# EXISTENCE RESULTS FOR A CLASS OF SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS

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*Abstract.* We prove existence results for the Dirichlet problem associated with an elliptic semilinear second-order equation of divergence form. Degeneracy in the ellipticity condition is allowed.

*Keywords*: weak subsolution, degenerate equation, critical point, fixed-point theorems

*MSC 2000*: 35A05, 35J70, 47H10

## 1. Introduction

We consider the semilinear boundary value problem

$$\begin{cases} -\sum\limits_{i,j=1}^{m} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial u}{\partial x_j}\big) = f(u) \text{ in } \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.0}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^m$, $f$ is a real valued function defined on $\mathbb{R}$, and the coefficients $a_{i,j}(x)$ satisfy the ellipticity condition

$$\sum_{i,j=1}^{m} a_{ij}(x)p_i p_j \geqslant \alpha \sum_{i=1}^{m} \nu_i(x) p_i^2 \quad \text{for a.e. } x \in \Omega \text{ and for any } p \in \mathbb{R}^m$$

with $\nu_i(x)$ satisfying sufficiently general hypotheses.

We obtain some results of existence, uniqueness and boundedness for weak solutions of problem (1.0) with minimal hypotheses on $f$. Similar results, when $f$ has a natural polynomial growth, have been obtained in [3], [5], [7] and in [8] by pseudomonotone operators' theory, while our proof uses fixed-point theorems. The paper is structured as follows. In Sections 2 and 3 we state hypotheses and results. In Section 4 we establish some useful lemmas and, finally, in Section 5 we prove our main theorems.

## 2. Functional spaces

Let $\mathbb{R}^m$ be the Euclidean $m$-space with a generic point $x = (x_1, x_2, \ldots, x_m)$, $\Omega$ a bounded open subset of $\mathbb{R}^m$. The notation $\operatorname{meas}_x$ will indicate the $m$-dimensional Lebesgue measure.

If $u(x)$ is a measurable function defined in $\Omega$, we will denote by $|u|_p$ $(1 \leqslant p \leqslant \infty)$ the usual norm in the space $L^p(\Omega)$.

Hypothesis 2.1. Let $\nu_i(x)$ $(i = 1, 2, \ldots, m)$ be a positive and measurable function defined in $\Omega$ such that

$$\nu_i(x) \in L^1(\Omega), \quad \nu_i^{-1}(x) \in L^{g_i}(\Omega)$$

where $\sum\limits_i^m \frac{1}{g_i} < 2$ $(g_i > 1)$ if $m \geqslant 3$ $\quad (m = 2)$.

The symbol $H^1(\nu_i, \Omega)$ stands for the completion of $C^1(\overline{\Omega})$ with respect to the norm

$$\|u\|_1 = \left( \int_\Omega \left( |u|^2 + \sum_{i=1}^m \nu_i(x) \left| \frac{\partial u}{\partial x_i} \right|^2 \right) \mathrm{d}x \right)^{\frac{1}{2}};$$

$H_0^1(\nu_i, \Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\nu_i, \Omega)$.

Finally, $H^{-1}(\nu_i^{-1}, \Omega)$ denotes the dual space of $H_0^1(\nu_i, \Omega)$ (see also [5], [6] and [10] for details concerning the weighted Sobolev spaces).

## 3. Hypotheses, problems and results

Hypothesis 3.1. The coefficients $a_{ij}(x)$ $(i, j = 1, 2, \ldots, m)$ are functions defined and measurable in $\Omega$ satisfying

$$a_{ij}(x) = a_{ji}(x),$$
$$\frac{a_{ij}(x)}{\sqrt{\nu_i(x)\nu_j(x)}} \in L^\infty(\Omega) \quad (i, j = 1, 2, \ldots, m).$$

Hypothesis 3.2. There exists $\alpha > 0$ such that for almost every $x$ in $\Omega$ we have

$$\sum_{i,j=1}^m a_{ij}(x) p_i p_j \geqslant \alpha \sum_{i=1}^m \nu_i(x) p_i^2 \quad \text{for any } p \in \mathbb{R}^m. \tag{3.1}$$

