o o f. Put $x_j = \tau^{-j}(x_0)$, $j = -1, 0, 1, \dots$ and denote $I_j = [x_{j-1}, x_j]$, where $j = 0, 1, 2, \dots$. We consider an increasing function $\alpha_0(x) \in C^0(I_0)$ such that

$$\alpha_0(x_{-1}) = \alpha_0(x_0) - 1.$$

Then the function

$$\alpha(x) = \alpha_0(\tau^n(x)) + n, \qquad x \in I_n, \quad n = 0, 1, 2, \dots$$

is a continuous increasing solution of (2.1).

Substituting $\varphi(x) = K^{\alpha(x)}$ into (1.3) it is easy to check that this function defines a solution of (1.3) with the required properties. □

R e m a r k 1. We note that the solutions of the Abel equation (2.1) can be given explicitly in some important cases (e.g., if $\tau(x) = x - r$, $\tau(x) = \lambda x$, $\tau(x) = x^\gamma$). For methods of solving the Abel equation and other functional equations we refer to [10].

To study the asymptotic behaviour at infinity of all solutions of (1.1) we first recall the following result which is due to I. Győri and M. Pituk [6]. The authors considered the equation

$$z'(x) = p(x)z(\tau(x)), \qquad x \in I. \tag{2.3}$$

For

$$p^-(x) = \max(0, -p(x)), \qquad x \in I$$

we have

Theorem 1. *Let $p(x), \tau(x) \in C^0(I)$, $\tau(x) < x$ for every $x \in I$. If*

$$\int_{x_0}^{\infty} |p(x)|\,\mathrm{d}x < \infty, \tag{2.4}$$

then every solution $z(x)$ of (2.3) tends to a finite (possibly zero) constant $L \in \mathbb{R}$. In addition to (2.4) assume that

$$\int_{x_0}^{\infty} p^-(x)\,\mathrm{d}x < 1. \tag{2.5}$$

Then for every $L \in \mathbb{R}$ there exists a solution $z^*(x)$ of (2.3) such that $\lim\limits_{x\to\infty} z^*(x) = L$.

Using Theorem 1 it is easy to prove

Lemma 1. *Let $a(x), \tau(x) \in C^0(I)$, $b(x) \in C^0(I^*)$, $\tau(x) < x$ for every $x \in I$ and let*

$$\int_{x_0}^{\infty} \left(|a(x)| \exp\left\{-\int_{\tau(x)}^{x} b(s)\,\mathrm{d}s\right\}\right)\mathrm{d}x < \infty. \tag{2.6}$$

If $y(x)$ is any solution of (1.1), then

$$\lim_{x\to\infty}\left(\exp\left\{-\int_{x_0}^{x} b(s)\,\mathrm{d}s\right\}y(x)\right) = L \in \mathbb{R}. \tag{2.7}$$

Conversely, we can choose $\sigma \geqslant x_0$ such that there exists a function $y^*(x)$ fulfilling (1.1) on $[\sigma, \infty)$ and

$$\lim_{x\to\infty}\left(\exp\left\{-\int_{x_0}^{x} b(s)\,\mathrm{d}s\right\}y^*(x)\right) = 1.$$

P r o o f. Put $z(x) = \exp\{-\int_{x_0}^{x} b(s)\,\mathrm{d}s\}y(x)$ in (1.1) to obtain equation (2.3) with

$$p(x) = a(x)\exp\Big\{-\int_{\tau(x)}^{x} b(s)\,\mathrm{d}s\Big\}, \qquad x \in I.$$

The first part of the statement follows immediately from Theorem 1. To prove the second part it is enough to consider $\sigma \geqslant x_0$ large enough so that (2.5) holds with x_0 replaced by σ. □

R e m a r k 2. If the integral condition (2.6) is fulfilled and, moreover,

$$\int_{x_0}^{\infty}\Big(a^-(x)\exp\Big\{-\int_{\tau(x)}^{x} b(s)\,\mathrm{d}s\Big\}\Big)\,\mathrm{d}x < 1,$$

where $a^-(x) = \max(0, -a(x))$, $x \in I$, then we can put $\sigma = x_0$. This case occurs, e.g., provided $a(x) > 0$ for every $x \in I$.

R e m a r k 3. The assumption $b(x) > 0$ for every $x \in I$ is not necessary to ensure the validity of (2.6). However, in the sequel we consider delays $\tau(x)$ with the property $0 < \tau'(x) \leqslant \lambda < 1$. Under such a requirement it is natural to assume positive values of $b(x)$ to satisfy (2.6). E.g., if $b(x) \geqslant \delta > 0$ and $\tau'(x) \leqslant \lambda < 1$ for every $x \in I$, then it is enough to assume $a(x) = O(\mathrm{e}^{\gamma x})$ as $x \to \infty$, $\gamma < \delta(1-\lambda)$, to fulfil condition (2.6).

