EXAMPLES FROM THE CALCULUS OF VARIATIONS III. LEGENDRE AND JACOBI CONDITIONS

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Abstract. We will deal with a new geometrical interpretation of the classical Legendre and Jacobi conditions: they are represented by the rate and the magnitude of rotation of certain linear subspaces of the tangent space around the tangents to the extremals. (The linear subspaces can be replaced by conical subsets of the tangent space.) This interpretation can be carried over to nondegenerate Lagrange problems but applies also to the degenerate variational integrals mentioned in the preceding Part II.

Keywords: Legendre condition, Jacobi condition, Poincaré-Cartan form, Lagrange problem, degenerate variational integral

MSC 2000: 49-01, 49K15, 58A10

This Part III of our article is devoted to the very suggestive and lacking in current textbooks geometrical interpretation of the familiar Legendre and Jacobi extremality conditions: a certain part of the Pfaffian system determining the admissible curves of the variational problem should rotate around the tangents of the extremals all the time in one direction but not too long. Owing to the Poincaré-Cartan forms, the interpretation can be carried over to all nondegenerate Lagrange problems. Moreover, in view of its generality, a new way is open to an investigation of the generic subcase $e^1_1 \neq 0$, $A \neq 0$ (see II 2 and comments in II 3) of the density

$$\begin{gathered}\alpha = f(x, w^1_0, w^2_0, w^1_1, w^2_1)\,\mathrm{d}x,\\ f^{11}_{11} f^{22}_{11} = (f^{12}_{11})^2.\end{gathered} \tag{1}$$

This is a long standing classical degenerate variational problem which was (to our best knowledge) absolutely omitted in literature. No wonder since the final result principally differs from naive expectations: together with a certain modification of the Legendre and Jacobi conditions which ensure the extremum only on very narrow classes of admissible curves, also some comparison of a "curvature" with an "area" resembling a curious eigenvalue problem is necessary to attain full generality. In accordance with our opinion all our reasoning is developed only for the lowest possible dimension to clarify the main ideas. Nonetheless, we conclude with a quite general variational formula (30) which is of independent interest.

The lowest dimension case

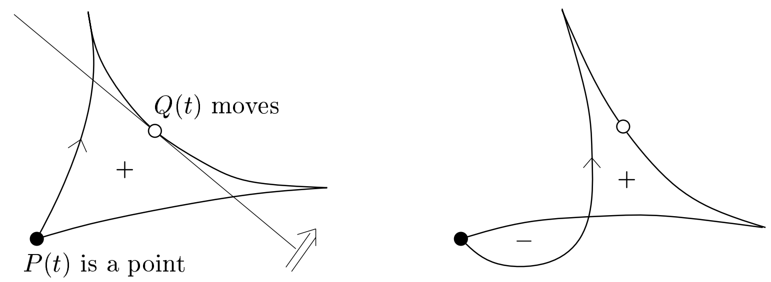
1. Rectified extremals (see also I 2 with $c = 1$, and I 4 slightly adapted to finite dimension)**.** Our reasoning will be developed in a three-dimensional space $\mathbf{N}$ with coordinates x, u, v. However, some arguments can be better interpreted by taking the projection into variables u, v and then x should be regarded as a parametr. We will deal with a density $\beta = u\,\mathrm{d}v - \mathrm{d}W \in \Phi(\mathbf{N})$ where $W = W(x,u,v) \in \mathcal{F}(\mathbf{N})$ is a given function. Recall that a curve $P(t) \in \mathbf{N}$ $(0 \leqslant t \leqslant 1)$ is *stationary* (to β) if and only if P^*u and P^*v are constants (alternatively, the projection reduces to a point). Then another curve $Q(t) \in \mathbf{N}$ $(0 \leqslant t \leqslant 1)$ with the same ends obviously satisfies

$$\tag{2} \begin{aligned} \int_Q \beta - \int_P \beta &= \int_Q u\,\mathrm{d}v - \int_P u\,\mathrm{d}v \\ &= \int_Q u\,\mathrm{d}v \end{aligned}$$

(cf. I (6)) and the value (2) can be made quite arbitrary, by an appropriate choice of Q. We therefore introduce *admissible* ($\mathcal{A}$-) *curves* Q which satisfy a fixed Pfaffian equation $U\,\mathrm{d}u = V\,\mathrm{d}v$. We will moreover deal only with curves *near enough* to the stationary curves in the sense that $\mathrm{d}Q^*u/\,\mathrm{d}t$ and $\mathrm{d}Q^*v/\,\mathrm{d}t$ are functions very close to zero (consequently Q^*u and Q^*v are close to constants). It follows that they can be *parametrized without loss of generality by the coordinate* x: $Q^*x = x(t)$ where $\mathrm{d}x/\,\mathrm{d}t > 0$.

2. Theorem. *If the subspaces (depending on the parameter x and defined by) $U\,\mathrm{d}u = V\,\mathrm{d}v$ of the tangent spaces to $\mathbf{N}$ rotate around the tangents (given by $\mathrm{d}u = \mathrm{d}v = 0$) of the stationary curves all the time in one direction, and if the total angle of the rotation is less than π, then the value (2) keeps its sign.*

Using projections into the space of variables u, v, the statement becomes rather transparent:



Short rotation ensures a constant sign of the area on the left, unlike the right hand figure with ambigous total sign.

Passing to the proof, we may suppose $V \neq 0$ by using a linear change of variables u, v together with the assumption of the total angle. Then clearly $\partial G/\partial x \neq 0$ where $G = U/V$. Denoting moreover $C = u(Q(0)) = u(P(0)) = u(P(1)) = u(Q(1))$, we obtain

$$
\begin{aligned}
\int_Q \beta - \int_P \beta &= \int_Q u \,\mathrm{d}v = \int_Q (u - C)\,\mathrm{d}v \\
&= \int_Q (u-C) G \,\mathrm{d}u = \frac{1}{2}\int_Q G\,\mathrm{d}(u-C)^2 = -\frac{1}{2}\int_Q (u-C)^2\,\mathrm{d}G,
\end{aligned} \tag{3}
$$

where

$$
\mathrm{d}G = \frac{\mathrm{d}G}{\mathrm{d}x}\,\mathrm{d}x, \quad \frac{\mathrm{d}G}{\mathrm{d}x} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial G}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x} \sim \frac{\partial G}{\partial x} \tag{4}
$$

since $\mathrm{d}u/\mathrm{d}x$, $\mathrm{d}v/\mathrm{d}x$ are close to zero. This concludes the proof.

N o t e. Using a linear change of u, v the above conditions $V \neq 0$, $(U/V)_x \neq 0$ can be replaced by the less restrictive

$$
CU + DV \neq 0, \quad \frac{\partial}{\partial x}\frac{AU + BV}{CU + DV} \neq 0 \tag{5}
$$

where A, B, C, D are constants satisfying $AD \neq BC$. We shall see that requirements (5) can be verified without use of favourable coordinates u, v. See Section 4 below.

3. Definitions. Let us introduce two general (and well-known) concepts concerning a submodule $\Theta \subset \Phi(\mathbf{M})$. First, the submodule

$$
\Theta^{\perp} = \{Z\colon\ \vartheta(Z) \equiv 0 \text{ for all } \vartheta \in \Theta\} \subset \mathcal{T}(\mathbf{M}) \tag{6}
$$

of the module of all vector fields on $\mathbf{M}$. (In particular, $\Theta^{\perp} = \mathcal{H}(\Omega)$ if $\Theta = \Omega$ is a diffiety, see I 3). Second, let a curve $Q(t) \in \mathbf{M}$ $(0 \leqslant t \leqslant 1)$ be a solution of the

Pfaffian system $\vartheta \equiv 0$ ($\vartheta \in \Theta$), hence $Q^*\vartheta \equiv 0$. Then a vector field $Z \in \mathcal{T}(\mathbf{M})$ satisfying

$$Q^*\mathfrak{L}_Z\vartheta = Q^*Z\rfloor\,\mathrm{d}\vartheta + \mathrm{d}Q^*\vartheta(Z) \equiv 0 \quad (\vartheta \in \Theta) \tag{7}$$

is called *a variation of* (the solution) $Q(t)$. One can observe that only the values of Z at the points of the curve $Q(t)$ are important in this definition. (In particular, if $\Theta = \Omega$ is a diffiety, then (7) is equivalent to the identity $Q^*\omega([Z, X]) \equiv 0$ for all $\omega \in \Omega$ and $Z \in \mathcal{H}(\Omega)$, that is, to the inclusion $[Z, \mathcal{H}(\Omega)] \subset \mathcal{H}(\Omega)$ valid along the curve Q.) Variations of a $\mathcal{C}$-surve $P(t)$ of a solution of an $\mathcal{EL}$ system are known under the name of *Jacobi vector fields*.

