ON THE EXTENSION OF EXPONENTIAL POLYNOMIALS

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Abstract. Exponential polynomials are the building bricks of spectral synthesis. In some cases it happens that exponential polynomials should be extended from subgroups to whole groups. To achieve this aim we prove an extension theorem for exponential polynomials which is based on a classical theorem on the extension of homomorphisms.

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Exponential polynomials are the building bricks of spectral synthesis [2]. In some cases it happens that exponential polynomials should be extended from subgroups to whole groups. The aim of this paper is to prove an extension theorem for exponential polynomials. The treatment is based on a well-known theorem from algebra: subgroup homomorphisms of an abelian group into a divisible abelian group can be extended to homomorphisms of the whole group.

In this paper $\mathbb{C}$ denotes the set of complex numbers.

Let G be an abelian group. Homomorphisms of G into the additive group of complex numbers are called *additive functions* and homomorphisms of G into the multiplicative group of nonzero complex numbers are called *exponential functions* or simply *exponentials*. Products of additive functions and exponentials are called *exponential monomials*. As the product of exponentials is an exponential, too, hence the general form of exponential monomials is

$$x \mapsto a_1^{\alpha_1}(x)a_2^{\alpha_2}(x)\dots a_n^{\alpha_n}(x)m(x),$$

where $a_1, a_2, \dots, a_n \colon G \to \mathbb{C}$ are additive functions, $m\colon G \to \mathbb{C}$ is an exponential and $n, \alpha_1, \alpha_2, \dots, \alpha_n$ are positive integers. If m is identically 1, then we call the function

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a *monomial.* Linear combinations of monomials are called *polynomials* and linear combination of exponential monomials are called *exponential polynomials.* Hence exponential polynomials are the elements of the algebra generated by the additive and the exponential functions.

In some cases we need a more general concept of exponential polynomials: the so-called generalized exponential polynomials. Here the point is that we use generalized polynomials instead of polynomials. A generalized polynomial originates from multi-additive functions. If $A\colon G^n \to \mathbb{C}$ is a function, which is a homomorphism in each variable and is symmetric, then it is called a *multi-additive (symmetric) function.* More precisely, it is called an *n-additive function.* The *diagonalization of A* is the function $A^*\colon G \to \mathbb{C}$ defined by

$$A^*(x) = A(x, x, \ldots, x)$$

for all x in G. A linear combination of diagonalizations of multi-additive functions is called a *generalized polynomial.* Finally, we define generalized exponential monomials as functions $f\colon G \to \mathbb{C}$ of the form

$$f(x) = \sum_{i=1}^{n} p_i(x) m_i(x),$$

where n is a positive integer, $p_i\colon G \to \mathbb{C}$ is a generalized polynomial and $m_i\colon G \to \mathbb{C}$ is an exponential $(i = 1, 2, \ldots, n)$. It is easy to see that polynomials are generalized polynomials and exponential polynomials are generalized exponential polynomials, but in general, the converse is not true. For more about exponential polynomials and generalized exponential polynomials we refer to [2].

Our main result is based on a classical theorem given below. We exhibit a simple proof, too, for the sake of completeness (see [1], Volume I., Theorem A.7).

Theorem 1. *Let G be an abelian group and let D be a divisible abelian group. Furthermore, let H be a subgroup of G, and $\psi\colon H \to D$ a homomorphism. Then ψ can be extended to a homomorphism of G into D, that is, there exists a homomorphism $\Psi\colon G \to D$ such that $\Psi(h) = \psi(h)$ for all h in H.*

P r o o f. Let x_0 be any element of G not contained in H. We have two possibilities. If nx_0 does not belong to H for any $n \geqslant 2$, then we define $\psi_0(nx_0 + h) = \psi(h)$ for any integer n and for any h in H. It is easy to see that this definition extends ψ to a homomorphism of the subgroup

$$H_0 = \{nx_0 + h\colon\ h \text{ is in } H \text{ and } n \text{ is an integer}\}.$$

In other words, $\psi_0 \colon H_0 \to D$ is well-defined, it is a homomorphism of H_0 into D and $\Psi_0(h) = \psi(h)$ holds for all h in H.

In the opposite case there exists $n \geqslant 2$ for which nx_0 belongs to H. In this case let k denote the smallest n with this property and let z denote a solution of the equation $kz = \psi(kx_0)$. The existence of a z with this property is guaranteed by the divisibility of D. Then we define $\psi_0(nx_0 + h) = nz + \psi(h)$ for any integer n and for any h in H. Here again we see that $\psi_0 \colon H_0 \to D$ is a well-defined homomorphism with the extension property: $\psi_0(h) = \psi(h)$ holds for any h in H.

Hence we have seen that if H is different from G then ψ can be extended to a homomorphism of a subgroup, which properly contains H. By applying Zorn's lemma the proof is complete. □

From this result we can derive the following one:

Theorem 2. *Let G be an abelian group, n a positive integer and let D be a divisible abelian group. Furthermore, let H be a subgroup of G, and $A \colon H^n \to D$ an n-additive symmetric function. Then A can be extended to an n-additive, symmetric mapping of G^n into D, that is, there exists an n-additive symmetric function $\mathscr{A} \colon G^n \to D$ such that $\mathscr{A}(h_1, h_2, \ldots, h_n) = A(h_1, h_2, \ldots, h_n)$ holds for all $h_1, h_2, \ldots, h_n$ in H.*

