SOME SUMS OF LEGENDRE AND JACOBI POLYNOMIALS

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Abstract. We prove identities involving sums of Legendre and Jacobi polynomials. The identities are related to Green's functions for powers of the invariant Laplacian and to the Minakshisundaram-Pleijel zeta function.

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1. Introduction

In [1] some new identities involving Legendre polynomials are given as applications of results for Green's functions for powers of the invariant Laplacian. We intend to give other proofs.

In Remark 4.3 in [1] a bilinear formula analogous to (4) below is indicated-but not proved. We give a closed form for

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n(n+1))^2} P_n(x) P_n(y).$$

Furthermore, we calulate

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n(n+1))^3} P_n(x)$$

(cf. Remark 4.3 in [1]).

For $m \geqslant 4$ the sums

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n(n+1))^m} P_n(x)$$

seem to be rather complicated and it seems unlikely that they can be expressed with the help of polylogarithms.

2. Prelimaries

The sums in [1] for which we will give alternative proofs are

$$\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} P_n(x) = \log 2 - 1 - \log(1-x), \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{2n+\nu+1}{n(n+\nu+1)} P_n^{(0,\nu)}(x) = -\sum_{j=1}^{\nu+1} \frac{1}{j} - \log\frac{1-x}{2}, \tag{2}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{2n+1}{(n(n+1))^2} P_n(x) \\ &= \log\frac{1-x}{1+x} \cdot \log\frac{2}{1+x} - \frac{1}{2}\log^2\frac{2}{1+x} + \mathrm{Li}_2\Big(-\frac{1-x}{1+x}\Big) + 1, \end{aligned} \tag{3}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} P_n(x) P_n(y) \\ &= 2\log 2 - 1 - \log(1-x)(1+y) \quad \text{if} \quad -1 < x \leqslant y < 1. \end{aligned} \tag{4}$$

For some simplifications we will need Landau's functional equation

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\Big(-\frac{z}{1-z}\Big) = -\frac{1}{2}\log^2(1-z) \tag{5}$$

and an equation by Euler

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log z \cdot \log(1-z), \tag{6}$$

see [2].

If we use (5) and (6) with $z = \frac{1-x}{2}$ we get

$$\begin{aligned} &\log\frac{1-x}{1+x} \cdot \log\frac{2}{1+x} - \frac{1}{2}\log^2\frac{2}{1+x} + \mathrm{Li}_2(-\frac{1-x}{1+x}) + 1 \\ &\quad = \Big(\log\frac{1-x}{2} - \log\frac{1+x}{2}\Big) \cdot \log\frac{2}{1+x} - \frac{1}{2}\log^2\frac{2}{1+x} - \mathrm{Li}_2\frac{1-x}{2} \\ &\quad - \frac{1}{2}\log^2\frac{1+x}{2} + 1 = \log\frac{1-x}{2} \cdot \log\frac{2}{1+x} - \mathrm{Li}_2\frac{1-x}{2} + 1 = 1 - \frac{\pi^2}{6} + \mathrm{Li}_2\frac{1+x}{2}. \end{aligned}$$

Thus we have simplified the right hand side of (3).

From [3] we have some basic facts for Jacobi polynomials. The defining equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big((1-x^2)\frac{\mathrm{d}P_n^{(\alpha,\beta)}(x)}{\mathrm{d}x}\Big)+\big(\beta-\alpha-(\lambda-1)x\big)\frac{\mathrm{d}P_n^{(\alpha,\beta)}(x)}{\mathrm{d}x}+n(n+\lambda)P_n^{(\alpha,\beta)}(x)=0 \tag{7}$$

where $\lambda=\alpha+\beta+1$.

Suppose formally that

$$f(x)=\sum_{n=0}^{\infty}c_nP_n^{(\alpha,\beta)}(x).$$

Then

$$c_n=\frac{1}{h_n}\int_{-1}^{1}f(x)P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x \tag{8}$$

where

$$h_n=\frac{2^{\lambda}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\lambda)n!\Gamma(n+\lambda)}. \tag{9}$$

We will also need

$$P_n^{(\alpha,\beta)}(1)=\frac{(\alpha+1)(\alpha+2)\cdot\ldots\cdot(\alpha+n)}{n!}. \tag{10}$$

