REMARKS ON COMMUTATIVE HILBERT ALGEBRAS

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Abstract. The paper shows that commutative Hilbert algebras introduced by Y. B. Jun are just J. C. Abbot's implication algebras.

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1. Introduction

Hilbert algebras are important tools for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value "true". As usual, the operation is denoted by "$\cdot$" instead of "$\Rightarrow$" although it has the same meaning.

The concept of Hilbert algebra was introduced in the 50-ties by L. Henkin and T. Skolem for investigations in intuitionistic and other non-classical logics. A. Diego [5] proved that Hilbert algebras form a variety which is locally finite.

They were studied from various points of view. Concerning congruence properties it is shown in [2] that Hilbert algebras form a congruence distributive variety the congruences in which are in a 1-1 correspondence with ideals [4]. Pseudocomplements as well as relative pseudocomplements of elements in lattices of ideals of Hilbert algebras were then described and studied in [3].

In [6] the notion of a commutative Hilbert algebra was introduced and studied. The aim of this short note is to show that this paper contains non-valid theorems as well as that commutative Hilbert algebras are exactly implication algebras treated by J. C. Abbott [1].

2. Preliminaries

Definition 1. A *Hilbert algebra* is a triplet $\mathcal{H} = (H; \cdot, 1)$, where H is a non-empty set, $\cdot$ is a binary operation on H and 1 is a fixed element of H (i.e. a nullary operation) such that the following axioms hold in $\mathcal{H}$:

(HA1) $x \cdot (y \cdot x) = 1$,
(HA2) $(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1$,
(HA3) $x \cdot y = 1$ and $y \cdot x = 1$ imply $x = y$.

For the proof of the following result, see e.g. [5].

Proposition 1. *Every Hilbert algebra satisfies the following properties:*
(1) $x \cdot x = 1$,
(2) $1 \cdot x = x$,
(3) $x \cdot 1 = 1$,
(4) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$,
(5) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$,
(6) $x \leqslant y \Rightarrow y \cdot z \leqslant x \cdot z$,
(7) $x \leqslant y \Rightarrow z \cdot x \leqslant z \cdot y$.

It can be easily verified that the relation $\leqslant$ defined in a Hilbert algebra $\mathcal{H} = (H; \cdot, 1)$ by

$$x \leqslant y \text{ if and only if } x \cdot y = 1$$

is a partial order relation on H with 1 as the greatest element. This order relation is called the *natural ordering* on H.

Example 1. It is of great importance that every partially ordered set $(P, \leqslant, 1)$ with the greatest element 1 can be regarded as a Hilbert algebra, namely, if we define for $x, y \in P$

$$x \cdot y = 1 \text{ whenever } x \leqslant y, \text{ and } x \cdot y = y \text{ otherwise,}$$

then $(P, \cdot, 1)$ is a Hilbert algebra the natural ordering on which coincides with the relation $\leqslant$.

Hilbert algebras generalize properties of implicative reducts of Boolean algebras (i.e. algebras corresponding to a classical logic), the so called implication algebras, treated by J. C. Abbott in [1]:

Definition 2. An *implication algebra* (IA) is an algebra $(A, \cdot, 1)$ of type (2,0) satisfying the following conditions:

(I1) $x \cdot x = 1$,
(I2) $(x \cdot y) \cdot x = x$,
(I3) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$,
(I4) $(x \cdot y) \cdot y = (y \cdot x) \cdot x$.

Of course, since implication algebras are a special case of Hilbert algebras, one can define a natural ordering $\leqslant$ on A in the same way as for Hilbert algebras.

Abbott has shown that implication algebras are a natural generalization of Boolean algebras in the following sense:

Proposition 2. (i) *Let $(A, \cdot, 1)$ be an implication algebra. Then each interval $[p, 1]$ in A is a Boolean algebra w.r.t. operations defined by*

$$\begin{aligned} x \vee y &= (x \cdot y) \cdot y, \\ x \wedge y &= ((x \cdot p) \vee (y \cdot p)) \cdot p, \\ x' &= x \cdot p. \end{aligned}$$

(ii) *Conversely, if $(A, \vee)$ is a $\vee$-semilattice each interval in which is a Boolean algebra w.r.t. the induced order, then A with the operation $\cdot$ defined by*

$$x \cdot y = (x \vee y)^y,$$

where $(x \vee y)^y$ is the relative pseudocomplement of $x \vee y$ in the Boolean algebra $[y, 1]$, is an implication algebra.

Proposition 2 says that there is a 1-1 correspondence between implication algebras and join semilattices having Boolean algebras for intervals.

By [6], a Hilbert algebra $\mathcal{H}$ is said to be *commutative* if it satisfies the axiom (I4). Hence $\mathcal{H}$ is then an implication algebra if and only if also (I3) is satisfied in $\mathcal{H}$.

Theorem 3.3. in [6] claims that commutative Hilbert algebras are just those which are join semilattices w.r.t. the natural ordering. A simple inspection shows that this does not hold:

Example 2. Let us consider a 4-element Boolean algebra $A = \{0, 1, a, a'\}$ with the corresponding order relation $\leqslant$. By Example 1, the operation $\cdot$ defined on A by

$$x \cdot y = 1 \text{ if and only if } x \leqslant y, \ \ x \cdot y = y \text{ otherwise},$$

defines on A a Hilbert algebra which is surely a join semilattice. On the other hand, it is not commutative, since e.g. $1 = (a \cdot 0) \cdot 0 \neq (0 \cdot a) \cdot a = a$.

In the next section we will show by using Proposition 2 that commutative Hilbert algebras are just the implication ones.