Let $a\colon H_0^1(\nu_i,\Omega)\times H_0^1(\nu_i,\Omega)\to\mathbb{R}$ be such that

$$a(u,v)=\int_\Omega\sum_{ij=1}^m a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,\mathrm{d}x,$$

and define

$$\tau=\inf_{u\in H_0^1(\nu_i,\Omega)\setminus\{0\}}\frac{a(u,u)}{|u|_2}.$$

In Section 4 we prove the following

**Lemma 4.4.** *Let us assume that* (2.1), (3.1), (3.2) *hold. Then* $\tau>0$ *and there exists* $u_0\in H_0^1(\nu_i,\Omega)$ *such that* $\tau=a(u_0,u_0)$ *and*

$$a(u,u_0)=\tau\int_\Omega uu_0\,\mathrm{d}x\quad\textit{for any } u\in H_0^1(\nu_i,\Omega);$$

*moreover, we can choose* $u_0\geqslant 0$.

**Definition 3.2.** Let $H$ be a Hilbert space, $f$, $g\in C^1(H,\mathbb{R})$, and let

$$E=\{u\in H\colon\ g(u)=0,\quad g'(u)\neq 0\}.$$

A point $u_0\in H$ is a *critical point* of $f|_E$ if $\frac{\mathrm{d}}{\mathrm{d}t}f(h(t))|_{t=0}=0$ for all $C^1$ paths $h(t)\colon\ ]-\varepsilon,\varepsilon[\to E$ such that $h(0)=u_0$.

Remark 3.3. If there exists $u_0\in E$ such that $f(u_0)=\min\{f(u)\colon\ u\in E\}$, then $(f|_E)'(u_0)=0$.

**Theorem 3.4** (see, e.g., [2]). *A point* $u_0\in E$ *is a critical point of* $f|_E$ *if and only if there exists* $\lambda\in\mathbb{R}$ *such that* $f'(u_0)=\lambda g'(u_0)$.

Now, if $f\in C(\mathbb{R})$ satisfies the condition

$$u\in H_0^1(\nu_i,\Omega)\Rightarrow f(u)\in H^{-1}(\nu_i,\Omega),$$

we obtain the following well posed problem

Problem. Find a function $u(x)\in H_0^1(\nu_i,\Omega)$ such that

$$\int_\Omega\sum_{ij=1}^m a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,\mathrm{d}x=(f(u),v)\,(1)\tag{3.1}$$

for any $v(x)\in H_0^1(\nu_i,\Omega)$.

(1) We denote by $(\cdot,\cdot)$ the duality pairing between $H_0^1(\nu_i,\Omega)$ and $H^{-1}(\nu_i,\Omega)$.

A function $u(x)$ satisfying (3.1) is a weak solution of Problem (1.0).

Remark 3.5. When $f$ does not depend on $u$, $f \in H^{-1}(\nu_i, \Omega)$, the hypotheses (2.1), (3.1), (3.2) are sufficient to ensure existence and uniqueness of a weak solution of problem (1.0), moreover we have

$$\|u\|_{1,0} \leqslant \|f\|_{H^{-1}(\nu_i,\Omega)}.$$

Proof follows from Lemma 4.1 and the Lax-Milgram theorem (see Remark 4.2 for the definition of $\|u\|_{1,0}$).

