DISTRIBUTIVITY OF LATTICES OF BINARY RELATIONS

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Abstract. We present a formal scheme which whenever satisfied by relations of a given relational lattice L containing only reflexive and transitive relations ensures distributivity of L.

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Distributivity of lattices of binary relations was treated by several authors, see e.g. [1] for lattices of tolerances and [2], [3] for lattices of congruences. H.-P. Gumm developed in [4] two schemes (the so called Shifting Lemma and Shifting Principle) to characterize modularity of congruence lattices in algebras and varieties. A certain scheme characterizing distributivity of congruence lattices can be found in [2]. The aim of this short note is to present a suitable scheme for characterizing distributivity in a more general case.

Let α be a binary relation on a set A. The fact that $\langle x,y\rangle \in \alpha$ will be visualized by an arrow going from x to y (where x,y are depicted by points in a plane) which is valuated by α, see Fig. 1.

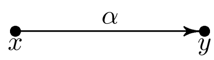


Fig. 1.

Definition. Let L be a lattice of binary relations on a set $A \neq \emptyset$. We say that L satisfies the *Corner Scheme* if for any $\alpha, \beta, \gamma \in L$ the following condition is satisfied:

$$\text{if } \alpha \cap \beta \subseteq \gamma \text{ and } \langle z,y\rangle \in \beta,\ \langle a,x\rangle \in \alpha \text{ and } \langle x,y\rangle \in \alpha \vee \gamma,\ \text{then } \langle z,y\rangle \in \gamma.$$

Remark. In our graphical convention, the Corner Scheme can be visualized as shown in Fig. 2.

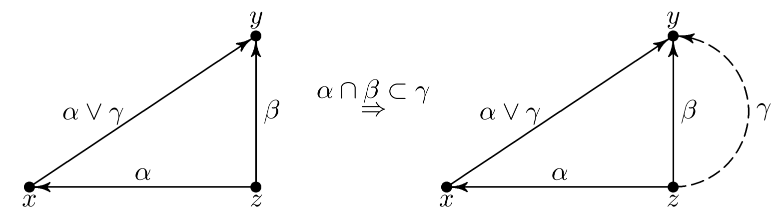


Fig. 2.

Lemma 1. *Let L be a lattice of transitive binary relations on a set $A \neq \emptyset$. If L is distributive than it satisfies the Corner Scheme.*

P r o o f. Let L be distributive, $\alpha, \beta, \gamma \in L$ and $\alpha \cap \beta \subseteq \gamma$. Suppose $\langle z, y\rangle \in \beta$, $\langle z, x\rangle \in \alpha$ and $\langle x, y\rangle \in \alpha \vee \gamma$. Due to transitivity, we have $\langle z, y\rangle \in \alpha \cdot (\alpha \vee \gamma) \subseteq (\alpha \vee \gamma) \cdot (\alpha \vee \gamma) \subseteq \alpha \vee \gamma$, thus also

$$\langle a, y\rangle \in \beta \cap (\alpha \vee \gamma) = (\beta \cap \alpha) \vee (\beta \cap \gamma) \subseteq \gamma \vee (\beta \cap \gamma) = \gamma,$$

so L satisfies the Corner Scheme. □

Lemma 2. *Let L be a lattice of reflexive binary relations on a set $A \neq \emptyset$. If L satisfies the Corner Scheme then it is distributive.*

P r o o f. Let L satisfy the Corner Scheme and suppose that it is not distributive. Then L contains a sublattice isomorphic to M_3 or N_5 as shown in Fig. 3.

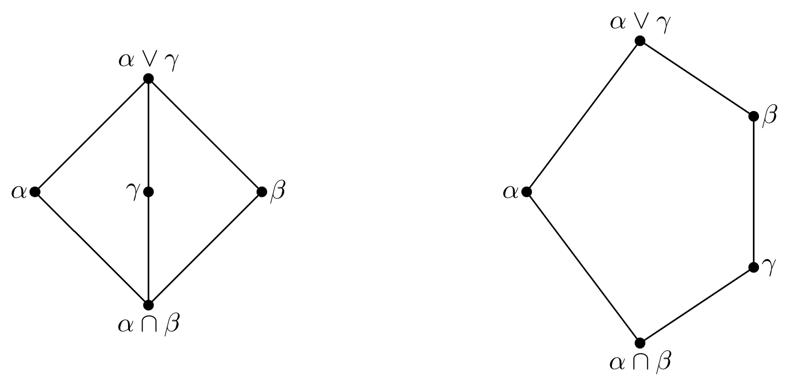


Fig. 3.