3. Some sums

Proof of (1) and (2). Combining (7) and (8) yields

$$\begin{aligned}-n(n+\lambda)c_nh_n&=\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\Big((1-x^2)\frac{\mathrm{d}P_n^{(\alpha,\beta)}(x)}{\mathrm{d}x}\Big)f(x)(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x\\&\quad+\int_{-1}^{1}\big(\beta-\alpha-(\lambda-1)x\big)\frac{\mathrm{d}P_n^{(\alpha,\beta)}(x)}{\mathrm{d}x}f(x)(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x.\end{aligned} \tag{11}$$

Integration by parts transforms the first integral to

$$\begin{aligned}&\Big[(1-x^2)\frac{\mathrm{d}P_n^{(\alpha,\beta)}(x)}{\mathrm{d}x}f(x)(1-x)^{\alpha}(1+x)^{\beta}\Big]_{-1}^{1}\\&\quad-\int_{-1}^{1}\frac{\mathrm{d}P_n^{(\alpha,\beta)}(x)}{\mathrm{d}x}\big(f'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\big)\,\mathrm{d}x\\&\quad-\int_{-1}^{1}\frac{\mathrm{d}P_n^{(\alpha,\beta)}(x)}{\mathrm{d}x}\big(f(x)(\beta-\alpha-(\lambda-1)x)(1-x)^{\alpha}(1+x)^{\beta}\big)\,\mathrm{d}x.\end{aligned}$$

Combining the two integrals in (11) yields

$$(12)\quad \begin{aligned} -n(n+\lambda)c_n h_n &= -\int_{-1}^{1} \frac{\mathrm{d}P_n^{(\alpha,\beta)}(x)}{\mathrm{d}x} f'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\,\mathrm{d}x \\ &= -\left[P_n^{(\alpha,\beta)}(x)f'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\right]_{-1}^{1} \\ &\quad + \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\frac{\mathrm{d}}{\mathrm{d}x}\big(f'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\big)\,\mathrm{d}x. \end{aligned}$$

To prove (1) we specify

$$f(x) = \log 2 - 1 - \log(1-x) = -1 - \log\frac{1-x}{2}, \quad \text{and} \quad \alpha = \beta = 0.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\big) = 1$$

and the second integral in (12) vanishes for $n \geqslant 1$. Since

$$\left[P_n^{(\alpha,\beta)}(x)f'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\right]_{-1}^{1} = P_n(1)\cdot 2 = 2$$

and

$$h_n = \frac{2}{2n+1}$$

we have

$$c_n = \frac{2n+1}{n(n+1)}, \quad n \geqslant 1$$

(see (9) and (10)). It remains to determine c_0. However,

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = 0.$$

Thus $c_0 = 0$ and we have proved (1).

In order to prove (2) we start with a redefinition of

$$f(x) = -\sum_{j=1}^{\nu+1}\frac{1}{j} - \log\frac{1-x}{2}.$$

Furthermore, put $\alpha = 0$ and $\beta = \nu$. Then—see (12)—

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\big) = \frac{\mathrm{d}}{\mathrm{d}x}\big(f'(x)(1-x)(1+x)^{\nu+1}\big) = (\nu+1)(1+x)^{\nu}$$

and the second integral in (12) vanishes for $n \geqslant 1$. Since

$$\bigl[P_n^{(\alpha,\beta)}(x) f'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\bigr]_{-1}^{1} = P_n^{(0,\nu)}(1) \cdot 2^{\nu+1} = 2^{\nu+1}$$

and

$$h_n = \frac{2^{\nu+1}}{2n+\nu+1}$$

we have

$$c_n = \frac{2n+\nu+1}{n(n+\nu+1)}, \quad n \geqslant 1.$$

To complete the proof of (2) we evaluate

$$\int_{-1}^{1} f(x)(1+x)^{\nu}\,\mathrm{d}x = -\frac{2^{\nu+1}}{\nu+1}\sum_{j=1}^{\nu+1}\frac{1}{j} - \int_{-1}^{1}\log\frac{1-x}{2}(1+x)^{\nu}\,\mathrm{d}x. \tag{13}$$