In Section 5 we prove

**Theorem 5.1** (Existence, uniqueness and boundedness). *Let us assume that* (2.1), (3.1), (3.2) *hold and let $f$ be Lipschitz continuous with a Lipschitz constant $L < \tau$.*

*Then there exists a unique weak solution $u(x)$ of problem* (1.0)*; moreover, $u(x) \in L^\infty(\Omega)$ and*

$$\|u\|_\infty \leqslant \gamma(L, g, m, \operatorname{meas}_x \Omega). \tag{5.0}$$

**Theorem 5.2.** *Let us assume that* (2.1), (3.1), (3.2) *hold and let $f$ be a bounded continuous function. Then Problem* (1.0) *has a weak solution $u(x)$. Moreover, $u(x) \in L^\infty(\Omega)$ and* (5.0) *holds.*

## 4. Preliminary lemmas

**Lemma 4.1.** *If the hypothesis* (2.1) *is satisfied then there exists a constant $C = C(m, g_i, |\nu_i^{-1}|_{g_i})$ such that*

$$|u|_{2^\star} \leqslant C\Big(\int_\Omega \sum_{i=1}^m \nu_i(x)\Big|\frac{\partial u}{\partial x_i}\Big|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \qquad \text{for all } u \in H_0^1(\nu_i, \Omega), \tag{4.1}$$

*where* $2^\star = 2m\big(m - 2 + \sum\limits_{i=1}^m \frac{1}{g_i}\big)^{-1}$.

Moreover, the imbedding of $H_0^1(\nu_i, \Omega)$ into $L^2(\Omega)$ is compact.

P r o o f. Let us fix $m_i = \frac{2g_i}{g_i+1}$. Then

$$\Big|\frac{\partial u}{\partial x_i}\Big|_{m_i} \leqslant |\nu_i^{-1}|_{g_i}^{\frac{1}{2}} \Big|\nu_i^{\frac{1}{2}} \frac{\partial u}{\partial x_i}\Big|_2. \tag{4.2}$$

Since $\sum\limits_{i=1}^{m} \frac{1}{m_i} = \sum\limits_{i=1}^{m} \frac{g_i+1}{2g_i} = \frac{1}{2}\Big(m + \sum\limits_{i=1}^{m} \frac{1}{g_i}\Big) > 1$, Sobolev's imbedding theorem yields (see, for instance, [12])

$$|u|_q \leqslant C(m, m_i, q) \prod_{i=1}^{m} \Big|\frac{\partial u}{\partial x_i}\Big|_{m_i}^{\frac{1}{m}} \tag{4.3}$$

where $q = m\big(-1 + \sum\limits_{i=1}^{m} \frac{1}{m_i}\big)^{-1}$.

From (4.2) and (4.3) we obtain

$$|u|_{2^\star} \leqslant C \prod_{i=1}^{m} \Big(|\nu_i^{-1}|_{g_i}^{\frac{1}{2m}} \Big|\nu_i^{\frac{1}{2}} \frac{\partial u}{\partial x_i}\Big|_2^{\frac{1}{m}}\Big).$$

Now, let $\{u_n\}$ be a sequence of functions of $H_0^1(\nu_i, \Omega)$ with equibounded norms and let $\{\Pi_k\}$ be a sequence of open intervals in $\Omega$ such that

1. $\Pi_k \subset \Pi_{k+1}$ for any $k \in \mathbb{N}$,
2. $\lim\limits_{k\to+\infty} \Pi_k = \Omega$,
3. for any closed, bounded subset C of $\Omega$ there exists $\bar{k}$: $C \subset \Pi_k$, $k \geqslant \bar{k}$.

Let us denote by $W^{1,1}(\Pi_1)$ the usual Sobolev space on the set $\Pi_1$.

It follows that the norms of $\{u_n\}$ in $W^{1,1}(\Pi_1)$ are equibounded; in fact, applying the Hölder inequality we obtain the following estimate:

$$\begin{aligned}\|u_n\|_{W^{1,1}(\Pi_1)} &= \int_{\Pi_1} |u_n| \, \mathrm{d}x + \int_{\Pi_1} \sum_{i=1}^{m} \Big|\frac{\partial u_n}{\partial x_i}\Big| \, \mathrm{d}x \\ &\leqslant \Big(\int_{\Pi_1} |u_n|^2 \, \mathrm{d}x\Big)(\operatorname{meas}\Pi_1)^{\frac{1}{2}} + \sum_{i=1}^{m} \Big(\int_{\Pi_1} \frac{1}{\nu_i(x)} \, \mathrm{d}x\Big)^{\frac{1}{2}} \|u_n\|_1 \\ &\leqslant \text{const } \|u_n\|_1.\end{aligned}$$

Due to the compact imbedding of $W^{1,1}(\Pi_1)$ into $L^1(\Pi_1)$ (see e.g. [1]) there is a subsequence $\{u_{1,n}\}$ from $\{u_n\}$ that converges a.e. in $\Pi_1$.

