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HOMOGENEOUSLY EMBEDDING STRATIFIED GRAPHS IN STRATIFIED GRAPHS

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Dedicated to Robert C. Brigham on the occasion of his retirement from the University of Central Florida

Abstract. A 2-stratified graph G is a graph whose vertex set has been partitioned into two subsets, called the strata or color classes of G. Two 2-stratified graphs G and H are isomorphic if there exists a color-preserving isomorphism φ from G to H. A 2-stratified graph G is said to be homogeneously embedded in a 2-stratified graph H if for every vertex x of G and every vertex y of H, where x and y are colored the same, there exists an induced 2-stratified subgraph H' of H containing y and a color-preserving isomorphism φ from G to H' such that $\varphi(x) = y$. A 2-stratified graph F of minimum order in which G can be homogeneously embedded is called a frame of G and the order of F is called the framing number $\mathrm{fr}(G)$ of G. It is shown that every 2-stratified graph can be homogeneously embedded in some 2-stratified graph. For a graph G, a 2-stratified graph F of minimum order in which every 2-stratification of G can be homogeneously embedded is called a fence of G and the order of F is called the fencing number $\mathrm{fe}(G)$ of G. The fencing numbers of some well-known classes of graphs are determined. It is shown that if G is a vertex-transitive graph of order n that is not a complete graph then $\mathrm{fe}(G) = 2n$.

Keywords: stratified graph, homogeneous embedding

MSC 2000: 05C10, 05C15

1. Introduction

A common problem in graph theory concerns embedding one graph in another subject to certain conditions. For example, in 1936 König [8] showed that for every graph G with maximum degree r, there exists an r-regular graph containing G as an induced subgraph. In 1963 Erdös and Kelly [7] determined for each graph G and each integer $r \geqslant \Delta(G)$, the minimum order of an r-regular graph containing G as an induced subgraph.

In 1992 a more restrictive embedding problem was introduced in [1]. A graph G is said to be *homogeneously embedded* in a graph H if for each vertex x of G and each vertex y of H, there exists an embedding of G in H as an induced subgraph with x at y. Equivalently, a graph G is *homogeneously embedded* in a graph H if for each vertex x of G and each vertex y of H there exists an induced subgraph H' of H containing y and an isomorphism φ from G to H' such that $\varphi(x) = y$. A graph F of minimum order in which G can be homogeneously embedded is called a *frame* of (or for) G and the order of F is called the *framing number* $\mathrm{fr}(G)$ of G. In [1] it was shown that every graph contains a frame and therefore a framing number.

For example, $\mathrm{fr}(P_3) = 4$ since P_3 can be homogeneously embedded in C_4 (but not in any graph of order less than 4). Figure 1 shows homogeneous embeddings of P_3 in C_4 for two non-similar vertices of P_3.

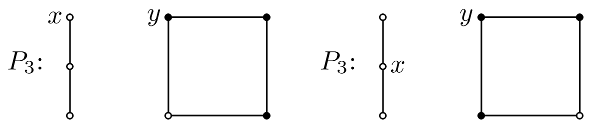

Figure 1. Homogeneously embedding P_3 in C_4

In 1995 the concept of stratified graphs was introduced, inspired by the observation that in VLSI design, computer chips are designed so that its nodes are divided into layers. A graph G whose vertex set has been partitioned is called a *stratified graph*. If $V(G)$ is partitioned into k subsets, then G is a *k-stratified graph*. The k subsets are called the *strata* or *color classes* of G. If $k = 2$, then we customarily color the vertices of one subset red and the vertices of the other subset blue. Two 2-stratified graphs G and H are *isomorphic* if there exists a color-preserving isomorphism φ from G to H. In this case, we write $G \cong H$.

In [4] it was shown that there is a connection among embeddings, stratified graphs, and the area of domination. A vertex v in a graph G *dominates* itself and all of its neighbors. A set S of vertices in a graph G is a *dominating set* of G if every vertex of G is dominated by some vertex in S. The minimum cardinality of a dominating set in G is the *domination number* $\gamma(G)$ of G. Although $\gamma(G)$ is the standard domination number of a graph G, there are many other domination parameters in graph theory, whose definitions depend on how the term *domination* is being interpreted in each case. For example, a vertex v in a graph G *openly dominates* (or *totally dominates*) each of its neighbors, but a vertex does not openly dominate itself. A set S of vertices in a graph G is an *open dominating set* if every vertex of G is openly dominated by some vertex of S. A graph G contains an open dominating set if and only if G contains no isolated vertices. The minimum cardinality of an open dominating set is the *open domination number* $\gamma_o(G)$ of G.