Finally, for the convenience of the reader, let us once more recall the submodule

$$\operatorname{Adj}\mathrm{d}\alpha = \{Z\rfloor\,\mathrm{d}\alpha\colon\ \text{all}\ \ Z \in \mathcal{T}(\mathbf{M})\} \subset \Phi(\mathbf{M})$$

for a density $\alpha \in \Phi(\mathbf{M})$. Then $\operatorname{Adj}\mathrm{d}\alpha = \{\mathrm{d}a^1, \dots, \mathrm{d}a^{2c}\}$ is generated by the differentials of certain (*adjoint*) functions $a^1, \dots, a^{2c}$. The form $\mathrm{d}\alpha$ can be expressed in terms of them, see I (4,5).

4. The rectified extremals again. We return to the topic of Section 1. Assuming the form $\mathrm{d}\beta$ to be known (in a certain quite arbitrary system of coordinates), it is possible to determine the module $\operatorname{Adj}\mathrm{d}\beta$. (Since $\mathrm{d}\beta = \mathrm{d}u \wedge \mathrm{d}v$, in fact $\operatorname{Adj}\mathrm{d}\beta = \{\mathrm{d}u, \mathrm{d}v\}$ but the functions u, v need not be explicitly known here.) Choose moreover $x \in \mathcal{F}(\mathbf{N})$ with $\mathrm{d}x \notin \operatorname{Adj}\mathrm{d}\beta$. Then the vector field $X \in \operatorname{Adj}\mathrm{d}\beta^\perp$ normalized by $Xx = 1$ can be easily found. (In fact $X = \partial/\partial x$ in terms of the unknown coordinates x, u, v.)

Assuming (the subspace determined by $U\,\mathrm{d}u = V\,\mathrm{d}v$, hence) the submodule $\Theta = \{U\,\mathrm{d}u - V\,\mathrm{d}v\} \subset \Phi(\mathbf{N})$ to be known, we may calculate $\Theta^\perp \subset \mathcal{T}(\mathbf{N})$. (Clearly $\Theta^\perp = \{X, Y\}$ where $Y = V\partial/\partial u + U\partial/\partial v$ in terms of the unknown coordinates.)

Finally recall that a stationary curve $P(t) \in \mathbf{N}$ $(0 \leqslant t \leqslant 1)$ is a solution of the system $\mathrm{d}u = \mathrm{d}v = 0$ equivalent to $P^*\varphi \equiv 0$ ($\varphi \in \operatorname{Adj}\mathrm{d}\beta$). One can then calculate the variations $Z \in \mathcal{T}(\mathbf{N})$. They are

$$Z = z\frac{\partial}{\partial x} + f\frac{\partial}{\partial u} + g\frac{\partial}{\partial v} \quad (z, f, g \in \mathcal{F}(\mathbf{N});\ P^*f = C,\ P^*g = D)$$

(where C, D are constants) in terms of the (unknown) coordinates.

Trivially $\vartheta(Z) = fU + gV$, and moreover $[X, Y] = V_x\partial/\partial u + U_x\partial/\partial v$ whence

$$\mathrm{d}\beta([X, Y], Y) = V_xU - U_xV = -\frac{(CU + DV)^2}{AD - BC}\frac{\partial}{\partial x}\frac{AU + BV}{CU + DV}$$

with arbitrary constants A, B, C, D satisfying $AD \neq BC$. It follows that the conditions

$$P^*\vartheta(Z) \neq 0 \ (\vartheta \in \Theta),\ \mathrm{d}\beta([X,Y],Y) \neq 0 \tag{8}$$

are respectively equivalent to those in (5). They can be verified without any use of functions u, v: it is sufficient to employ *arbitrary nonvanishing vector fields*

$$X \in \operatorname{Adj} \mathrm{d}\beta^{\perp},\ Y \in \Theta^{\perp},\ Y \notin \operatorname{Adj} \mathrm{d}\beta^{\perp} \tag{9}$$

and search for *appropriate* Z satisfying $(7, 8_1)$.

5. Towards the classical theory. We apologize for the following easy example but it will be useful in several respects. Our primary goal is to verify that (8) turns into the familiar Legendre-Jacobi conditions for the particular case $m = 1$ of II 1. So we find ourselves in the space $\mathbf{M}(1)$ equipped with coordinates x, w^1_r, contact forms $w^1_r \equiv \mathrm{d}w^1_r - w^1_{r+1}\,\mathrm{d}x$ (cf. I 3), a vector field $X \in \Omega(1)^{\perp}$ (cf. I (8_2)), and a density $\alpha = f(x, w^1_0, w^1_1)\,\mathrm{d}x$. Then the $\mathcal{PC}$ form $\breve{\alpha} = f\,\mathrm{d}x + f^1_1\omega^1_0$ (cf. II 1) gives two generators

$$\omega^1_0,\ e^1\,\mathrm{d}x - f^{11}_{11}\omega^1_1 = E\,\mathrm{d}x - f^{11}_{11}\,\mathrm{d}w^1_1 \tag{10}$$

(where $e^1 = f^1_0 - Xf^1_1$, hence $E = f^1_0 - f^1_{1x} - w^1_1 f^{11}_{01}$) of the module $\operatorname{Adj} \mathrm{d}\breve{\alpha}$ in the *nondegenerate case* $f^{11}_{11} \neq 0$ (see formulae II (2), (3) with $m = 1$). It follows that $\mathrm{d}\breve{\alpha} = \mathrm{d}u \wedge \mathrm{d}v$, hence $\breve{\alpha} = u\,\mathrm{d}v + \mathrm{d}W$ for appropriate (unknown) functions u, v, W of the variables x, w^1_0, w^1_1.

The $\mathcal{EL}$ subspace $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}(1)$ consists of all points satisfying

$$X^k e^1 = -f^{11}_{11} w^1_{k+2} + \ldots \equiv 0 \quad (k = 0, 1, \ldots),$$

therefore x, w^1_0, w^1_1 (and alternatively x, u, v) provide coordinates on $\mathbf{E}$. In accordance with the above reasoning, we may put

$$\mathbf{N} = \mathbf{E},\ \beta = \breve{\alpha},\ \vartheta = \omega^1_0 \tag{11}$$

and apply Theorem 2. Let us calculate vector fields (9) in terms of the coordinates x, w^1_0, w^1_1. Clearly

$$X = \frac{\partial}{\partial x} + w^1_1 \frac{\partial}{\partial w^1_0} + \frac{E}{f^{11}_{11}} \frac{\partial}{\partial w^1_1}, \quad Y = \frac{\partial}{\partial w^1_1}$$

whence $[X,Y] = -\partial/\partial w_0^1 \pmod{\partial/\partial w_1^1}$ and therefore

$$\mathrm{d}\beta([X,Y],Y) = \mathrm{d}\breve{\alpha}(-\partial/\partial w_0^1, -\partial/\partial w_1^1) = f_{11}^{11}$$

by virtue of II (2). Consequently, (7_2) turns into the familiar *Legendre condition.*