The same procedure can be done on each $\Pi_j$ for $j = 2, 3, \ldots$ . Hence we get a system of sequences $\{u_{j,n}\}$, $n, j = 1, 2, \ldots$ (where $\{u_{j,n}\}$ is a subsequence of $\{u_{j-1,n}\}$) such that $\{u_{j,n}\}$ is convergent a.e. in $\Pi_j$ for $j = 1, 2, \ldots$.

By the diagonals method we obtain that $\{u_{n,n}\}$ converges a.e. in $\Omega$ and, by virtue (4.1), in $L^2(\Omega)$.

R e m a r k 4.2. If the hypothesis (2.1) holds, then $\Big(\int_\Omega \sum_{i=1}^m \nu_i(x)|\frac{\partial u}{\partial x_i}|^2\,\mathrm{d}x\Big)^{1/2}$ constitutes an equivalent norm in $H_0^1(\nu_i,\Omega)$. We will denote this norm by $\|u\|_{1,0}$.

**Lemma 4.3.** *Let* $u(x)\in H_0^1(\nu_i,\Omega)$ *and* $k\geqslant 0$, *then the function* $\min(u,k)$ *belongs to* $H_0^1(\nu_i,\Omega)$.

P r o o f. Define $v=\min(u,k)$ for $u\in H_0^1(\nu_i,\Omega)$ and let $\{\varphi_n\}$ be a sequence of functions of $C_0^\infty(\Omega)$ such that

$$\lim_{n\to+\infty}\|\varphi_n-u\|_1=0.$$

Let $\psi_n=\min(\varphi_n,k)$ for any $n\in\mathbb{N}$.

By regularization, we can prove that $\psi_n$ belongs to $H_0^1(\nu_i,\Omega)$; moreover, because the norms of $\{\psi_n\}$ are equibounded in $H_0^1(\nu_i,\Omega)$, there exists a subsequence that weakly converges in $H_0^1(\nu_i,\Omega)$. On the other hand,

$$|v(x)-\psi_n(x)|\leqslant|u(x)-\varphi_n(x)|\qquad\text{a.e. in }\Omega,$$

so $\{\psi_n\}$ converges to $v$ in $L^2(\Omega)$. □

The conclusion now follows easily.

P r o o f o f L e m m a 4.4. We observe that

$$\tau=\inf\Big\{a(u,u)\colon\ u\in H_0^1(\nu_i,\Omega),\quad \int_\Omega u^2\,\mathrm{d}x=1\Big\},\tag{4.4}$$

and we define $f,g\colon H_0^1(\nu_i,\Omega)\to\mathbb{R}$ as

$$f(u)=a(u,u),\qquad g(u)=\int_\Omega u^2\,\mathrm{d}x-1.$$

Let

$$E=\{u\in H_0^1(\nu_i,\Omega)\colon\ g(u)=0\}.$$

Then

$$\tau=\inf_{u\in E}f(u).$$

Let $\{u_n\}$ be a sequence such that $a(u_n,u_n)\to\tau$; from (3.2) and Remark 4.2 we have that $\{u_n\}$ is bounded in $H_0^1(\nu_i,\Omega)$, so there exist $\{u_{n_k}\}$, $u_0\in H_0^1(\nu_i,\Omega)$ such that $u_{n_k}\rightharpoonup u_0$ weakly in $H_0^1(\nu_i,\Omega)$. By the compact imbedding of $H_0^1(\nu_i,\Omega)$ into $L^2(\Omega)$ (Lemma 4.1), $u_{n_k}\to u_0$ strongly in $L^2(\Omega)$, which gives $\int_\Omega u_0^2\,\mathrm{d}x=1$. Therefore $u_0\in E$.