In the last integral in (13) we make the substitution $x = \frac{1-t}{1+t}$. Then

$$\begin{aligned}\int_{-1}^{1}\log\frac{1-x}{2}(1+x)^{\nu}\,\mathrm{d}x &= \int_{0}^{\infty}\log\frac{t}{1+t}\cdot\frac{2^{\nu+1}}{(1+t)^{\nu+2}}\,\mathrm{d}t\\ &= 2^{\nu+1}\Bigl(\int_{0}^{\infty}\frac{\log t}{(1+t)^{\nu+2}}\,\mathrm{d}t - \int_{0}^{\infty}\frac{\log(1+t)}{(1+t)^{\nu+2}}\,\mathrm{d}t\Bigr)\\ &= 2^{\nu+1}\Bigl(-\frac{1}{\nu+1}\sum_{j=1}^{\nu}\frac{1}{j} - \frac{1}{(\nu+1)^2}\Bigr) = -\frac{2^{\nu+1}}{\nu+1}\sum_{j=1}^{\nu+1}\frac{1}{j} \qquad (14)\end{aligned}$$

where we have used the fact that

$$\frac{1}{\nu+1}\Bigl(-\frac{\log t}{(1+t)^{\nu+1}} + \sum_{j=1}^{\nu}\frac{1}{j(1+t)^{j}} + \log t - \log(1+t)\Bigr)$$

is a primitive function to $\frac{\log t}{(1+t)^{\nu+2}}$. Combining (13) and (14) we get

$$c_0 h_0 = \int_{-1}^{1} f(x)(1+x)^{\nu}\,\mathrm{d}x = 0.$$

We have proved (2).

Remark 1. The technique is also applicable to sums involving other Jacobi polynomials, e.g.

$$\sum_{n=1}^{\infty}\frac{(2n+3)(n+2)}{6n(n+3)}P_n^{(1,1)}(x) = -\frac{7}{9} - \frac{1}{3}\log\frac{1-x}{2} + \frac{1}{3(1-x)}.$$

P r o o f of (3). To simplify the notation we define—or redefine—some functions:

$$f(x) = -1 - \log\frac{1-x}{2},$$
$$F(x) = (1-x)\log\frac{1-x}{2},$$
$$g(x) = \frac{F(x)}{x^2-1},$$
$$G(x) = 1 - \frac{\pi^2}{6} + \mathrm{Li}_2\,\frac{1+x}{2}.$$

Then $F'(x) = f(x)$ and $G'(x) = g(x)$. According to (1) we have

$$\begin{aligned}\frac{2n+1}{n(n+1)}h_n &= \int_{-1}^{1} f(x)P_n(x)\,\mathrm{d}x = \underbrace{\big[F(x)P_n(x)\big]_{-1}^{1}}_{=0} - \int_{-1}^{1} F(x)P_n'(x)\,\mathrm{d}x \\ &= -\int_{-1}^{1} g(x)(x^2-1)P_n'(x)\,\mathrm{d}x = -\big[\underbrace{G(x)(x^2-1)}_{\to 0 \text{ as } x\to\pm 1}P_n'(x)\big]_{-1}^{1} \\ &\quad + \int_{-1}^{1} G(x)\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}((x^2-1)P_n'(x))}_{n(n+1)P_n(x)}\,\mathrm{d}x = n(n+1)\int_{-1}^{1} G(x)P_n(x)\,\mathrm{d}x.\end{aligned}$$

We have proved that

$$\frac{1}{h_n}\int_{-1}^{1} G(x)P_n(x)\,\mathrm{d}x = \frac{2n+1}{(n(n+1))^2} \quad \text{if} \quad n = 1, 2, \ldots$$

After integration by parts we can complete the proof with

$$\int_{-1}^{1} G(x)\,\mathrm{d}x = 0.$$

A n e x t e n s i o n of (3). We intend to prove that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(2n+1)}{(n(n+1))^3}P_n(x) \\ &= \frac{\pi^2}{6} - 2 + 2\zeta(3) + \log\frac{1-x}{2}\cdot\mathrm{Li}_2\,\frac{1-x}{2} - \mathrm{Li}_2\,\frac{1+x}{2} - 2\,\mathrm{Li}_3\,\frac{1-x}{2}.\end{aligned} \tag{15}$$

We use the same technique as above.