Lemma 2. *Let $b(x) \in C^0(I)$, $\tau(x) \in C^1(I)$, let $b(x)$ be positive and nondecreasing on I, $|a(x)| = Kb(x)$ for every $x \in I$ and a constant $K > 0$, $\tau(x) < x$ and $0 < \tau'(x) \leqslant \lambda < 1$ for every $x \in I$. Assume that $\varphi(x)$ is a continuous positive solution of (1.3) given by (2.2). If $y(x)$ is a solution of (1.1) satisfying*

$$y(x) = o\Big(\exp\Big\{\int_{x_0}^{x} b(s)\,\mathrm{d}s\Big\}\Big) \qquad \text{as } x \to \infty,$$

then

$$y(x) = O(\varphi(x)) \qquad \text{as } x \to \infty.$$

P r o o f. Multiply both sides of equation (1.1) by $\exp\{-\int_{x_0}^{x} b(s)\,\mathrm{d}s\}$ to get

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\exp\Big\{-\int_{x_0}^{x} b(s)\,\mathrm{d}s\Big\}y(x)\Big] = a(x)\exp\Big\{-\int_{x_0}^{x} b(s)\,\mathrm{d}s\Big\}y(\tau(x)).$$

Integrating this equality over $[x,\infty)$ we obtain

$$y(x) = -\exp\Big\{\int_{x_0}^{x} b(s)\,\mathrm{d}s\Big\}\int_{x}^{\infty}\Big(a(t)\exp\Big\{-\int_{x_0}^{t} b(s)\,\mathrm{d}s\Big\}y(\tau(t))\Big)\,\mathrm{d}t$$

by using the relation $\lim_{x\to\infty} (y(x)\exp\{-\int_{x_0}^x b(s)\,\mathrm{d}s\}) = 0$.

Put $x_n = \tau^{-n}(x_0)$, $n = 0, 1, 2, \dots$ and assume that $M > 0$ is such that

$$|y(x)| \leqslant M \exp\left\{\int_{x_0}^x b(s)\,\mathrm{d}s\right\}, \qquad x \geqslant x_0.$$

Then

$$\begin{aligned}
|y(x)| &\leqslant M \exp\left\{\int_{x_0}^x b(s)\,\mathrm{d}s\right\} \int_x^\infty \left(|a(t)| \exp\left\{-\int_{\tau(t)}^t b(s)\,\mathrm{d}s\right\}\right)\mathrm{d}t \\
&= MK \exp\left\{\int_{x_0}^x b(s)\,\mathrm{d}s\right\} \int_x^\infty \left(b(t) \exp\left\{-\int_{\tau(t)}^t b(s)\,\mathrm{d}s\right\}\right)\mathrm{d}t \\
&\leqslant MK \exp\left\{\int_{x_0}^x b(s)\,\mathrm{d}s\right\} \\
&\quad \times \int_x^\infty \left(\frac{b(t)}{-b(t) + b(\tau(t))\tau'(t)} \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left\{-\int_{\tau(t)}^t b(s)\,\mathrm{d}s\right\}\right]\right)\mathrm{d}t \\
&\leqslant MK \exp\left\{\int_{x_0}^x b(s)\,\mathrm{d}s\right\} \frac{1}{1-\lambda} \exp\left\{-\int_{\tau(x)}^x b(s)\,\mathrm{d}s\right\} \\
&= \frac{MK}{1-\lambda} \exp\left\{\int_{x_0}^{\tau(x)} b(s)\,\mathrm{d}s\right\}, \qquad x \geqslant x_1.
\end{aligned}$$

Further, repeating this we can deduce that

$$|y(x)| \leqslant \frac{MK^n}{(1-\lambda)\dots(1-\lambda^n)} \exp\left\{\int_{x_0}^{\tau^n(x)} b(s)\,\mathrm{d}s\right\}, \qquad x \geqslant x_n,$$

$n = 1, 2, \dots$. Since

$$\exp\left\{\int_{x_0}^{\tau^n(x)} b(s)\,\mathrm{d}s\right\} \leqslant \exp\left\{\int_{x_0}^{x_1} b(s)\,\mathrm{d}s\right\}, \qquad x \leqslant x_{n+1},$$

$n = 1, 2, \dots$, we can estimate $y(x)$ as

$$|y(x)| \leqslant M_n K^n, \qquad x_n \leqslant x \leqslant x_{n+1}, \tag{2.8}$$

where $M_n = \frac{M}{(1-\lambda)\dots(1-\lambda^n)} \exp\{\int_{x_0}^{x_1} b(s)\,\mathrm{d}s\}$.