Finally, by virtue of

$$\tau \leqslant a(u_0, u_0) \leqslant \liminf_{k \to +\infty} a(u_{n_k}, u_{n_k}) = \tau$$

we obtain

$$\tau = a(u_0, u_0)$$

and $f$ attains its minimum at $u_0 \in E$. By Remark 3.3 we have

$$(f|_E)'(u_0) = 0.$$

Accordingly, Theorem 3.4 yields

$$(f)'(u_0) = \lambda(g)'(u_0) \text{ for some } \lambda \in \mathbb{R}$$

or

$$a(u, u_0) = \lambda \int_\Omega u u_0 \, \mathrm{d}x \quad \text{ for any } u \in H_0^1(\nu_i, \Omega).$$

Choosing $u = u_0$ we have

$$\tau = a(u_0, u_0) = \lambda \int_\Omega u_0^2 \, \mathrm{d}x \Rightarrow \tau = \lambda.$$

Obviously $u_0 \in H_0^1(\nu_i, \Omega)$ is such that

$$a(u, u_0) = \tau \int_\Omega u u_0 \, \mathrm{d}x \quad \text{ for any } u \in H_0^1(\nu_i, \Omega).$$

Next, Lemma 4.3 implies that if $u$ satisfies (4.4) then $|u|$ also satisfies (4.4), therefore we can choose $u_0$ to be non-negative.

## 5. Proof of main results

Define $G\colon H^{-1}(\nu_i^{-1}, \Omega) \to H_0^1(\nu_i, \Omega)$ as

$$G(g) = w \quad \text{where } w \text{ is a weak solution of } \begin{cases} -\sum\limits_{i,j=1}^{m} \frac{\partial}{\partial x_i}\big(a_{ij}(x) \frac{\partial w}{\partial x_j}\big) = g \text{ in } \Omega \\ w = 0 \quad \text{on } \partial\Omega \end{cases}$$

Remark 3.5 ensures that $G$ is a linear continuous map. For $u \in H_0^1(\nu_i, \Omega)$ define $F(u) = G(f(u))$. Then a fixed point $u$ of $F$ is a solution of problem (1.0).

P r o o f o f T h e o r e m 5.1. We claim that

$$u \in L^2(\Omega) \Rightarrow f(u) \in L^2(\Omega).$$

Indeed,

$$|f(u)| \leqslant |f(u) - f(0)| + |f(0)| \leqslant L|u| + |f(0)|,$$

thus

$$\int_\Omega |f(u)|^2 \, dx \leqslant 2L^2 \int_\Omega |u|^2 \, dx + 2|f(0)|^2 \operatorname{meas}_x \Omega.$$

We proceed to show that $F$ is a contractive mapping. We see at once that

$$|f(u) - f(v)|_2 \leqslant L|u - v|_2 \quad \text{ for any } u, v \in H_0^1(\nu_i, \Omega).$$

By (3.1) and Remark 4.2 we deduce that

$$\alpha \|u\|_{1,0}^2 \leqslant a(u, u) = (f(u), u) \leqslant c|f(u)|_2 \|u\|_{1,0}$$

or

$$\|u\|_{1,0} \leqslant \frac{c}{\alpha} |f(u)|_2.$$

Consequently, $G$ is continuous from $L^2(\Omega) \to L^2(\Omega)$. Therefore

$$\begin{aligned} |F(u) - F(v)|_2 = |G(f(u) - f(v))|_2 &\leqslant \|G\|_\star |f(u) - f(v)|_2 \\ &\leqslant L\|G\|_\star |u - v|_2. \end{aligned} \tag{5.1}$$

Since $\tau|u|_2^2 \leqslant a(u, u) = \int_\Omega f(u) u \, dx \leqslant |f(u)|_2 |u|_2$ or

$$\frac{|G(f(u))|_2}{|f(u)|_2} \leqslant \frac{1}{\tau},$$

it results that

$$\|G\|_\star \leqslant \frac{1}{\tau}.$$

We conclude from (5.1) that

$$|F(u) - F(v)|_2 \leqslant \frac{L}{\tau} |u - v|_2$$

and finally that, since $L < \tau$, $F$ has a fixed point in $H_0^1(\nu_i, \Omega)$.