Proof. We fix the elements $h_2, h_3, \ldots, h_n$ and consider the homomorphism

$$h \mapsto A(h, h_2, h_3, \ldots, h_n)$$

of H into D. By virtue of the previous theorem, there is an extension of this function to a homomorphism of G into D. In other words, there exists a function $A_1 \colon G \times H \times H \times \ldots \times H \to D$ which is additive in each variable and satisfies $A_1(h, h_2, h_3, \ldots, h_n) = A(h, h_2, h_3, \ldots, h_n)$ for all $h, h_2, h_3, \ldots, h_n$ in H. Now we continue this process. We fix the elements g_1 in G and $h_3, h_4, \ldots, h_n$ in H and consider the homomorphism $h \mapsto A_1(g_1, h, h_3, h_4, \ldots, h_n)$ of H into D. Applying Theorem 1 again we get an extension of this homomorphism to a homomorphism of G into D. It is obvious that continuing this process we arrive at a function $A_n \colon G^n \to D$ which is additive in each variable and satisfies $A_n(h_1, h_2, \ldots, h_n) = A(h_1, h_2, \ldots, h_n)$ for all $h_1, h_2, \ldots, h_n$ in H. To achieve symmetry, we define

$$\mathscr{A}(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma} A_n(h_{\sigma(1)}, h_{\sigma(2)}, \ldots, h_{\sigma(n)}),$$

where σ runs through all permutations of the set $\{1, 2, \ldots, n\}$. Then $\mathscr{A}$ possesses all the desired properties. □

As the additive group of complex numbers is divisible, by the definition of generalized polynomials we immediately have a corollary.

Corollary 3. *Let G be an abelian group, let H be a subgroup of G and let $P\colon G \to \mathbb{C}$ be a generalized polynomial. Then P can be extended to a generalized polynomial on G^n, that is, there exists a generalized polynomial $\mathscr{P}\colon G \to \mathbb{C}$ such that $\mathscr{P}(h) = P(h)$ holds for all h in H.*

To treat exponential polynomials we need a similar extension theorem for exponential functions. This is a simple corollary of Theorem 1, because the multiplicative group of nonzero complex numbers is divisible.

Corollary 4. *Let G be an abelian group and let H be a subgroup. If $m\colon H \to \mathbb{C}$ is an exponential function, then it has an extension to G, that is, there exists an exponential function $\mathscr{M}\colon G \to \mathbb{C}$ such that $\mathscr{M}(h) = m(h)$ holds for all h in H.*

Now we can summarize our results in the following theorem.

Theorem 5. *Let G be an abelian group and let H be a subgroup of G. Let $f\colon H \to \mathbb{C}$ be a generalized exponential polynomial. Then there exists a generalized exponential polynomial extension of f to G, that is, a generalized exponential polynomial $\mathscr{F}\colon G \to \mathbb{C}$ such that $\mathscr{F}(h) = f(h)$ for all h in H.*

We can apply our extension results to functional equations. Here we present a result on the extension of the solution of a linear functional equation.

Theorem 6. *Let G be an abelian group and let H be a subgroup of G. Let further n be a positive integer, a_i, b_i integers with the property that $a_i b_j \neq a_j b_i$ for any $i \neq j$, and let c_i be nonzero real numbers ($i = 1, 2, \dots, n$). If $f\colon H \to \mathbb{C}$ satisfies the functional equation*

$$\sum_{i=1}^{n+2} c_i f(a_i x + b_i y) = 0 \tag{1}$$

for all x, y in H, then there exists a function $F\colon G \to \mathbb{C}$ satisfying

$$\sum_{i=1}^{n+2} c_i F(a_i x + b_i y) = 0 \tag{2}$$

for all x, y in G and $F(h) = f(h)$ for all h in H.

P r o o f. From Theorem 9.5 and Theorem 9.1 in [2] it follows that for $k = 0, 1, \ldots, n$ there exist k-additive symmetric functions $A_k\colon\ H^k \to \mathbb{C}$ such that

$$f(x) = \sum_{k=0}^{n} A_k^*(x)$$

and

$$A_k(x, x, \ldots, x; y, y, \ldots, y) \sum_{i=1}^{n+2} c_i a_i^j b_i^{k-j} = 0 \tag{3}$$

for all x, y in H and for $j = 0, 1, \ldots, n$, $k = j, j+1, \ldots, n$. (We remark that for $k = 0$ $H^k = H$ and A_k is a constant. In the argument of A_k in the latter equation there are j x's and $k - j$ y's.) It follows that for any k we have either

$$A_k(x, x, \ldots, x; y, y, \ldots, y) = 0$$

for all x, y in H, or

$$\sum_{i=1}^{n+2} c_i a_i^j b_i^{k-j} = 0.$$

In the first case we have that

$$A_k(x_1, x_2, \ldots, x_k) = 0$$

for all $x_1, x_2, \ldots, x_k$ in H, because the diagonalization determines the multiadditive function uniquely ([2], Lemma 1.6). In the first case we let $\mathscr{A}_k(x_1, x_2, \ldots, x_k) = 0$ for any $x_1, x_2, \ldots, x_k$ in G. In the latter case we apply Theorem 2, and we denote by $\mathscr{A}_k$ an arbitrary k-additive symmetric extension of A_k to G^k. Then obviously the functions $\mathscr{A}_k$, $k = 0, 1, \ldots, n$ satisfy the system of equations (3) for all x, y in G. Then by Theorem 2.5 in [2], the function defined by

$$F(x) = \sum_{k=0}^{n} \mathscr{A}_k^*(x)$$

for all x in G satisfies the functional equation (2) for all x, y in G, and it is clear that $F(h) = f(h)$ for all h in H. The theorem is proved. □

Of course, the coefficients a_i, b_i can also be rational numbers provided the group G is uniquely divisible, that is, if it is a linear space over the rationals.

References

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