Passing to (7_1), we have to find $Z \in \mathcal{T}(\mathbf{N})$ satisfying (7) with $\vartheta = w_0^1$. This is possible but a little clumsy. In order to obtain the result with less effort, we shall calculate in the ambient space $\mathbf{M}(1)$ assuming $Z = \sum z^r \partial/\partial w_r^1$ *tangent to* $\mathbf{E}$ *along a given* $\mathcal{C}$*-curve* $P(t) \in \mathbf{E} \subset \mathbf{M}(1)$. In particular,

$$\begin{aligned} 0 = P^* Z e^1 &= P^* Z f_0^1 - \frac{\mathrm{d}}{\mathrm{d}x} P^* Z f_1^1 \\ &= \bar{z}^0 P^* f_{00}^{11} + \bar{z}^1 P^* f_{01}^{11} - \frac{\mathrm{d}}{\mathrm{d}x}(\bar{z}^0 P^* f_{01}^{11} + \bar{y}^1 P^* f_{11}^{11}) \end{aligned} \tag{12}$$

with unknown functions $\bar{z}^r \equiv P^* z^r$. However, (7) reads $\bar{z}^1 \,\mathrm{d}x = \mathrm{d}\bar{z}^0$, hence $\bar{z}^1 = \mathrm{d}\bar{z}^0/\,\mathrm{d}z$. Altogether taken, (12) turns into the familiar Jacobi equation and (8_1) requires the existence of a nonvanishing solution in full accordance with the classical results.

***6. A Lagrange problem.** Among all examples mentioned in Part I, only I 7 belongs to the lowest dimension case. Then we may again choose (11), however, with coordinates x, w_0^1, w_0^2 on $\mathbf{E}$, another $\mathcal{PC}$ form $\breve{\alpha} = f\,\mathrm{d}x + (f_0^1/g_0^1)\omega_0^1$, but (formally) the same ϑ. Alternative (unknown) coordinates x, u, v are determined by

$$\operatorname{Adj} \mathrm{d}\breve{\alpha} = \{\omega_0^1, e\,\mathrm{d}x - (f_0^1/g_0^1)\omega_0^2\} = \{\mathrm{d}u,\, \mathrm{d}v\},\ \mathrm{d}x \notin \operatorname{Adj} \mathrm{d}\breve{\alpha},$$

and the (generalized) Legendre condition (cf. I 7) follows from the identity

$$\mathrm{d}\beta([X,Y],Y) = \mathrm{d}\breve{\alpha}(g_0^1 \partial/\partial w_0^2, \partial/\partial w_0^1) = (f_0^1/g_0^1)_0^1\, g_0^1$$

where $X \simeq \partial/\partial x + g\partial/\partial w_0^2 \pmod{\partial/\partial w_0^1}$, $Y = \partial/\partial w_0^1$. We omit the easy discussion concerning the Jacobi condition here.$_*$

***7. An anxious note.** Using the opportunity, we would like to point out the deceitful nature of the naive degeneracy concept: some seemingly dissimilar variational problems can be in reality identified regardless of whether they are degenerate or not.

(i) A Lagrange problem. Let us look at the opening example I 6 for the particular choice

$$g = G(x, w_0^1, w_0^2) w_1^1 \text{ (hence } w_1^2 = G w_1^1),\ \ f = -w_0^2 w_1^1.$$

Trivially $af^{11}_{11} = bg^{11}_{11} = 0$ and we deal with the case where the general procedure subsequently developed in I 6 fails. However, one can find the $\mathcal{PC}$ form $\breve{\alpha} = -w^2_0\,\mathrm{d}w^1_0$ and denoting $w^1_0 = u, w^2_0 = v, W = uv$, we have identified $\breve{\alpha} = \beta = u\,\mathrm{d}v - \mathrm{d}W$. As concerns the Pfaffian equation $U\,\mathrm{d}u = V\,\mathrm{d}v$ (hence $U\,\mathrm{d}w^1_0 = V\,\mathrm{d}w^2_0$), it is equivalent to the ordinary differential equation $Uw^1_1 = Vw^2_1$ identical with the above mentioned $w^2_1 = Gw^1_1$. In this sense, *Theorem 2 concerns a certain degenerate Lagrange problem.*

(ii) T h e c l a s s i c a l e x a m p l e . In the above Section 5, we have proved that *the case of nondegenerate density* $\alpha = f(x, w^1_0, w^1_1)\,\mathrm{d}x$ *with trivial in the classical sense constraint* Ω (1) can be reduced to Theorem 2.

(iii) C o n t i n u a t i o n . We shall mention this example once more in the "opposite direction". In the space $\mathbf{N}$ equipped with coordinates x, u, v, there exist functions $w^1_0, w^1_1 \in \mathcal{F}(\mathbf{N})$ such that a given form $\vartheta = U\,\mathrm{d}u - V\,\mathrm{d}v$ is proportional to $\omega^1_0 = \mathrm{d}w^1_0 - w^1_1\,\mathrm{d}x$. (Hint: choose a first integral of the system $\mathrm{d}x = \vartheta = 0$ for the function w^1_0 and then w^1_1 is uniquely determined.) Since x, w^1_0, w^1_1 may be taken for new coordinates on $\mathbf{N}$, the space $\mathbf{N}$ may be regarded as a factorspace of $\mathbf{M}(1)$. Using the common jet coordinates x, w^1_r, clearly $\beta \simeq f\,\mathrm{d}x \pmod{\omega^1_0, \omega^1_1}$ where

$$f = uv_x - W_x + w^1_1(uv^1_0 - W^1_0) + w^1_2(uv^1_1 - W^1_1).$$

If we choose W satisfying $uv^1_1 = W^1_1$, we have explicitly

$$W = \int uv^1_1\,\mathrm{d}w^1_1 = (1 - w^1_1)\int u\,\mathrm{d}w^1_1,$$

then $f = f(x, w^1_0, w^1_1)$ is independent of w^1_2. So we have the density $\alpha = f\,\mathrm{d}x$ of Section 6. Theorem 2 was reversely transferred into the *classical problem.*

(iv) A n u n d e r d e t e r m i n e d c a s e . In quite an analogous manner, even the subcase of the degenerate problem (1) when $e = 0$ is identically vanishing can be involved, see II 7 and especially II (27). In this sense, Theorem 2 concerns an *underdetermined variational integral.*

(v) R e m a r k . If $\partial G/\partial x \neq 0$, $G = U/V$ may be chosen for the coordinate x in the calculations of (iii), then the equation $\vartheta = 0$ reads $\mathrm{d}v - x\,\mathrm{d}u = 0$, hence $\mathrm{d}(v - xu) + u\,\mathrm{d}x = 0$. This may be identified with the contact equation $\omega^1_0 = 0$ if the functions $v - xu = w^1_0$, $-u = w^1_1$ are regarded as jet coordinates. Then

$$\begin{aligned}\beta &= u\,\mathrm{d}v - \mathrm{d}W \simeq -w^1_1 x\,\mathrm{d}u - \mathrm{d}W = w^1_1 x\,\mathrm{d}w^1_1 - \mathrm{d}W\\ &= -\tfrac{1}{2}(w^1_1)^2\,\mathrm{d}x - \mathrm{d}(W - \tfrac{1}{2}x(w^1_1)^2) \pmod{\vartheta}.\end{aligned}$$

The form $\beta = -\frac{1}{2}(w^1_1)^2\,\mathrm{d}x$ is identified with a fixed "standard" classical density if we moreover choose $W = \frac{1}{2}x(w^1_1)^2$.$_*$

*__8. Rectified extremals.__ We retain $\mathbf{N}$ and β, hence the same stationary curves $P(t)$ as in Section 1; however, the new $\mathcal{A}$-curves $Q(t)$ will be more general here: they satisfy a Monge equation

$$g(x,u,v,u',v') = 0 \qquad (u' = \mathrm{d}u/\,\mathrm{d}t, v' = \mathrm{d}v/\,\mathrm{d}t), \tag{13}$$

by definition. Here g is a function such that $g(x,u,v,0,0) = 0$ is vanishing (the stationary curves are admissible) and $\partial g/\partial u'$, $\partial g/\partial v'$ do not simultaneously vanish.