A *red-blue coloring* of a graph G is an assignment of the colors red and blue to the vertices of G, one color to each vertex. If there is at least one red vertex and at least one blue vertex, then a 2-stratified graph results. Let F be a 2-stratified graph, where some blue vertex v of F has been designated as the root. An *F-coloring* of a graph G is a red-blue coloring of G such that every blue vertex v of G belongs to a copy of F rooted at v. The *F-domination number* $\gamma_F(G)$ of G is the minimum number of red vertices in an F-coloring of G. For the 2-stratified rooted graphs F_0, F_1, and F_2 shown in Figure 2, it was shown in [4] that for every graph G of order at least 3 containing no isolated vertices,

$$\gamma_{F_0}(G) = \gamma_{F_1}(G) = \gamma(G) \quad \text{and} \quad \gamma_{F_2}(G) = \gamma_o(G).$$

Other domination parameters can be expressed as $\gamma_F(G)$ for some 2-stratified rooted graph F. Furthermore, for every 2-stratified graph F, there is a domination theory corresponding to F.

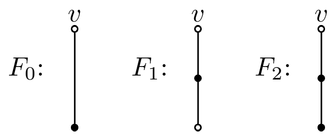

Figure 2. Three 2-stratified rooted graphs

This suggests the idea of homogeneously embedding one 2-stratified graph in another. A 2-stratified graph G is said to be *homogeneously embedded* in a 2-stratified graph H if for every vertex x of G and every vertex y of H, where x and y are colored the same, there exists an induced 2-stratified subgraph H' of H containing y and a color-preserving isomorphism φ from G to H' such that $\varphi(x) = y$. A 2-stratified graph F of minimum order in which G can be homogeneously embedded is called a *frame* of (or for) G and the order of F is called the *framing number* $\mathrm{fr}(G)$ of G.

2. Frames

First we show that every 2-stratified graph has a frame and therefore a framing number.

Theorem 1. *Every 2-stratified graph can be homogeneously embedded in some 2-stratified graph.*

Proof. Let G be a 2-stratified graph of order n, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $v_1, v_2, \ldots, v_r$ are red and $v_{r+1}, v_{r+2}, \ldots, v_{r+b}$ are blue, where $r + b = n$. We may assume that $r \geqslant b$. We construct a 2-stratified graph H in which G can be homogeneously embedded. We begin with $2r-1$ copies $G_1, G_2, \ldots G_{2r-1}$ of G with $V(G_j) = \{v_{1,j}, v_{2,j}, \ldots, v_{n,j}\}$ for $1 \leqslant j \leqslant 2r-1$, as shown below, where $v_{i,j}$ $(1 \leqslant i \leqslant n)$ denotes the vertex v_i of G in the graph G_j.

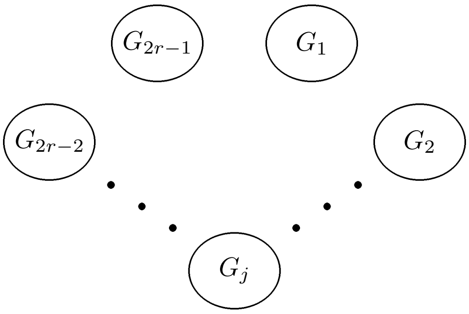

Figure 3. The $2r-1$ copies of G

The vertex set of H is $\bigcup\limits_{j=1}^{2r-1} V(G_j)$ and every edge in G_j $(1 \leqslant j \leqslant 2r-1)$ is an edge of H. Additional edges are added to complete the construction of H. For each vertex $v_{i,j}$ where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant 2r-1$, the vertex $v_{i,j}$ is joined to vertices of H not in G_j as follows:

(1) First, suppose that $v_{i,j}$ is a red vertex, that is, $1 \leqslant i \leqslant r$. For each integer k with $1 \leqslant k < i$, the vertex $v_{i,j}$ is joined to the neighbors of $v_{k,j+k}$ in G_{j+k}. For each integer k with $i < k \leqslant r$, the vertex $v_{i,j}$ is joined to the neighbors of $v_{k,j+k-1}$ in G_{j+k-1}. (The subscripts $j+k$ and $j+k-1$ are expressed modulo $2r-1$.)

(2) Next, suppose that $v_{i,j}$ is a blue vertex, that is, $r+1 \leqslant i \leqslant n$. For each integer k with $r+1 \leqslant k < i$, the vertex $v_{i,j}$ is joined to the neighbors of $v_{k,j+k-r}$ in G_{j+k-r}. For each integer k with $i < k \leqslant n$, the vertex $v_{i,j}$ is joined to the neighbors of $v_{k,j+k-r-1}$ in $G_{j+k-r-1}$. (Again, the subscripts $j+k-r$ and $j+k-r-1$ are expressed modulo $2r-1$.)