On the other hand,

$$|\varphi(x)| \geqslant NK^n, \qquad x_n \leqslant x \leqslant x_{n+1}, \tag{2.9}$$

where $N > 0$ is a constant. Summarizing (2.8) and (2.9) we have

$$y(x) = O(\varphi(x)) \qquad \text{as } x \to \infty.$$

□

Lemmas 1 and 2 yield

Theorem 2. *Let $b(x) \in C^0(I)$, $\tau(x) \in C^1(I)$, let $b(x)$ be positive and nondecreasing on I, $|a(x)| = Kb(x)$ for every $x \in I$ and a constant $K > 0$, $\tau(x) < x$ and $0 < \tau'(x) \leqslant \lambda < 1$ for every $x \in I$. Further, assume that $\varphi(x)$ is a continuous positive solution of* (1.3) *given by* (2.2). *Then for any solution $y(x)$ of* (1.1) *there exists a constant $L \in \mathbb{R}$ and a function $g(x)$ such that*

$$y(x) = Ly^*(x) + g(x), \qquad x \geqslant \sigma, \tag{2.10}$$

where L, $y^*(x)$ and $\sigma \geqslant x_0$ are given by Lemma 1 *and $g(x) = O\left(\varphi(x)\right)$ as $x \to \infty$.*

R e m a r k 4. In the sequel we wish to show that the O-estimate of a function $g(x)$ given in Theorem 2 is strong enough. We introduce a change of variables

$$z(t) = \frac{y(h(t))}{\psi(h(t))}, \tag{2.11}$$

where $\psi(x) \in C^1(I)$, $|\psi(x)| > 0$ on I, is a solution of the functional equation

$$a(x)\psi(\tau(x)) + b(x)\psi(x) = 0, \qquad x \in I \tag{2.12}$$

and $h(t) = \alpha^{-1}(t)$ on $\alpha(I)$, $\alpha(x) \in C^1(I)$ being a solution of the Abel equation (2.1) such that $\alpha'(x) > 0$ for every $x \in I$. We note that the existence of a solution $\alpha(x)$ of (2.1) with such properties is ensured provided $\tau(x) \in C^1(I)$, $\tau(x) < x$ and $\tau'(x) > 0$ for every $x \in I$ (for more information about the transformation theory of functional differential equations see [11]).

If we assume $|a(x)| = Kb(x)$ for every $x \in I$ and a constant $K > 0$, then equation (2.12) admits the solution $\psi(x) = \overline{K}^{\alpha(x)}$, where $\overline{K} = -K \operatorname{sign} a(x_0)$.

Transformation (2.11) converts equation (1.1) into the form

$$w(t)\dot{z}(t) + p(t)z(t) - z(t-1) = 0, \tag{2.13}$$

where

$$w(t) = \frac{1}{-b(h(t))\dot{h}(t)}, \; p(t) = 1 + \frac{\dot{\psi}(h(t))\dot{h}(t)}{\psi(h(t))} w(t) = 1 + \ln \overline{K} w(t)$$

and thus equation (1.1) becomes the type discussed by N. G. de Bruijn in [2]. The relevant theorem reads as follows:

Let B and ϱ be positive constants, $\varrho > 1$, and suppose that for $t \geqslant 1$ the functions $w^n(t)$ and $p^{(n)}(t)$, $n = 0, 1, 2, \ldots$, are continuous and satisfy

$$|w^{(n)}(t)| < B^{n+1} n^n t^{-n-\varrho}, \quad |\{p(t) - 1\}^{(n)}| < B^{n+1} n^n t^{-n-\varrho} \quad (0^0 = 1). \tag{2.14}$$

Then, if $z(t)$ is a solution of (2.13) *and* $\lim_{t\to\infty} z(t) = 0$, *we have* $z(t) \equiv 0$.

Now we substitute back transformation (2.11) to obtain (with respect to $\varphi(x) = |\psi(x)|$) the following result:

In addition to the assumptions of Theorem 2 we assume that conditions (2.14) with the above specified $w(t)$ and $p(t)$ are fulfilled for $t \geqslant 1$. Then all conclusions of Theorem 2 remain valid and, moreover, if the function $g(x)$ satisfies $g(x) = o\left(\varphi(x)\right)$ as $x \to \infty$, then $g(x)$ is the identically zero function on $[\sigma, \infty)$.

We note that both inequalities contained in (2.14) coincide provided $|a(x)| = Kb(x)$.

3. Applications

In this section we give two examples to illustrate the above results.