We are again interested in the sign of the value (2) for a given stationary curve $P(t)$ and a (variable) $\mathcal{A}$-curve $Q(t)$ with the same ends and near enough to this $P(t)$. We will see that the final result is just the same as above if the rotations of the tangent planes

$$U\,\mathrm{d}u - V\,\mathrm{d}v \quad \Big(U = \frac{\partial g}{\partial u'}(x,u,v,0,0), V = \frac{\partial g}{\partial v'}(x,u,v,0,0)\Big) \tag{14}$$

to the Monge cone (13_1) are engaged. In brief:

9. Theorem. *Theorem* 2 *remains true.*

Passing to the proof, we may assume $V \neq 0, \partial G/\partial x \neq 0$ where $G = U/V$ (by virtue of the same argument as in Section 2: use an appropriate linear change of coordinates u, v). Then, owing to the implicit function theory, equation (13_1) implies

$$v' = Gu' + o(u') \qquad (o(u')/u' \to 0 \text{ as } u' \to 0)$$

(where o depends on x, u, v as mere parameters). Simulating (3), we obtain

$$\int_Q \beta - \int_P \beta = \int_Q (u-C)\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int_Q (u-C)\Big(G\frac{\mathrm{d}u}{\mathrm{d}x} + o\Big(\frac{\mathrm{d}u}{\mathrm{d}x}\Big)\Big)\,\mathrm{d}x.$$

Assuming $P^*G > 0$ for a moment (hence $Q^*G > 0$ for the near curve Q), then the o-summand is negligible and we are done. In general, the inequality $P^*G > 0$ can be ensured by a choice of coordinates (e.g., with x retained but v replaced with $v + Mu$ where M is a constant large enough).

The above conditions $V \neq 0$, $G_x \neq 0$ can be replaced by the less restrictive (5) which can be effectively verified without any knowledge of the coordinates u, v by using only the form $\mathrm{d}\beta$ and (a multiple of) the equation (13).$_*$

10. A survey of the task. Passing to the crucial part of the paper, we will deal with the generic subcase $e_1^1 \neq 0$, $A \neq 0$ (see II 2) of degenerate densities (1) where moreover $f_{11}^{11} \neq 0$. Recall that the primary $\mathcal{EL}$ equations $e^1 = e^2 = 0$ can be adapted to obtain a first order system $e = \tilde{e} = 0$ which may be brought into the canonical shape $w_1^i \equiv g^i(x, w_0^1, w_0^2)$ with the derivatives w_1^1, w_1^2 separated on the left. It follows that the functions x, w_0^1, w_0^2 provide coordinates on the $\mathcal{EL}$ subspace $\mathbf{e}\colon\ \mathbf{E} \subset \mathbf{M}(2)$. Following II 3, let a $\mathcal{C}$-curve $P(t) \in \mathbf{E}$ $(0 \leqslant t \leqslant 1)$ be a solution of the $\mathcal{EL}$ system, and consider variable and near enough $\mathcal{A}$-curves $Q(t) \in \mathbf{M}(2)$ $(0 \leqslant t \leqslant 1)$ with the same ends. If $R(t) \in \mathbf{E}$ $(0 \leqslant t \leqslant 1)$ is the projection of $Q(t)$ into $\mathbf{E}$ given by

$$R(t) = (x(t), w_0^1(t), w_0^2(t), g^1(x(t), w_0^1(t), w_0^2(t)), g^2(x(t), w_0^1(t), w_0^2(t)), \ldots) \in \mathbf{E},$$

then we are interested in the increment (see II (14, 15))

$$\int_Q \alpha - \int_P \alpha = \int \mathcal{E}\,\mathrm{d}x + \iint A\omega_0^1 \wedge \omega_0^2. \tag{15}$$

It is expressed by means of the Weierstrass function

$$\mathcal{E} = f(\ldots, w_1^1, w_1^2) - f(\ldots, g^1, g^2) - \sum f_1^i(\ldots, g^1, g^2)(w_1^i - g^i) \in \mathcal{F}(\mathbf{M}(2))$$

(where $\ldots = x, w_0^1, w_0^2$) and the *area coefficient*

$$A = f_{10}^{21} - f_{01}^{21} + f_{11}^{11}(e_0^2 - be_0^1)/e_1^1 \in \mathcal{F}(\mathbf{E})$$

(where g^i are substituted for the respective variables w_1^i). Let us moreover recall the identity II(17):

$$\iint A\omega_0^1 \wedge \omega_0^2 = \iint A(\,\mathrm{d}w_0^1 - g^1\,\mathrm{d}x) \wedge (\,\mathrm{d}w_0^2 - g^2\,\mathrm{d}x) = \iint \mathrm{d}u \wedge \mathrm{d}v, \tag{16}$$

where appropriate first integrals $u, v, \in \mathcal{F}(\mathbf{E})$ of the $\mathcal{EL}$ system $w_1^i \equiv g^i$ are employed.

To transparently demonstrate the sense of some future results without clumsy formulae, they will be represented quite explicitly only for the case of functions $g^1 = g^2 = 0$ identically vanishing (equivalently: for the $\mathcal{EL}$ system $w_1^1 = w_1^2 = 0$). This causes no loss of generality since this favourable state can be achieved by a change of coordinates. (Choosing $w_0^1 = u$, $w_0^2 = v$ for new jet coordinates, one can moreover ensure $A = 1$.)

11. On the Weierstrass function. Identities II (8) and $e_1^1 \neq 0$ imply that $b = \bar{b}(\ldots, e)$ may be regarded as a function of e if the variables $\ldots = x, w_0^1, w_0^2$ are kept fixed. Consequently, the vector field $Z = \partial/\partial w^1 - b\partial/\partial w_1^2 \in \mathcal{T}(\mathbf{M}(2))$ is constant along every level set $e = \text{const.}$ However, $Ze = 0$ (use II (8) again) and we conclude that the level sets mentioned are certain lines

$$\overline{P} + \overline{Q}w_1^1 + \overline{R}w_1^2 = 0 \qquad (\overline{Q} \neq 0, \overline{R}/\overline{Q} = \bar{b}) \tag{17}$$

in the plane of the variables w_1^1, w_1^2 (fixed x, w_0^1, w_0^2). Coefficients $\overline{P}$, $\overline{Q}$, $\overline{R}$ depend on the parameters x, w_0^1, w_0^2 and on the choice of the level set (hence on the value $e = \text{const.}$). Denoting $\bar{c} = \overline{R}/\overline{Q}$, it follows that the requirements

$$e(\ldots, w_1^1, w_1^2) = \text{const.}, \;\; w_1^1 + \bar{b}(\ldots, \text{const.})w_1^2 + \bar{c}(\ldots, \text{const.}) = 0 \tag{18}$$

are equivalent. (Cf. also II (20) with the choice $z = e$ for an alternative proof.) Trivially $Zf_1^1 = Zf_1^2 = 0$ (hence $Z^2 f = 0$), which means that the tangent planes of the surface $f = f(\ldots, w_1^1, w_1^2)$ over every line (18_2) are identical. (We find ourselves in the space of the variables f, w_1^1, w_1^2 for a moment.) In other words, the graph $f = f(\ldots, w_1^1, w_1^2)$ where $\ldots = x, w_0^1, w_0^2$ are kept fixed is a *developable surface* with generating straight lines given by (18_2).