We now show that G can be homogeneously embedded in H. It suffices to show that for each vertex v_k of G, where $1 \leqslant k \leqslant n$, and each vertex y of H such that v_k and y are colored the same, the graph G can be embedded as an induced subgraph of H with v_k at y. We may assume that $y = v_{i,j}$, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant 2r-1$.

Thus, if $1 \leqslant i \leqslant r$, define

$$U = \begin{cases} V(G_{j+k}) \cup \{v_{i,j}\} - \{v_{k,j+k}\} & \text{if} \quad 1 \leqslant k < i \\ V(G_j) \quad \text{if} \quad i = k \\ V(G_{j+k-1}) \cup \{v_{i,j}\} - \{v_{k,j+k-1}\} & \text{if} \quad i < k \leqslant r; \end{cases}$$

while if $r+1 \leqslant i \leqslant n$, define

$$U = \begin{cases} V(G_{j+k-r}) \cup \{v_{i,j}\} - \{v_{k,j+k-r}\} & \text{if} \quad r+1 \leqslant k < i \\ V(G_j) \quad \text{if} \quad i = k \\ V(G_{j+k-r-1}) \cup \{v_{i,j}\} - \{v_{k,j+k-r-1}\} & \text{if} \quad i < k \leqslant n. \end{cases}$$

In each case, $\langle U \rangle_H \cong G$, as desired. $\square$

Figure 4 illustrates the construction of the 2-stratified graph H described in Theorem 2.1 for a given graph G. Since G has two red vertices and two blue vertices, the 2-stratified graph H is constructed from three copies G_1, G_2, G_3 of G.

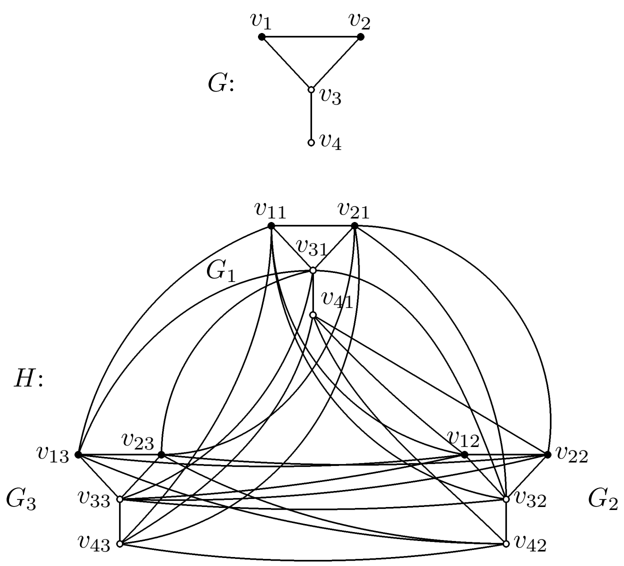

Figure 4. Constructing a 2-stratified graph H in which G can be homogeneously embedded

The construction of the 2-stratified graph H in Theorem 2.1 gives the following upper bound for $\mathrm{fr}(G)$ in terms of the number of red vertices and the number of blue vertices in a 2-stratified graph G.

Corollary 2.2. *Let G be a 2-stratified graph with r red vertices and b blue vertex. Then*

$$\mathrm{fr}(G) \leqslant \max\{2r-1, 2b-1\}|V(G)|.$$

The upper bound in Corollary 2.2 can be improved. In order to show this, we need some additional definitions. Let G be a 2-stratified graph with coloring c. Two vertices u and v with $c(u) = c(v)$ in G are *similar* if there exists a color-preserving automorphism φ of G such that $\varphi(u) = v$. A 2-stratified graph G is *color vertex-transitive* if every two vertices of G having the same color are similar. Similarity is an equivalence relation on the vertex set of G and the resulting equivalence classes are referred to as the *orbits* of G. Clearly, every orbit contains vertices of a single color. Suppose that G is 2-stratified graph with k_r red orbits and k_b blue orbits, where say $k_r \geqslant k_b$. By an argument similar to the one described in Theorem 2.1, we can construct a 2-stratified graph H from the $2k_r - 1$ copies G in which G can be homogeneously embedded. Therefore, we have the following.

Corollary 2.3. *Let G be a 2-stratified graph with k_r red orbits and k_b blue orbits. Then*

$$\mathrm{fr}(G) \leqslant \max\{2k_r-1, 2k_b-1\}|V(G)|.$$

Corollary 2.4. *If G is a graph with two orbits and G' is the 2-stratification of G in which the vertices of one orbit are colored red and the vertices of the other orbit are colored blue, then G' is a frame of itself.*

By Theorem 2.1, for every 2-stratified graph G, there exists a 2-stratified graph in which G can be homogeneously embedded. In fact, more can be said.