Now, let us fix $k \geqslant 0$, then from (3.1) for $v = u - \min(u, k)$ we get

$$\alpha \|v\|_{1,0}^2 \leqslant L \int_\Omega |u||v| \, dx + \int_\Omega |f(0)||v|. \tag{5.2}$$

Lemma 4.1 and the definition of $v$ imply

$$\begin{aligned}
\int_{\Omega} |u||v| \,\mathrm{d}x &\leqslant \int_{\Omega(u>k)} v^2 \,\mathrm{d}x + k \int_{\Omega(u>k)} v \,\mathrm{d}x \\
&\leqslant |v|_{2^\star}^2 [\operatorname{meas}_x \Omega(u>k)]^{1-\frac{2}{2^\star}} + k \int_{\Omega(u>k)} v \,\mathrm{d}x \\
&\leqslant c^2 \|v\|_{1,0}^2 [\operatorname{meas}_x \Omega(u>k)]^{1-\frac{2}{2^\star}} + k \int_{\Omega(u>k)} v \,\mathrm{d}x.
\end{aligned}$$

Therefore from (5.2) we obtain

$$\|v\|_{1,0}^2 (\alpha - Lc^2 [\operatorname{meas}_x \Omega(u>k)]^{1-\frac{2}{2^\star}}) \leqslant (Lk + |f(0)|) \int_{\Omega(u>k)} v \,\mathrm{d}x. \tag{5.3}$$

Recalling that

$$\lim_{k \to +\infty} \operatorname{meas}_x \Omega(u>k) = 0$$

we can certainly choose $\tilde{k} \geqslant 0$ such that for any $k \geqslant \tilde{k}$ we have

$$Lc^2 [\operatorname{meas}_x \Omega(u>k)]^{1-\frac{2}{2^\star}} \leqslant \frac{\alpha}{2}.$$

We apply this inequality to (5.3) obtaining

$$\|v\|_{1,0} \leqslant \frac{2c}{\alpha} [\operatorname{meas}_x \Omega(u>k)]^{1-\frac{1}{2^\star}} (|f(0)| + Lk) \text{ for any } k \geqslant \tilde{k}. \tag{5.4}$$

Let $h$, $k$ be real numbers, $h > k \geqslant \tilde{k}$. Then one has

$$|v|_{2^\star} = \left[ \int_{\Omega(u>k)} |u-k|^{2^\star} \,\mathrm{d}x \right]^{\frac{1}{2^\star}} \geqslant (h-k) [\operatorname{meas}_x \Omega(u>h)]^{\frac{1}{2^\star}};$$

furthermore, (5.4) and Lemma 4.1 yield

$$[\operatorname{meas}_x \Omega(u>h)]^{\frac{1}{2^\star}} \leqslant \frac{2c^2}{\alpha(h-k)} (|f(0)| + Lk) [\operatorname{meas}_x \Omega(u>k)]^{1-\frac{1}{2^\star}}. \tag{5.5}$$

Next, if $k > 0$, we get

$$\begin{aligned}
\operatorname{meas}_x \Omega(u>k) \leqslant \frac{1}{k^{2^\star}} \int_{\Omega(u>k)} u^{2^\star} \,\mathrm{d}x,\ & \frac{2c^2}{\alpha k} (|f(0)| + 2Lk) 2^{\frac{2^\star - 1}{2^\star - 2}} [\operatorname{meas}_x \Omega(u>k)]^{1-\frac{2}{2^\star}} \\
&\leqslant \frac{2c^2}{\alpha k^{2^\star - 1}} (|f(0)| + 2Lk) 2^{\frac{2^\star - 1}{2^\star - 2}} \left( \int_{\Omega(u>k)} u^{2^\star} \,\mathrm{d}x \right)^{1-\frac{2}{2^\star}}.
\end{aligned}$$