Let us turn to the function $\mathcal{E}$ which clearly represents the vertical distance between the tangent plane to the graph at the point $w_1^i \equiv g^i$ and the graph itself. The distance is taken over a general point w_1^i. (This is the common interpretation, of course.) It follows that $\mathcal{E} = 0$ is vanishing if the points (g^1, g^2) and (w_1^1, w_1^2) are lying on the same straight line (18). However, $e(\ldots, g^1, g^2) = 0$, hence $\mathcal{E} = 0$ if (18) is true with const. $= 0$. Still more explicitly: $\mathcal{E}$ differs from any of the values

$$(e)^2, \quad \left(w_1^1 + \bar{b}(\ldots, 0)w_2^1 + \bar{c}(\ldots, 0)\right)^2 \tag{19}$$

by a factor. To determine it, we may employ the Taylor formula which yields

$$\mathcal{E} = f_{11}^{11}(\ldots, \xi^1, \xi^2)\left(w_1^1 - g^1 - b\;(\ldots, \xi^1, \xi^2)(w_1^2 - g^2)\right)^2 \tag{20}$$

(direct verification) with ξ^i lying between w_1^i and g^i. It follows that the factor is near to f_{11}^{11} as (19_2) is concerned, and near to $f_{11}^{11}(e_1^1)^2$ for the value (19_1).

12. On the area summand. We find ourselves in the space $\mathbf{E}$ with a fixed $\mathcal{C}$-curve $P(t)$ and projections $R(t)$ of variable $\mathcal{A}$-curves. In principle these $R(t)$ may be quite arbitrary curves near enough to $P(t)$ and satisfying the boundary conditions $R(0) = P(0), R(1) = P(1)$. They are parametrizable by the variable x.

The double integral (16) can be alternatively expressed as the line integral $\int u\,\mathrm{d}v$ over the curve $R(t)$. Every such curve individually satisfies (a rather ambigous but nontrivial) Pfaffian equation $U\,\mathrm{d}u = V\,\mathrm{d}v$ where U, V are certain functions of u, v, x depending on the choice of $R(t)$. In the particular case when $R(t)$ is the projection of such an $\mathcal{A}$-curve $Q(t)$ that fulfils $Q^*e = 0$, this equation can be explicitly found: obviously

$$e(x, w_0^1, w_0^2, \mathrm{d}w_0^1/\,\mathrm{d}x,\ \mathrm{d}w_0^2/\,\mathrm{d}x) = 0 \tag{21}$$

for the curve $Q(t)$ hence for $R(t)$, and this is equivalent to

$$\mathrm{d}w_0^1 + \bar{b}(\ldots, 0)\,\mathrm{d}w_0^2 + \bar{c}(\ldots, 0)\,\mathrm{d}x = 0. \tag{22}$$

(see 18). Using alternative coordinates x, u, v, (22) can be expressed as a certain equation $\overline{U}\,\mathrm{d}u = \overline{V}\,\mathrm{d}v$. The summand with $\mathrm{d}x$ is absent since $\mathcal{C}$-curves (given by $\mathrm{d}u = \mathrm{d}v = 0$) are solutions: they satisfy (21), hence (22).

13. The main results. We are passing to the extremality conditions: for a fixed $\mathcal{C}$-curve $P(t) \in \mathbf{E} \subset \mathbf{M}(2)$, and variable but near enough $\mathcal{A}$-curves $Q(t) \in \mathbf{M}(2)$ with the same ends, we are interested in the sign of the value (15). Let us deal with the *minimum* for certainty, i.e., (15) should be a *nonnegative* number.

(i) Oscillating curves. Consider curves $Q(t)$ such that the projections $R(t) \in \mathbf{E}$ satisfy a certain "stationary" equation $U\,\mathrm{d}u = V\,\mathrm{d}v$ with constant U, V. In the plane with variables u, v they can be visualized as a point "oscillating" on the straight line $U(u - u(P(0)) = V(v - v(P(0))$ over the equilibrium $P(t) \equiv P(0)$. The area summand vanishes and (15) reduces to the integral $\int E\,\mathrm{d}x$, which implies $f_{11}^{11} > 0$ for the case of a minimum, see (20). This is a *necessary* condition which is sufficient for the *oscillating curves*.

(ii) Degenerate curves. Consider curves $Q(t)$ satisfying (21), hence $\overline{U}\,\mathrm{d}u = \overline{V}\,\mathrm{d}v$ for the projections $R(t)$. Then $\mathcal{E} = 0$ and (15) reduces to the area summand and the reasoning of Section 2 can be repeated (with $\overline{U}$, $\overline{V}$, $\overline{G} = \overline{U}/\overline{V}$ at the places of previous U, V, G). By virtue of (3), (4), we obtain the condition $\partial\overline{G}/\partial x < 0$. In more generality, conditions (5) with $AD > BC$ and the inequality $<$ in (5_2) are sufficient, too, and they can be effectively verified (see below). According to Section 5 and also 7 (ii), (iii), they may be regarded as *sufficient* Legendre-Jacobi conditions in the above mentioned restricted class of "degenerate" curves $Q(t)$.

(iii) Continuation. To express conditions (5) in more explicit terms, we shall employ the coordinates of the ambient space $\mathbf{M}(2)$ but the calculations will be made at the points of the subspace $\mathbf{E} \subset \mathbf{M}(2)$. Concerning vector fields (9), clearly we have

$$X = \partial/\partial x + \sum^{\infty} w^i_{s+1}\partial/\partial w^i_s, \quad Y = \partial/\partial w^2_0 - b\partial/\partial w^1_0.$$

(Indeed, $\beta = \breve{\alpha}$ and $X \in \operatorname{Adj} \mathrm{d}\breve{\alpha}$ along $\mathbf{E}$, moreover ϑ can be identified with the form (20) where $\bar{b} = b$ along $\mathbf{E}$.) It follows that $[X, Y] = -Xb\,\partial/\partial w^1_0$ and

$$\mathrm{d}\beta = \mathrm{d}\breve{\alpha} = A\omega_0 \wedge \omega^2_0 = A\omega^1_0 \wedge \omega^2_0$$

along $\mathbf{E}$ according to II (10) with $e = \tilde{e} = 0$ substituted. Consequently

$$\mathrm{d}\beta([X, Y], Y) = A\omega^1_0 \wedge \omega^2_0(-Xb\,\partial/\partial w^1_0, \partial/\partial w^2_0) = -AXb$$

and the *Legendre condition* (5_2) for the case of a minimum reads $AXb > 0$. To verify the *Jacobi condition* (5_1), it is necessary to prove the existence of the Jacobi vector field $Z = z^1\partial/\partial w^1_0 + z^2\partial/\partial w^2_0$ (i.e., variation of solutions of the $\mathcal{E}L$ system $e = \tilde{e} = 0$) which realize the inequality $\vartheta(Z) \neq 0$, that is, $z^1 + bz^2 \neq 0$. We may refrain from more comments.

(iv) If the area dominates. We are passing to general curves $Q(t)$. Such $Q(t)$ arbitrarily near to $P(t)$ may occur that the inequality

$$\int \mathcal{E}\,\mathrm{d}x + \iint A\omega^1_0 \wedge \omega^2_0 < 0 \tag{23}$$

which excludes a minimum is satisfied. Owing to (i) and (20), we may assume $\int \mathcal{E}\,\mathrm{d}x \geqslant 0$ here so that the double integral should be negative. However, a large positive double integral excludes a minimum, too, since the orientation of the boundary loop (hence of the area) can be reversed (see the next point).

