GRAPH AUTOMORPHISMS OF MULTILATTICES

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Abstract. In the present paper we generalize a result of a theorem of J. Jakubík concerning graph automorphisms of lattices to the case of multilattices of locally finite length.

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1. Introduction

Inspired by a problem proposed G. Birkhoff ([1], Problem 6) J. Jakubík investigated graph automorphisms of modular lattices [4], semimodular lattices [10] and lattices [5].

The present author studied graph isomorphisms of multilattices [7], [8], [11]. We will apply some results [4], [5] and our results [7], [8] for dealing with graph automorphisms of multilattices of locally finite length. We obtain a generalization of a theorem of J. Jakubík [4], [5].

2. Preliminaries

The notion of a multilattice was introduced by Benado [2]. It is defined as follows. Let P be a partially ordered set. For $x, y, \in P$ we denote by $L(x, y)$ and $U(x, y)$ the system of all lower bounds and all upper bounds of the set $\{x, y\}$ in P, respectively. Let $x \wedge y$ be the system of all maximal elements of $L(x, y)$; similarly we denote by $x \vee y$ the system of all minimal elements of $U(x, y)$. If P is directed then both $x \wedge y, x \vee y$ are nonempty. P is said to be a multilattice if whenever $x, y \in P$ and $z \in L(x, y)$ then there is z_1 in $L(x, y)$ such that $z_1 \geqslant z, z_1$ is a maximal element of

$L(x,y)$ (this case we will write down as $z_1 \in (x \wedge y)_z = \{u \in x \wedge y\colon\ u \geqslant z\}$) and if the corresponding dual condition concerning $U(x,y)$ also holds.

In what follows M is a directed multilattice of locally finite length. For $a, b \in M$ with $a \leqslant b$, the interval $[a,b]$ is the set $\{x \in M\colon\ a \leqslant x \leqslant b\}$. If $[a,b] = \{a,b\}$ and $a \neq b$ then $[a,b]$ is said to be a prime interval and we put $a \prec b$.

By a graph $G(M)$ we mean an unoriented graph whose vertices are elements of M: two vertices are joined by an edge (a,b) iff $[a,b]$ is a prime interval. A graph automorphism of M is a one-to-one maping $\varphi\colon M$ onto M such that whenever $x, y \in M$ and $x \prec y$, then either $\varphi(x) \prec \varphi(y)$ or $\varphi(y) \prec \varphi(x)$.

The following assertion (A) was proved in [2].

(A) A multilattice M of locally finite length is modular iff it fulfils the following covering condition (σ') and the condition (σ'') dual to σ'.

(σ') If $a, b, u, v \in M$ are such that $[u,a], [u,b]$ are prime intervals and $v \in a \vee b$, then $[a,v], [b,v]$ are prime intervals.

3. Cells in partially ordered sets

Let M be a multilattice. Assume that $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n, u, v$ are distinct elements of M such that

(i) $u \prec x_1 \prec x_2 \prec \ldots \prec x_m \prec v, \quad u \prec y_1 \prec \ldots \prec y_n \prec v$;
(ii) either $v \in x_1 \vee y_1$ or $u \in x_m \wedge y_n$.

Then the set $\{u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\} = C$ is called a cell in M. The cell C in M is said to be proper if either $m > 1$ or $n > 1$. A cell C in M such that $m = n = 1$ will be called an elementary square. We will say that an elementary square $C = \{u, v, x_1, y_1\}$ in M is broken by a graph automorphism φ if either $\varphi(u) \prec \varphi(x_1)$, $\varphi(u) \prec \varphi(y_1)$, $\varphi(v) \prec \varphi(x_1)$, $\varphi(v) \prec \varphi(y_1)$ or dually.

A cell C is called regular under a graph automorphism φ if either each prime interval $[a,b] \in C$ is preserved by the graph automorphism φ (i.e. $\varphi(a) \prec \varphi(b)$) or each prime interval $[a,b] \in C$ is reversed by the graph automorphism φ (i.e. $\varphi(b) \prec \varphi(a)$).

The present author proved the following results.

3.1. Theorem (Cf. [7].). *Let M, M' be directed modular multilattices of locally finite length. Then the following conditions are equivalent:*

(α_1) *There exists a graph isomorphism φ of M onto M' such that no elementary square of M or M' is broken by φ or φ^{-1}, respectively.*
(α_2) *There are multilattices A, B and direct representations $f\colon M \to A \times B$, $g\colon M' \to A \times B^d$ such that $\varphi = g^{-1} f$ (B^d is the dual to B).*

3.2. Theorem (Cf. [8].). *Let M, M' be directed multilattices of locally finite length and let $\varphi\colon M \to M'$ be a bijection. Then the condition (α_2) is equivalent to the following condition.*

(β_1) *φ is a graph isomorphism of the multilattice M onto M' such that no elementary square of M or M' is broken under φ or φ^{-1}, respectively, and all proper cells of M, M' are regular under φ or φ^{-1}, respectively.*

For a multilattice M we denote by

$A(M)$—the set of all graph automorphisms of M;

$A_s(M)$—the set of all $\varphi \in A(M)$ such that no elementary square of M is broken by φ and by φ^{-1};

$A_c(M)$—the set of all $\varphi \in A_s(M)$ such that each proper cell in M is regular under φ or φ^{-1}.

Futher, let $C, (C_0$ and $C_1)$ be the class of multilattices M such that whenever $\varphi \in A(M)$ (or $\varphi \in A_s(M), \varphi \in A_c(M)$) then φ is a lattice automorphism on M.