Example 1. We consider the equation

$$y'(x) = axy(\lambda x) + bxy(x), \qquad x \in [1, \infty), \tag{3.1}$$

where $a \neq 0$, $b > 0$, $0 < \lambda < 1$. Functional equation (2.12) becomes

$$ax\psi(\lambda x) + bx\psi(x) = 0, \qquad x \in I$$

and has a solution $\psi(x) = x^\beta$, $\beta = \dfrac{\log \frac{a}{-b}}{\log \lambda^{-1}}$. Then

$$\varphi(x) = |\psi(x)| = x^{|\beta|}, \quad |\beta| = \frac{\log \left|\frac{a}{b}\right|}{\log \lambda^{-1}}$$

is a solution of (1.3), where $a(x) = ax$, $b(x) = bx$, $\tau(x) = \lambda x$. The Abel equation (2.1) can be read as

$$\alpha(\lambda x) = \alpha(x) - 1, \qquad x \in [1, \infty)$$

and admits a solution $\alpha(x) = \dfrac{\log x}{\log \lambda^{-1}}$ with positive derivative on $[1, \infty)$. Then $h(t) = \alpha^{-1}(t) = \lambda^{-t}$. Now it is easy to verify that the assumptions of Theorem 2 and Remark 4 imposed on $a(x) = ax$, $b(x) = bx$, $\tau(x) = \lambda x$ are satisfied and we may summarize the results as follows:

Consider equation (3.1), where $a \neq 0$, $b > 0$ and $0 < \lambda < 1$. Then there exists a $\sigma \geqslant x_0$ and a function $y^*(x)$ fulfilling (3.1) on $[\sigma, \infty)$ such that

$$y^*(x) \sim \exp\left\{\frac{b}{2}x^2\right\} \qquad \text{as } x \to \infty.$$

Furthermore, for any solution $y(x)$ of (3.1) there exists a constant $L \in \mathbb{R}$ and a function $g(x)$, $g(x) = O\left(x^{|\beta|}\right)$ as $x \to \infty$, $\beta = \dfrac{\log \frac{a}{-b}}{\log \lambda^{-1}}$, such that

$$y(x) = Ly^*(x) + g(x), \qquad x \geqslant \sigma.$$

If $g(x) = o\left(x^{|\beta|}\right)$ as $x \to \infty$, then $g(x)$ is the zero function on $[\sigma, \infty)$, i.e., $y(x)$ is a constant multiple of $y^*(x)$.

Example 2. We apply our asymptotic results to equation (1.1) with $a(x) = -b(x)$, i.e., we consider the equation

$$y'(x) = b(x)[y(x) - y(\tau(x))], \qquad x \in I, \tag{3.2}$$

where $b(x) \in C^0(I)$, $\tau(x) \in C^1(I)$, $b(x)$ is positive and nondecreasing on I, $\tau(x) < x$ and $0 < \tau'(x) \leqslant \lambda < 1$ for every $x \in I$. Equations (2.12) and (1.3) with $a(x) = -b(x)$ admit a constant solution. Then we get the following statement:

Let the above introduced assumptions on $b(x)$ and $\tau(x)$ be fulfilled. Then there exists a $\sigma \geqslant x_0$ and a function $y^*(x)$ fulfilling (3.2) on $[\sigma, \infty)$ such that

$$y^*(x) \sim \exp\left\{\int_{x_0}^{x} b(s)\,\mathrm{d}s\right\} \qquad \text{as } x \to \infty.$$

Furthermore, any solution $y(x)$ of (3.2) can be represented in the form

$$y(x) = Ly^*(x) + g(x), \qquad x \geqslant \sigma, \tag{3.3}$$

where $L \in \mathbb{R}$ is a constant depending on $y(x)$ and $g(x)$ is a bounded function fulfilling (3.2) on $[\sigma, \infty)$. Assume that conditions (2.14) specified in Remark 4 are fulfilled. If the bounded function $g(x)$ tends to zero, then $g(x)$ must be identically zero on $[\sigma, \infty)$.

Equation (3.2) has been studied by several authors, usually under the assumption $\tau(x) = x - r$ or, more generally, $\tau(x) = x - r(x)$, $r(x)$ being bounded (see, e.g., Atkinson and Haddock [1], J. Diblík [5] and S. N. Zhang [12]). We mention the result derived in [5], where equation (3.2) has been considered under the assumptions $b(x)$, $\tau(x) \in C^0(I)$, $b(x) > 0$, $\tau(x) < x$, where $\tau(x)$ is increasing and $r(x) = x - \tau(x)$ is bounded for every $x \in I$. It is interesting that the structure formula derived in [5] for solutions $y(x)$ of (3.2) coincides with formula (3.3) including the boundedness of $g(x)$ even if our assumption $\tau'(x) \leqslant \lambda < 1$ implies that $r(x) = x - \tau(x)$ is unbounded. Therefore our approach enables us to extend some asymptotic results to a wider class of equations (3.2).

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