(v) Continuation. Assume $g^1 = g^2 = 0$, hence the $\mathcal{C}$-curves satisfy $\mathrm{d}w^1_0 \equiv 0$. To simplify a little some formulae, these curves will be considered on the interval $0 \leqslant x \leqslant 2\pi$. Let $Q(t) \in \mathbf{M}(2)$ ($0 \leqslant t \leqslant 1$) be given by

$$x = 2\pi t, w^1_0 = \pm\varepsilon \sin x + \text{const.},\ w^2_0 = \varepsilon(1 - \cos x) + \text{const.},$$

the sign $\pm$ being chosen according to whether $A \lessgtr 0$. Then

$$\iint A\omega^1_0 \wedge \omega^2_0 = \iint A\,\mathrm{d}w^1_0 \wedge \mathrm{d}w^2_0 \geqslant \pi\varepsilon^2 \min|A|,$$

$$\int \mathcal{E}dz = \int_0^{2\pi} f^{11}_{11}(w^1_1 - bw^2_1)^2\,\mathrm{d}x \leqslant 2\pi\varepsilon^2 \int_0^{2\pi} (\cos x - b\sin x)^2\,\mathrm{d}x \max f^{11}_{11},$$

where $\int_0^{2\pi} \ldots \leqslant \pi \max(1 + b^2)$. The inequality (23) is satisfied if

$$2\pi\varepsilon^2.\ \pi\max(1 + b^2) f^{11}_{11} < \pi\varepsilon^2 \min|A|,$$

and $\varepsilon \to 0$ yields the condition

$$2\pi \max_x (1 + b^2) f_{11}^{11} < \min |A| \qquad (w_1^1 = w_1^2 = 0) \tag{24}$$

ensuring the *nonexistence* of a minimum.

(vi) If the area is small. We wish to obtain the minimum, hence a non-negative value (15) for a large class of curves $Q(t)$. Recalling alternative coordinates x, u, v and the Pfaffian equation $\overline{U}\,\mathrm{d}u = \overline{V}\,\mathrm{d}v$ for the "degenerate" curves from point (ii), we will assume $\overline{V} \neq 0$ and $\overline{G}_x < 0$ to ensure the Legendre-Jacobi conditions. Recalling the curves $Q(t)$ and the function $\mathcal{E}$, we know that this $\mathcal{E}$ is a multiple of (19_2), hence of $(\overline{U}\,\mathrm{d}u/\,\mathrm{d}x - \overline{V}\,\mathrm{d}v/\,\mathrm{d}x)^2$ and the factor does not essentially depend on the choice of the curve $Q(t)$. (The factor can be expressed in terms of u, v, f but we need not state it here.) *We shall deal only with "short" curves* $Q(t)$ in the sense that each of them individually satisfies a Pfaffian equation $U\,\mathrm{d}u = V\,\mathrm{d}v$ with $V \neq 0$. Then

$$\overline{U}\frac{\mathrm{d}u}{\mathrm{d}x} - \overline{V}\frac{\mathrm{d}v}{\mathrm{d}x} = \overline{V}\frac{\mathrm{d}u}{\mathrm{d}x}(\overline{G} - G) \quad (G = U/V, \overline{G} = \overline{U}/\overline{V})$$

and it follows that $\mathcal{E}$ is a multiple of $(\overline{V}\frac{\mathrm{d}u}{\mathrm{d}x}(\overline{G} - G))^2$. We may write $\overline{V} = 1$ to simplify some formulae; this is a mere change of notation.

Our aim is to compare the summand $\int \mathcal{E}\,\mathrm{d}x$ in (15_2) with the summand

$$\mathbf{A} = \iint \mathrm{d}u \wedge \mathrm{d}v = \int_Q u\,\mathrm{d}v = -\frac{1}{2}\int_Q (u - C)^2 \frac{\mathrm{d}G}{\mathrm{d}x}\,\mathrm{d}x$$

stated in (16). Since $\mathbf{A} > 0$ trivially ensures the desired sign of the value (15), we may assume $\mathbf{A} \leqslant 0$ from now on. In this case

$$\begin{aligned}\int_Q (u - C)^2\,\mathrm{d}x \int_Q \Big(\frac{\mathrm{d}u}{\mathrm{d}x}(\overline{G} - G)\Big)^2\,\mathrm{d}x &\geqslant \Big(\int_Q (u - C)\frac{\mathrm{d}u}{\mathrm{d}x}(\overline{G} - G)\,\mathrm{d}x\Big)^2 \\ = \Big(\frac{1}{2}\int_Q (u - C)^2 \frac{\mathrm{d}(\overline{G} - G)}{\mathrm{d}x}\,\mathrm{d}x\Big)^2 &= \frac{1}{4}(\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2) \geqslant |\mathbf{AB}|\end{aligned}$$

by virtue of the Schwartz inequality, integration by parts, and elementary inequality concerning the numbers $\mathbf{A}$ and

$$\mathbf{B} = \int_Q (u - C)^2 \frac{\mathrm{d}\overline{G}}{\mathrm{d}x}\,\mathrm{d}x$$

of the same sign. By using the trivial estimate

$$|\mathbf{B}| \geqslant \int_Q (u - C)^2\,\mathrm{d}x \min |\,\mathrm{d}\overline{G}/\,\mathrm{d}x|$$

where $|\,\mathrm{d}\overline{G}/\,\mathrm{d}x| \geqslant$ const. > 0, one can easily obtain the final result

$$\int_Q \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 (\overline{G} - G)^2 \,\mathrm{d}x \geqslant \text{const. } |\mathbf{A}| \qquad \left(\text{const.} = \min\left|\frac{\mathrm{d}\overline{G}}{\mathrm{d}x}\right|\right) \tag{25}$$

which can be rewritten as

$$\int \mathcal{E} \,\mathrm{d}x \geqslant \text{Const. } |\mathbf{A}| \qquad (\text{Const.} > 0) \tag{26}$$

for a certain Constant (not explicitly stated here) expressible in terms of functions $f, \overline{U}, u, v$. If this Constant is at least 1, the *minimum is ensured*.

(vii) C o n t i n u a t i o n . Assume $g^1 = g^2 = 0$ to express the Constant in explicit terms. Then we may even choose $u = w_0^1$, $v = w_0^2$ and $\mathcal{E}$ becomes a multiple of

$$f_{11}^{11}(w_1^1 - bw_1^2)^2 = f_{11}^{11}\left(\frac{\mathrm{d}v}{\mathrm{d}x} - b\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 = f_{11}^{11}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 (\overline{G} - G)^2$$

with an (already quite simple) factor *near to* 1; see also (20). Clearly $b = \overline{G}$ whence

$$\int \mathcal{E} \,\mathrm{d}x \sim \int f_{11}^{11}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 (\overline{G} - G)^2 \,\mathrm{d}x \geqslant \int_Q \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 (\overline{G} - G)^2 \,\mathrm{d}x \min f_{11}^{11}$$
$$> \text{const. } |\mathbf{A}| \min f_{11}^{11}$$

by virtue of (25_1) with any fixed choice of the constant satisfying $|b_x| >$ const. > 0. Then (26) yields the condition

$$\min_x |f_{11}^{11} b_x| > 1 \qquad (w_1^1 = w_1^2 = 0) \tag{27}$$

ensuring Const. > 1, hence the minimum.

(viii) S u m m a r y . We have the *necessary* condition $f_{11}^{11} > 0$ (sufficient for oscillating curves), and the *Legendre condition* $AXb > 0$ which together with the *Jacobi condition* (see iii) ensure the minimum for "degenerate" perturbations. In general, the extremality properties are depending on certain *inequalities between the "curvature" of the Weierstrass function* $\mathcal{E}$ *and the "magnitude" of the area constant* A.