Of, course, we have $\alpha \cap \beta \subseteq \gamma$ in the both cases. Suppose $\langle z, y\rangle \in \beta$. Then $\langle z, y\rangle \in \alpha \vee \gamma$ and, due to reflexivity and the property $\alpha \subseteq \alpha \vee \gamma$, also $\langle z, y\rangle \in \alpha \cdot (\alpha \vee \gamma)$. Thus there is $x \in A$ with $\langle z, x\rangle \in \alpha$ and $\langle x, y\rangle \in \alpha \vee \gamma$. By the Corner Scheme we conclude $\langle z, y\rangle \in \gamma$. We have shown $\beta \subseteq \gamma$ which contradicts $\beta \| \gamma$ in M_3 or $\gamma \subset \beta$ in N_5. □

Theorem. *Let L be a lattice of reflexive and transitive binary relations on a set $A \neq \emptyset$. Then L is distributive if and only if L satisfies the Corner Scheme.*

This is an immediate consequence of Lemma 1 and Lemma 2. Since $\beta \cap \gamma \subseteq \beta \cap (\alpha \vee \gamma)$ for any α, β, γ of any lattice L, we can state the following conclusion of the Corner Scheme.

Corollary 1. *Any lattice of reflexive and transitive relations on a set $A \neq \emptyset$ is distributive if and only if it satisfies the quasiidentity:*

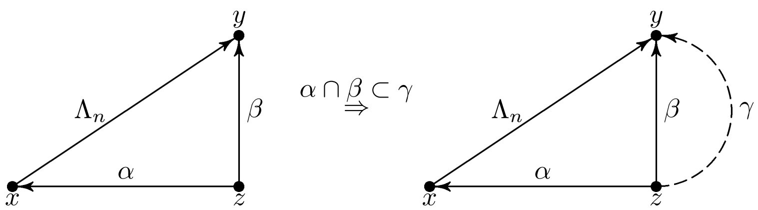
$$\alpha \cap \beta \subseteq \gamma \Rightarrow \beta \cap \gamma = \beta \cap (\alpha \vee \gamma).$$

R e m a r k. It is well-known and easy to check that in any lattice L of reflexive and transitive relations we have

$$\alpha \vee \gamma = \bigcup \{\alpha \cdot \gamma \cdot \alpha \cdot \ldots (n \text{ factors});\ n \in \mathcal{N}\}.$$

Denote by $\Lambda_n = \gamma \cdot \alpha \cdot \gamma \cdot \ldots (n \text{ factors})$ for $n \in \mathcal{N}_0$ (if $n = 0$ then Λ_0 is the identity relation on A). Then our Corner Scheme can be reformulated as follows:

Corollary 2. *Let L be any lattice of reflexive and transitive binary relations on a set $A \neq \emptyset$. Then L is distributive if and only if it satisfies the following scheme for all $n \in \mathcal{N}_0$.*



Let us note that the last scheme was first used for characterizing distributivity of congruence lattices in [2] under the name of Triangular Scheme.

We say that the lattice L of binary relations on a set A is *permutable* if

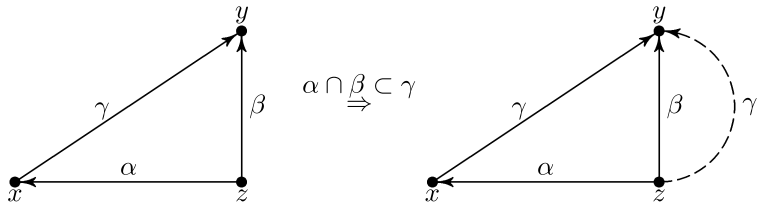
$$\alpha \cdot \gamma = \gamma \cdot \alpha$$

for every $\alpha, \gamma \in L$.

Of course, if L is a permutable lattice of reflexive and transitive relations on a set $A \neq \emptyset$ then $\alpha \vee \gamma = \alpha \cdot \gamma$.

Hence, we can take $n = 1$ in Corollary 2 to prove the last result:

Corollary 3. *Let L be a permutable lattice of reflexive and transitive binary relations on a set $A \neq \emptyset$. Then L is distributive if and only if it satisfies the following scheme:*



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