The following two lemmas were proved in [3] for a lattice L. The proofs of these lemmas remain valid if the assumption that L is a modular lattice is replaced by the assumption that L is a multilattice of locally finite length.

3.3. Lemma (Cf. [4].). *Let ψ be an isomorphism of the multilattice M onto the direct product $A \times B$. Further suppose that γ is an isomorphism of B onto B^d.*

For each $x \in M$ we put $\varphi(x) = y$ where $\psi(x) = (a, b)$ $y = \psi^{-1}(a, \gamma, (b))$.

Then φ is a graph automorphism of M.

3.4. Lemma (Cf. [4].). *Let the assumption of* 3.3 *be satisfied. Further suppose that B has more than one element. Then φ fails to be a multilattice automorphism on M.*

3.5. Lemma. *Let the assumption of* 3.3 *be valid. Then no elementary square of M is broken by the graph automorphism φ and by φ^{-1}; consequently $\varphi \in A_s(M)$.*

P r o o f. Let $\{a, b, u, v\}$ be an elementary square in M such that $a \prec v, b \prec v, u \prec a, u \prec b$. If $\psi(a) = (a_1,\ a_2)$, $\psi(b) = (b_1, b_2)$, $\psi(u) = (u_1, u_2)$, $\psi(v) = (v_1, v_2)$ then the relation $\psi(a) \prec \psi(v)$ is valid if and only if either

(i) $a_1 \prec v_1$ and $a_2 = v_2$,

or

(ii) $a_1 = v_1$ and $a_2 \prec v_2$.

From this and $a \prec v$ it follows that $\varphi(a) \prec \varphi(v)$ if and only if the case (i) is valid and $\varphi(v) \prec \varphi(a)$ if and only if the case (ii) is valid. Suppose that $\varphi(u) \prec \varphi(a)$,

$\varphi(u) \prec \varphi(b)$, $\varphi(v) \prec \varphi(a)$, $\varphi(v) \prec \varphi(b)$. From the relations $\varphi(u) \prec \varphi(a)$, $\varphi(u) \prec \varphi(b)$ we have $a_2 = u_2 = b_2$. The relations $\varphi(v) \prec \varphi(a)$, $\varphi(v) \prec \varphi(b)$ imply $a_1 = v_1 = b_1$.

Thus $\psi(a) = \psi(b)$, which is a contradiction.

If we consider $\varphi(a) \prec \varphi(u)$, $\varphi(b) \prec \varphi(u)$, $\varphi(a) \prec \varphi(v)$, $\varphi(b) \prec \varphi(v)$ then we obtain $\psi(a) = \psi(b)$ by a similar argument.

In the same way we arrive at a contradiction if we suppose that an elementary square of M is broken by the graph automorphism φ^{-1}. □

3.6. Lemma. *Let the assumptions of* 3.3 *be satisfied. Then each proper cell of M is regular under the graph automorphism φ and under φ^{-1}; consequently $\varphi \in A_c(M)$.*

Proof. Assume that $C = \{u, v, x_1, \ldots, x_m, y_1, \ldots, y_n\}$ is a proper cell in M such that $m > 1$ and $v \in x_1 \vee y_1$ (if $u \in (x_m \wedge y_n)$ we can apply the dual method). If $x \in M$ and $\psi(x) = (a, b)$ then we denote $a = x(A)$, $b = x(B)$.

Since $u \prec x_1$ we have either

(i) $u(A) \prec x_1(A)$ and $u(B) = x_1(B)$,

or

(ii) $u(A) = x_1(A)$ and $u(B) \prec x_1(B)$.

Similar relations hold for u and y_1; let us denote them by (i_1) and (ii_1). Consider the case when (i) is valid.

If (ii_1) holds, then $x_1 = \psi^{-1}(x_1(A), u(B))$, $y_1 = \psi^{-1}(u(A), y_1(B))$ and $(x_1(A), u(B)) \vee (u(A), y_1(B)) = \{(x_1(A), y_1(B))\}$. From this it follows that $\psi(v) = (x_1(A), y_1(B)) \prec (x_1(A), u(B)) = \psi(x_1)$ and thus $v \prec x_1$, which is a contradiction.

Hence (i_1) must hold and we have $\psi(x_1) \vee \psi(y_1) = (x_1(A), u(B)) \vee (y_1(A), u(B))$. From this it follows that $v(B) = u(B)$.

For each x_i and y_j we have $u \leqslant x_i \leqslant v$, $u \leqslant y_j \leqslant v$ whence $x_i(B) = u(B) = y_j(B)$ and therefore we get $\varphi(u) \prec \varphi(x_1) \prec \ldots \prec \varphi(x_m) \prec \varphi(v)$, $\varphi(u) \prec \varphi(y_1) \prec \ldots \prec \varphi(y_n) \prec \varphi(v)$.

Thus C is regular.

The proof for the case (ii) is analogous. □

By the same method as 1.3, 3.1 in [4] (with the only distinction that instead of [3] we now apply 3.2) we have

3.7. Lemma. *If a multilattice M belongs to C_1 then no direct factor of M having more than one element is self-dual.*

3.8. Lemma. *If no direct factor of M having more than one element is self-dual then M belongs to C_1.*

These lemmas yield the following assertion.

3.9. Theorem. *Let M be a directed multilattice of locally finite length. Then the following conditions are equivalent:*

(i) *M belongs to C_1;*

(ii) *no direct factor of M having more than one element is self-dual.*

Analogously as above (by applying 3.1) we obtain

3.10. Theorem. *Let M be a directed modular multilattice of locally finite length. Then the following conditions are equivalent:*

(i′) *M belongs to C_0;*

(ii) *no direct factor of M having more than one element is self-dual.*

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