14. A development of the increment. Leaving the three dimensional spaces, we find ourselves in the space $\mathbf{N}$ equipped with a differential form $\beta = u^1\,\mathrm{d}v^1 + \ldots + u^c\,\mathrm{d}v^c - \mathrm{d}W$ where $x, u^1, v^1, \ldots, u^c, v^c$ are coordinates on $\mathbf{N}$ and $W = W(x, u^1, v^1, \ldots, u^c, v^c) \in \mathcal{F}(\mathbf{N})$ is a given function. Choosing a *stationary curve* $P(t) \in \mathbf{N}$ $(0 \leqslant t \leqslant 1)$ defined by the property $(P^*Z\rfloor\,\mathrm{d}\beta \equiv 0$ for all $Z \in \mathcal{T}(\mathbf{N})$ or, equivalently) that P^*u^i and P^*v^i are all constant, we shall moreover deal with other curves $Q(t) \in \mathbf{N}$ $(0 \leqslant t \leqslant 1)$ with the same ends $Q(0) = P(0)$ and $Q(1) = P(1)$. Then the value of the increment $\int_Q \beta - \int_P \beta$ can be made arbitrary, by an appropriate choice of the curve Q. However, let us consider only *admissible* ($\mathcal{A}$-) curves that satisfy a Pfaffian system

$$\mathrm{d}v^i \equiv \sum G^{ij}\,\mathrm{d}u^j \qquad (i = 1, \ldots, c;\ G^{ij} \in \mathcal{F}(\mathbf{N})). \tag{28}$$

Then, denoting $u^i(Q(0)) = u^i(P(0)) = u^i(P(1)) = u^i(Q(1)) \equiv C^i$, we obtain

$$\begin{aligned}\int_Q \beta - \int_P \beta &= \sum \int_Q (u^i - C^i)\,\mathrm{d}v^i = \sum \int_Q G^{ij}\bar{u}^i\,\mathrm{d}u^j = \sum \int_Q G^{ij}\bar{u}^i\,\mathrm{d}u^j \\ &= -\sum \int_Q \bar{u}^i\bar{u}^j\,\mathrm{d}G^{(ij)} + \frac{1}{2}\sum \int_Q G^{ij}(\bar{u}^i\,\mathrm{d}u^j - \bar{u}^j\,\mathrm{d}u^i)\end{aligned} \tag{29_1}$$

where $G^{(ij)} \equiv \frac{1}{2}(G^{ij} + G^{ji})$ and $\bar{u}^i \equiv u^i - C^i$. The last "area summand" will be retained, however,

$$\mathrm{d}G^{(ij)} \equiv \frac{\partial G^{(ij)}}{\partial x}\,\mathrm{d}x + \sum G^{ijk}\,\mathrm{d}u^k\ \left(G^{ijk} \equiv \frac{\partial G^{(ij)}}{\partial u^k} + \sum \frac{\partial G^{(ij)}}{\partial v^l} G^{lk}\right) \tag{30_1}$$

may be substituted whence

$$\sum \int_Q \bar{u}^i\bar{u}^j\,\mathrm{d}G^{(ij)} = \sum \int_Q \bar{u}^i\bar{u}^j \frac{\partial G^{(ij)}}{\partial x}\,\mathrm{d}x + \sum \int_Q G^{ijk}\bar{u}^i\bar{u}^j\,\mathrm{d}u^k.$$

The first summand on the right hand side will be retained, however, we adapt the second:

$$\begin{aligned}\sum \int_Q G^{ijk}\bar{u}^i\bar{u}^j\,\mathrm{d}u^k &= -\sum \int_Q \bar{u}^i\bar{u}^j\bar{u}^k\,\mathrm{d}G^{(ijk)} \\ &+ \frac{1}{3}\sum \int_Q G^{ijk}(\bar{u}^i(\bar{u}^j\,\mathrm{d}u^k - \bar{u}^k\,\mathrm{d}u^j) + \bar{u}^j(\bar{u}^i(\bar{u}^j\,\mathrm{d}u^k - \bar{u}^k\,\mathrm{d}u^i))\end{aligned} \tag{29_2}$$

where $G^{(ijk)} = \frac{1}{3}(G^{ijk} + G^{jki} + G^{kij})$. The last "area summand" will be retained, however,

$$\mathrm{d}G^{(ijk)} = \frac{\partial G^{(ijk)}}{\partial x}\,\mathrm{d}x + \sum G^{ijkl}\,\mathrm{d}u^l \ \Big(G^{ijkl} = \frac{\partial G^{(ijk)}}{\partial u^l} + \sum \frac{\partial G^{(ijk)}}{\partial v^m} G^{ml}\Big) \tag{30_2}$$

may be substituted to obtain a more precise formula, and so on. Then, at least symbolically, the final result reads

$$\begin{aligned}\int_Q \beta - \int_P \beta = & -\sum \int_Q \bar{u}^i \bar{u}^j \frac{\partial G^{(ij)}}{\partial x}\,\mathrm{d}x + \sum \int_Q \bar{u}^i \bar{u}^j \bar{u}^k \frac{\partial G^{(ijk)}}{\partial x}\,\mathrm{d}x - \dots \\ & + \frac{1}{2}\sum \int_Q G^{ij}(\bar{u}^i\,\mathrm{d}u^j - \bar{u}^j\,\mathrm{d}u^i) \\ & - \frac{1}{3}\sum \int_Q G^{ijk}(\bar{u}^i(\bar{u}^j\,\mathrm{d}u^k - \dots - \bar{u}^k\,\mathrm{d}u^j)) + \dots\end{aligned} \tag{30}$$

and provides a very unusual information concerning the increment since the functions $\bar{u}^i \equiv u^i - C^i = Q^* u^i - C^i$ are (as a rule) assumed close to zero. (The convergence can be verified in the analytical case.)

15. Some quadratic forms. We will deal only with the quadratic terms of the above formula (30) which are well-sufficient in current applications. They admit a geometrical interpretation.

In a space $\mathbf{M}$, consider a density $\beta \in \Phi(\mathbf{M})$ which has a *characteristic vector field* $X \in \mathcal{T}(\mathbf{M})$ defined by the property $X \rfloor \mathrm{d}\beta = 0$ (equivalently: $X \in \operatorname{Adj} \mathrm{d}\beta^{\perp}$). Then, for any $Y, Z \in \mathcal{T}(\mathbf{M})$, we put $\mathbf{Q}(Y,Z) = \mathrm{d}\beta([X,Y],Z)$.

To establish the most important property of $\mathbf{Q}$, let us recall the exterior derivative $\mathrm{d}\varphi$ of a differential two-form: for any $U, V, W \in \mathcal{T}(\mathbf{M})$ we have $\mathrm{d}\varphi(U,V,W) =$

$$U\varphi(V,W) - V\varphi(U,W) + W\varphi(U,V) - \varphi([U,V],W) + \varphi([U,W],V) - \varphi([V,W],U).$$

Then, by using the identity $\mathrm{d}\varphi = 0$ valid for the choice $\varphi = \mathrm{d}\beta$, we obtain

$$X\,\mathrm{d}\beta(Y,Z) = \mathrm{d}\beta([X,Y],Z) - \mathrm{d}\beta([X,Z],Y) = \mathbf{Q}(Y,Z) - \mathbf{Q}(Z,Y)$$

and it follows that $\mathbf{Q}(Y,Z)$ is a *symmetric* (and consequently $\mathcal{F}(\mathbf{M})$-bilinear) *function* if the domain of the arguments Y, Z is restricted to an isotropic submodule of $\mathcal{T}(\mathbf{M})$. (Recall two concepts of the symplectic geometry: a submodule $\mathcal{I} \subset \mathcal{T}(\mathbf{M})$ is called *isotropic* to the form $\mathrm{d}\beta$ if $\mathrm{d}\beta(Y,Z) \equiv 0$ for all $Y, Z \in \mathcal{I}$, and a submodule $\mathcal{L} \subset \mathcal{T}(\mathbf{M})$ is called *Lagrangian* (to $\mathrm{d}\beta$) if it is isotropic and of the maximal possible dimension. In particular a characteristic vector field is lying in all Lagrangian submodules.)

Roughly speaking, $\mathbf{Q}(Y,Z)$ indicates those shifts $[X,Y]$ of Y by means of X which fall out of every Lagrangian submodule involving both Y and Z. In other words, for a given isotropic submodule $\mathcal{I} \subset \mathcal{T}(\mathbf{M})$, the vanishing $\mathbf{Q}(Y,Z) \equiv 0$ $(Y,Z \in \mathcal{I})$ means that "moving in direction X" again lies in a certain greater isotropic submodule over $\mathcal{I}$. This permits to introduce "higher order" forms.