Redefine

$$\begin{aligned}
f(x) &= 1 - \frac{\pi^2}{6} + \operatorname{Li}_2 \frac{1+x}{2}, \\
F(x) &= -\frac{\pi^2}{6}(1+x) + (1+x)\operatorname{Li}_2 \frac{1+x}{2} - (1-x)\log\frac{1-x}{2}, \\
g(x) &= \frac{F(x)}{x^2-1}, \\
G(x) &= \frac{\pi^2}{6} - 2 + 2\zeta(3) + \log\frac{1-x}{2}\cdot\operatorname{Li}_2\frac{1-x}{2} - \operatorname{Li}_2\frac{1+x}{2} - 2\operatorname{Li}_3\frac{1-x}{2}.
\end{aligned}$$

Then $F'(x) = f(x)$ and $G'(x) = g(x)$. According to (3) we have

$$\begin{aligned}
\frac{2n+1}{(n(n+1))^2}h_n &= \int_{-1}^{1} f(x)P_n(x)\,\mathrm{d}x = \underbrace{\big[F(x)P_n(x)\big]_{-1}^{1}}_{=0} - \int_{-1}^{1} F(x)P_n'(x)\,\mathrm{d}x \\
&= -\int_{-1}^{1} g(x)(x^2-1)P_n'(x)\,\mathrm{d}x = -\big[\underbrace{G(x)(x^2-1)}_{\to 0 \text{ as } x\to\pm 1}P_n'(x)\big]_{-1}^{1} \\
&\quad + \int_{-1}^{1} G(x)\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}((x^2-1)P_n'(x))}_{n(n+1)P_n(x)}\,\mathrm{d}x = n(n+1)\int_{-1}^{1} G(x)P_n(x)\,\mathrm{d}x.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{-1}^{1} G(x)\,\mathrm{d}x &= [(x-1)G(x)]_{-1}^{1} - \int_{-1}^{1}\frac{F(x)}{x+1}\,\mathrm{d}x \\
&= \frac{\pi^2}{3} - 4 - \int_{-1}^{1}\Big(-\frac{\pi^2}{6} + \operatorname{Li}_2\frac{1+x}{2} - \frac{1-x}{1+x}\log\frac{1-x}{2}\Big)\,\mathrm{d}x \\
&= \frac{\pi^2}{3} - 4 + \frac{\pi^2}{3} - \Big(\frac{\pi^2}{3} - 2\Big) + \Big(-\frac{\pi^2}{3} + 2\Big) = 0
\end{aligned}$$

we get $c_0 = 0$. We have proved (15).

P r o o f of (4). Now we redefine f as

$$f(x) = \begin{cases} -1 - \log\big(\frac{1-x}{2}\frac{1+y}{2}\big) & \text{if } \quad -1 \leqslant x \leqslant y \\ -1 - \log\big(\frac{1+x}{2}\frac{1-y}{2}\big) & \text{if } \quad y \leqslant x \leqslant 1. \end{cases}$$

If

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

then $c_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x = 0$ and for $n \geqslant 1$

$$\begin{aligned} h_n c_n = &-\int_{-1}^{y}\Big(1+\log\Big(\frac{1-x}{2}\frac{1+y}{2}\Big)\Big)P_n(x)\,\mathrm{d}x \\ &-\int_{y}^{1}\Big(1+\log\Big(\frac{1+x}{2}\frac{1-y}{2}\Big)\Big)P_n(x)\,\mathrm{d}x \\ = &-\frac{1}{n(n+1)}\int_{-1}^{y}\Big(1+\log\Big(\frac{1-x}{2}\frac{1+y}{2}\Big)\Big)\frac{\mathrm{d}}{\mathrm{d}x}((x^2-1)P_n'(x))\,\mathrm{d}x \\ &-\frac{1}{n(n+1)}\int_{y}^{1}\Big(1+\log\Big(\frac{1+x}{2}\frac{1-y}{2}\Big)\Big)\frac{\mathrm{d}}{\mathrm{d}x}((x^2-1)P_n'(x))\,\mathrm{d}x. \end{aligned}$$

Thus integration by parts yields

$$\begin{aligned} n(n+1)h_n c_n = &-\Big[\Big(1+\log\Big(\frac{1-x}{2}\frac{1+y}{2}\Big)\Big)(x^2-1)P_n'(x)\Big]_{-1}^{y} \\ &+\int_{-1}^{y}(x+1)P_n'(x)\,\mathrm{d}x - \Big[\Big(1+\log\Big(\frac{1+x}{2}\frac{1-y}{2}\Big)\Big)(x^2-1)P_n'(x)\Big]_{y}^{1} \\ &+\int_{y}^{1}(x-1)P_n'(x)\,\mathrm{d}x = [(x+1)P_n(x)]_{-1}^{y} + [(x-1)P_n(x)]_{y}^{1} \\ &-\int_{-1}^{1}P_n(x)\,\mathrm{d}x = 2P_n(y). \end{aligned}$$

We have proved that

$$c_n = \frac{(2n+1)P_n(y)}{n(n+1)},$$

which completes the proof of (4).