Now, the first term of the above inequality goes to zero as $k \to +\infty$, so we can fix $k_1$ ($\geqslant \tilde{k}$) such that

$$\frac{2c^2}{\alpha}\left(|f(0)| + 2Lk_1\right)\left[\operatorname{meas}_x \Omega(u > k_1)\right]^{1-\frac{2}{2^\star}} 2^{\frac{2^\star-1}{2^\star-2}} \leqslant k_1. \tag{5.6}$$

Moreover, one has

$$\frac{2c^2}{\alpha(h-k)}\left(|f(0)| + Lk\right) \leqslant \frac{2c^2}{(h-k)}\left(|f(0)| + 2Lk_1\right) \ \text{ if } 0 \leqslant k \leqslant k_1. \tag{5.7}$$

Combining (5.5) and (5.7) we obtain

$$[\operatorname{meas}_x \Omega(u > h)]^{\frac{1}{2^\star}} \leqslant \frac{2c^2}{\alpha(h-k)}\left(|f(0)| + 2Lk_1\right)[\operatorname{meas}_x \Omega(u > k)]^{1-\frac{1}{2^\star}}$$

for any $h, k \in \mathbb{R}$ such that $k_1 \leqslant k < h \leqslant 2k_1$.

Assuming in $[k_1, +\infty[$ that

$$\varphi(k) = \begin{cases} [\operatorname{meas}_x \Omega(u > k)]^{\frac{1}{2^\star}} \text{ if } k_1 \leqslant k \leqslant 2k_1 \\ 0 \ \text{ if } \ k > 2k_1 \end{cases}$$

we get

$$\varphi(h) \leqslant \frac{2c^2}{\alpha(h-k)}\left(|f(0)| + 2Lk_1\right)[\varphi(k)]^{2^\star-1}$$

for any $h, k \in \mathbb{R}$ such that $k_1 \leqslant k < h \leqslant 2k_1$, and from Stampacchia's lemma (see [11], p. 212) we deduce

$$\varphi(k_1 + d) = 0,$$

where $d$ is the first term of (5.6).

We can obtain the same conclusion for $-u$, so the proof of the theorem is complete.

Proof of Theorem 5.2. Set $F$ as in Theorem 5.1. Since the imbedding of $H^1_0(\nu_i, \Omega)$ into $L^2(\Omega)$ is compact, we have that $F$ is also compact from $L^2(\Omega)$ into $L^2(\Omega)$; therefore, by Schaefer's fixed point theorem, it will be sufficient to prove that the set of all solutions of the equation

$$u = \mu F(u) \quad \text{ for } 0 < \mu < 1 \tag{5.8}$$

is unbounded.

Indeed, if $u$ satisfies (5.8), then $u$ is solution of

$$\begin{cases} -\sum\limits_{i,j=1}^{m} \dfrac{\partial}{\partial x_i}\Big(a_{ij}(x)\dfrac{\partial u}{\partial x_j}\Big) = \mu f(u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

therefore

$$\tau |u|_2^2 \leqslant a(u,u) = \mu \int_\Omega f(u) u \,\mathrm{d}x \leqslant M\,(\mathrm{meas}_x\,\Omega)^{\frac{1}{2}}\,|u|_2$$

or

$$|u|_2 \leqslant \frac{M\,(\mathrm{meas}_x\,\Omega)^{\frac{1}{2}}}{\tau}.$$

Now, if we fix in (3.1) $v = u - \min(u,k)$, $k \geqslant 0$ we get

$$\alpha \|u\|_{1,0}^2 \leqslant M \int_\Omega v \,\mathrm{d}x \leqslant M |v|_{2^\star}\,[\mathrm{meas}_x\,\Omega(u > k)]^{\frac{2^\star - 1}{2^\star}}.$$

This inequality, as in the previous theorem, implies

$$\|u\|_\infty < +\infty.$$

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