Assuming $\mathbf{Q}(Y,Z) \equiv 0$ $(Y,Z \in \mathcal{I})$, we may introduce $\mathbf{Q}'(Y,Z) = \mathbf{Q}([X,Y],Z) = \mathrm{d}\beta([X,[X,Y]],Z)$, which is a *skewsymmetric* (hence $\mathcal{F}(\mathbf{M})$-bilinear) function of $Y,Z \in \mathcal{I}$:

$$\mathbf{Q}'(Y,Z) = \mathbf{Q}([X,Y],Z) = \mathbf{Q}(Z,[X,Y]) = \mathrm{d}\beta([X,Z],[X,Y]) = -\mathbf{Q}'(Z,Y).$$

Analogously, assuming $\mathbf{Q}'(Y,Z) \equiv 0$ $(Y,Z \in \mathcal{I})$, we may introduce the *symmetric form* $\mathbf{Q}''(Y,Z) = \mathbf{Q}'([X,Y],Z) = -\mathbf{Q}([X,Y],[X,Z])$, and so on.

16. Two concluding examples. At this point, all the previous reasonings concerning the lowest dimension case could be carried over to higher dimensions, in particular all examples of Part I should be revised from this point of view. To please the reader, we however shall not follow this way and will close our story with a few remarks concerning simple interrelations with the classical theory.

First of all, let us mention the obvious generalization of the main part of Theorem 2: *for the particular symmetrical case $G^{ij} \equiv G^{ji}$, the definiteness of the quadratic form*

$$\sum \frac{\partial G^{(ij)}}{\partial x}\xi^i\xi^j \qquad (G^{(ij)} = \tfrac{1}{2}(G^{ij} + G^{ji})) \tag{31}$$

ensures the constant sign of the difference $\int_Q \beta - \int_P \beta$ in (29_1). (The proof may be omitted.) One can observe that the symmetry is ensured if and only if $\mathrm{d}\beta \simeq 0$ modulo relations (28); this is the well-known Cartan's lemma. The definiteness can be interpreted by means of certain rotations. (Hint: assuming $\xi^2 = \ldots = \xi^c = 0$, we have the condition $\partial G^{11}/\partial x \neq 0$ which indicates a certain rotation of the line $\mathrm{d}v^1 = G^{11}\,\mathrm{d}u^1$. The particular choice of coordinates ξ^i does not matter much here.)

The above result does not exactly correspond to Theorem 2. To obtain the full agreement, the more general Pfaffian system

$$\sum U^{ik}\,\mathrm{d}u^k = \sum V^{il}\,\mathrm{d}v^l \;\; (i = 1,\ldots,c) \tag{32}$$

should be taken instead of (28). Then the condition det $(V^{il}) \neq 0$ ensuring the equivalence with the original system (28) could be interpreted by saying that the "magnitude of the total rotations" are small. By using a symplectical change $u \to$

$Au + Bv$, $v \to Cu + Dv$ of coordinates $u = (u^1, \dots, u^c)$, $v = (v^1, \dots, v^c)$, one can obtain a generalization of (5): denoting $U = (U^{ik}), V = (V^{il})$, the condition (5_1) will be replaced by the invertibility of the matrix $CU + DV$, the condition (5_2) turns into the definiteness of the matrix $(AU + VB)/(CU + DV)^{-1}$.

Finally, the definiteness can be verified in terms of the form $\mathrm{d}\beta$ and the submodule

$$\Theta = \left\{ \mathrm{d}v^i - \sum G^{ij}\, \mathrm{d}u^j \right\} = \left\{ \sum U^{ik}\, \mathrm{d}u^k - \sum V^{il}\, \mathrm{d}v^l \right\} \subset \mathcal{T}(\mathbf{N}).$$

One can easily verify the identity

$$\mathrm{d}\beta([X, Y], Y) = \sum A^i A^j \partial G^{(ij)}/\partial x = \mathbf{Q}(Y, Y), \tag{33}$$

where X is a nonvanishing characteristic vector field to $\mathrm{d}\beta$ and

$$Y = \sum A^i \left(\partial/\partial v^i + \sum G^{ij} \partial/\partial u^j \right) \in \Theta^{\perp} \ (A^i \in \mathcal{F}(\mathbf{N})).$$

This yields the *Legendre criterions*. Omitting any mention of the *Jacobi criterion*, let us eventually pass to concluding examples.

Quite analogously as in Section 5, the nondegenerate case $\det(f_{11}^{ij}) \neq 0$ of the density

$$\alpha = f(x, w_0^1, \dots, w_0^m, w_1^1, \dots, w_1^m)\, \mathrm{d}x \in \Phi(\mathbf{M}(m))$$

in the space $\mathbf{M}(m)$ equipped with the diffiety $\Omega(m)$ (see I3) can be included for the choice

$$\mathbf{N} = \mathbf{E},\ \beta = \breve{\alpha},\ \Theta = \{\omega_0^1, \dots, \omega_0^m\}$$

generalizing (11). Since $\mathrm{d}\beta \simeq 0 \pmod{\Theta}$ according to II(2), we have the symmetric subcase. In terms of jet coordinates, the characteristic vector field $X = \partial/\partial x + \overset{\infty}{\sum} w_{r+1}^i \partial/\partial w_r^i$ (restricted to $\mathbf{E}$) and the form $Y = \sum A^i \partial/\partial w_1^i \in \Theta^{\perp}$ substituted into (32) yield the familiar *Legendre condition*: the definiteness of the quadratic form $\sum f_{11}^{ij} \xi^i \xi^j$.

The second example concerns the density $\alpha = f(x, w_0^1, w_1^1, w_2^1)\, \mathrm{d}x \in \Phi(\mathbf{M}(1))$ and diffiety $\Omega(1)$. One can easily find the $\mathcal{PC}$ form and the $\mathcal{EL}$ expression

$$\breve{\alpha} = f\, \mathrm{d}x + (f_1^1 - X f_2^1)\omega_0^1 + f_2^1 \omega_1^1, \quad e^1 = f_0^1 - X f_1^1 + X^2 f_2^1. \tag{34}$$

Since $e^1 = f_{22}^{11} w_4^1 + \dots$, the functions $x, w_0^1, \dots, w_3^1$ may be used for coordinates on the $\mathcal{EL}$ subspace $\mathbf{E} \subset \mathbf{M}(1)$ for the nondegenerate case $f_{22}^{11} \neq 0$ (which we assume). Methods of Part I can be applied and yield the Hilbert-Weierstrass criterion (not stated here) together with the familiar and classical *Legendre condition* $f_{22}^{11} > 0$ for the minimum. One can then choose $\mathbf{N} = \mathbf{E}$, $\beta = \breve{\alpha}$, $\Theta = \{\omega_0^1, \omega_0^1\}$ and verify the formula

$$\mathrm{d}\beta([X,[X,[X,Y]]],Y) = f_{22}^{11} = \mathbf{Q}''(Y,Y)$$

for the choice $X = \partial/\partial x + \sum^{\infty} w_{r+1}^1 \partial/\partial w_r^1, Y = \partial/\partial w_3^1 \in \Theta^{\perp}$.

C o m m e n t s. Our analysis of the degenerate problem is worth two notes. The necessary condition in 13 (iv, v) involves some "long" perturbations and it would be interesting to determine an analogous but less restrictive result for "short" curves. On the contrary, the sufficient condition in 13 (vi, vii) is valid only for "short" perturbations (which may be however regarded as small in a certain sense). It would be interesting to examine the "long" curves. Concerning the higher dimensions, the structure of the development (30) can be related to the theory of decomposition of tensor fields into irreducible factors, the symmetrical and the next "symmetrical-but-area one" with the Young diagram $(2, k)$. We have included only the simplest possible ilustrative examples here and refer to the concluding Part IV for the adaptation of the Weirstrass-Hilbert method to other variational problems.

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