An extension of (4). We intend to prove that

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{2n+1}{(n(n+1))^2}P_n(x)P_n(y) \\ &= \begin{cases} 1-\log\frac{1+y}{2}\cdot\log\big(\frac{1-x}{2}\frac{1-y}{2}\big)+\mathrm{Li}_2\,\frac{1+x}{2}-\mathrm{Li}_2\,\frac{1+y}{2} & \text{if}\quad -1<x\leqslant y<1 \\ 1-\log\frac{1+x}{2}\cdot\log\big(\frac{1-x}{2}\frac{1-y}{2}\big)+\mathrm{Li}_2\,\frac{1+y}{2}-\mathrm{Li}_2\,\frac{1+x}{2} & \text{if}\quad -1<y\leqslant x<1. \end{cases} \end{aligned}$$

A formula like this is wanted in Remark 4.3 in [1].

In this proof we will recognize the pattern of the above proofs. However, first we need the notation

$$f(x) = \begin{cases} -1-\log\big(\frac{1-x}{2}\frac{1+y}{2}\big) & \text{if}\quad -1\leqslant x\leqslant y \\ -1-\log\big(\frac{1+x}{2}\frac{1-y}{2}\big) & \text{if}\quad y\leqslant x\leqslant 1, \end{cases}$$

$$F(x)=\begin{cases} F_1(x)=(1-x)\log\frac{1-x}{2}-(x+1)\log\frac{1+y}{2} & \text{if}\quad -1\leqslant x\leqslant y\\ F_2(x)=-(1+x)\log\frac{1+x}{2}-(x-1)\log\frac{1-y}{2} & \text{if}\quad y\leqslant x\leqslant 1,\end{cases}$$

$$g(x)=\begin{cases} \frac{F_1(x)}{x^2-1} & \text{if}\quad -1\leqslant x\leqslant y\\ \frac{F_2(x)}{x^2-1} & \text{if}\quad y\leqslant x\leqslant 1,\end{cases}$$

$$G(x)=\begin{cases} 1-\log\frac{1+y}{2}\cdot\log\left(\frac{1-x}{2}\frac{1-y}{2}\right)+\mathrm{Li}_2\,\frac{1+x}{2}-\mathrm{Li}_2\,\frac{1+y}{2} & \text{if}\quad -1\leqslant x\leqslant y\\ 1-\log\frac{1+x}{2}\cdot\log\left(\frac{1-x}{2}\frac{1-y}{2}\right)+\mathrm{Li}_2\,\frac{1+y}{2}-\mathrm{Li}_2\,\frac{1+x}{2} & \text{if}\quad y\leqslant x\leqslant 1.\end{cases}$$

Then the functions f, F, g and G are continuous at $x=y$ and furthermore $F'=f$ and $G'=g$.

Again integration by parts is useful. We get

$$\int_{-1}^{1} G(x)\,\mathrm{d}x = 0.$$

Finally, (4) yields

$$\begin{aligned}\frac{2n+1}{n(n+1)}P_n(y)h_n &= \int_{-1}^{1} f(x)P_n(x)\,\mathrm{d}x = \underbrace{\big[F(x)P_n(x)\big]_{-1}^{1}}_{=0 \text{ because } F_1(-1)=F_2(1)=0}\\ &-\int_{-1}^{1} g(x)(x^2-1)P_n'(x)\,\mathrm{d}x = -\big[\underbrace{G(x)(x^2-1)}_{\to 0 \text{ as } x\to\pm 1}P_n'(x)\big]_{-1}^{1}\\ &+\int_{-1}^{1} G(x)\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}((x^2-1)P_n'(x))}_{n(n+1)P_n(x)}\,\mathrm{d}x.\end{aligned}$$

We complete the proof with

$$\frac{1}{h_n}\int_{-1}^{1} G(x)P_n(x)\,\mathrm{d}x = \frac{2n+1}{(n(n+1))^2}P_n(y).$$

References

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