3. Commutative Hilbert algebras

First we show that commutative Hilbert algebras form a join semillatice w.r.t. the natural ordering:

Lemma 1. *If $\mathcal{H} = (H, \cdot, 1)$ is a commutative Hilbert algebra then the natural ordering $\leqslant$ on H is a semilattice and $x \vee y = (x \cdot y) \cdot y$.*

Proof. According to (HA1) and commutativity it is clear that the element $(x \cdot y) \cdot y = (y \cdot x) \cdot x$ is an upper bound of x and y. Suppose that $x \leqslant q$, $y \leqslant q$ for some $q \in H$. Then Proposition 1(6) yields $q \cdot y \leqslant x \cdot y$ and $(x \cdot y) \cdot y \leqslant (q \cdot y) \cdot y = (y \cdot q) \cdot q = 1 \cdot q = q$, proving that $(x \cdot y) \cdot y$ is the least upper bound of x and y. $\square$

Lemma 2. *Let $\mathcal{H} = (H, \cdot, 1)$ be a commutative Hilbert algebra and let $a, b, p \in H$. Then*

(1) *$p \leqslant a$ yields $(a \cdot p) \cdot a = a$;*
(2) *$p \leqslant b$ yields $a \cdot b = (a \cdot p) \vee b$.*

Proof. (1) Suppose $p \leqslant a$. Then $p \cdot a = 1$ and

$$(p \cdot a) \cdot a = 1 \cdot a = a = a \vee p = (a \cdot p) \cdot p.$$

Hence

$$(a \cdot p) \cdot a = (a \cdot p) \cdot [(a \cdot p) \cdot p] = [(a \cdot p) \cdot (a \cdot p)] \cdot [(a \cdot p) \cdot p] = 1 \cdot [(a \cdot p) \cdot p] = a.$$

(2) We compute

$$(a \cdot p) \vee b = [b \cdot (a \cdot p)] \cdot (a \cdot p) = [a \cdot (b \cdot p)] \cdot (a \cdot p) = a \cdot [(b \cdot p) \cdot p] = a \cdot (b \vee p) = a \cdot b.$$

$\square$

The foregoing theorem describes intervals in commutative Hilbert algebras:

Theorem. *Let $\mathcal{H} = (H, \cdot, 1)$ be a commutative Hilbert algebra. For every $p \in H$ the interval $[p, 1]$ is a Boolean algebra where for $a, b \in [p, 1]$ we have $a \vee b = (a \cdot b) \cdot b$, $a \wedge b = [a \cdot (b \cdot p)] \cdot p$, and the complement of a is $a^p = a \cdot p$.*

Proof. The first assertion follows from Lemma 1. Let us prove that $a \wedge b = [a \cdot (b \cdot p)] \cdot p$. Evidently, $[a \cdot (b \cdot p)] \cdot p \in [p, 1]$. By Lemma 2(2) we have $a \cdot (b \cdot p) = (a \cdot p) \vee (b \cdot p)$. Since $a \cdot p \leqslant (a \cdot p) \vee (b \cdot p)$, by using Proposition 1(7) we get

$$[a \cdot (b \cdot p)] \cdot p = [(a \cdot p) \vee (b \cdot p)] \cdot p \leqslant (a \cdot p) \cdot p = a \vee p = a,$$

thus $(a\cdot(b\cdot p))\cdot p \leqslant a$. Analogously we can show $(a\cdot(b\cdot p))\cdot p \leqslant b$ and hence $(a\cdot(b\cdot p))\cdot p$ is a lower bound of both a and b. Suppose $q \in [p,1]$, $q \leqslant a$, $q \leqslant b$. Then applying Proposition 1(6) again we have $a\cdot p \leqslant q\cdot p$, $b\cdot p \leqslant q\cdot p$, hence $(a\cdot p)\vee(b\cdot p) \leqslant q\cdot p$. Further, this gives

$$q \leqslant q\vee p = (q\cdot p)\cdot p \leqslant [(a\cdot p)\vee(b\cdot p)]\cdot p = [a\cdot(b\cdot p)]\cdot p,$$

thus $[a\cdot(b\cdot p)]\cdot p$ is the least upper bound of a and b in $[p,1]$. Let us prove that $a^p = a\cdot p$ is a complement of $a \in [p,1]$ in this interval. By Lemma 2(1) we have also

$$a\vee(a\cdot p) = [(a\cdot p)\cdot a]\cdot a = a\cdot a = 1.$$

Since $p \leqslant a\cdot p$, we have

$$a\wedge(a\cdot p) = [a\cdot((a\cdot p)\cdot p)]\cdot p = (a\cdot a)\cdot p = 1\cdot p = p.$$

Moreover,

$$a^{pp} = (a\cdot p)\cdot p = a\vee p = a.$$

If we prove that a^p is simultaneously a pseudocomplement of a in $[p,1]$, then by the previous property every element of this interval is Boolean and so $[p,1]$ is a Boolean algebra. Suppose that $b \in [p,1]$ is such that $a\wedge b = p$, hence $[a\cdot(b\cdot p)]\cdot p = p$. Then

$$a^p = a\cdot p = a\cdot[(a\cdot(b\cdot p))\cdot p] = [a\cdot(b\cdot p)]\cdot(a\cdot p) = a\cdot[(b\cdot p)\cdot p] = a\cdot(b\vee p) = a\cdot b,$$

henceforth $b\cdot a^p = b\cdot(a\cdot b) = 1$, or $b \leqslant a^p$. □

Comparing this Theorem with Proposition 2 we immediately get

Corollary. *Every commutative Hilbert algebra is an implication algebra.*

References

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