Corollary 2.5. *For every 2-stratified graph G, there exists a positive integer N such that for every integer $n \geqslant N$, there exists a 2-stratified graph H of order n in which G can be homogeneously embedded, while for each positive integer $n < N$, no such graph H of order n exists.*

Proof. Suppose that $\mathrm{fr}(G) = N$. Then there exists a 2-stratified graph F of order N in which G can be homogeneously embedded. Let v be a red vertex of F. Define F_1 be the 2-stratified graph of order $N+1$ by adding a new red vertex v_1 to F and joining v_1 to the neighbors of v. Then v and v_1 are color-similar vertices and G can be homogeneously embedded in F_1. Proceeding inductively, we see that for each integer $n \geqslant N$, there is 2-stratified graph H of order n in which G can be homogeneously embedded. On the other hand, by the definition of $\mathrm{fr}(G)$, there exists no 2-stratified graph H of order $n < N$ in which G can be homogeneously embedded. $\square$

Using the construction devised by König to produce a regular graph containing a given graph as an induced subgraph, we are able to show the following.

Theorem 2.6. *Every 2-stratified graph can be homogeneously embedded in some 2-stratified regular graph.*

Proof. Let G be a 2-stratified graph. We show that G can be homogeneously embedded in a 2-stratified regular graph R. By Theorem 2.1, the graph G can be homogeneously embedded in some 2-stratified graph H. If H is regular, then let $H = R$. Thus, we may assume that H is not an regular graph. Suppose that H has order n and $V(H) = \{v_1, v_2, \ldots, v_n\}$. Let H' be another copy of H with $V(H') = \{v'_1, v'_2, \ldots, v'_n\}$, where each vertex v'_i in H' corresponds to v_i in H for $1 \leqslant i \leqslant n$. Construct the graph H_1 from H and H' by adding the edges $v_i v'_i$ for all vertices v_i $(1 \leqslant i \leqslant n)$ such that $\deg v_i < \Delta(H)$. Then H is an induced subgraph of H_1 and $\delta(H_1) = \delta(H) + 1$. If H_1 is regular, then we let $R = H_1$. If not, then we continue this procedure until we obtain a regular graph H_k, where $k = \Delta(H) - \delta(H)$. It is routine to verify that G can be homogeneously embedded in H_k. □

We now determine frames and the framing numbers of the 2-stratifications of some familiar graphs, beginning with a simple example.

Proposition 2.7. *Every 2-stratification G of a complete graph K_n is its own frame and so $\mathrm{fr}(G) = n$.*

We now turn to complete bipartite graphs.

Proposition 2.8. *Let G be a 2-stratification of $K_{s,t}$ with partite sets V_1 and V_2, where $|V_1| = s$ and $|V_2| = t$. For $i = 1, 2$, let r_i be the number of red vertices in V_i and b_i the number of blue vertices in V_i and let*

$$r = \max\{r_1, r_2\} \quad \text{and} \quad b = \max\{b_1, b_2\}.$$

Then $\mathrm{fr}(G) = s + t$ if the vertices of each set V_i, $i = 1, 2$, are colored the same and $\mathrm{fr}(G) = 2(r + b)$ otherwise.

Proof. If the vertices of V_1 are colored the same and the vertices of V_2 are colored the same, then G is the frame of itself by Corollary 2.4 and so $\mathrm{fr}(G) = s + t$. Thus, we may assume that there are vertices in either V_1 or V_2 that are colored differently. Furthermore, we may assume, without loss of generality, that either V_1 or V_2 has all its vertices colored the same and this color is red.

Let F be a frame of G. Since G can be homogeneously embedded in F, every red vertex of F is (1) adjacent to at least r red vertices in F and not adjacent to at least $r-1$ red vertices in F and (2) adjacent to at least b blue vertices in F and not adjacent to at least b blue vertices in F. Hence F contains at least $2r$ red vertices and at least $2b$ blue vertices and so $\mathrm{fr}(G) \geqslant 2(r+b)$. On the other hand, let F' be the 2-stratification of the complete bipartite graph $K_{r+b,r+b}$ in which each partite sets of F' contains r red vertices and b blue vertices. Since G can be homogeneously embedded in F', it follows that $\mathrm{fr}(G) \leqslant 2(r+b)$. Therefore, $\mathrm{fr}(G) = 2(r+b)$. □

This gives us the framing numbers of all stars.

Corollary 2.9. *For each integer $n \geqslant 2$, the framing number of a 2-stratification of $K_{1,n-1}$ is either n or $2(n-1)$.*

We now determine frames and the framing numbers of all connected 2-stratified graphs of order 4 or less. Since every connected graph of order 3 or less is either complete or a star, we know the framing numbers of the 2-stratifications of all such graphs. The following result will be useful in determining the framing numbers of 2-stratifications of connected graphs of order 4.

Theorem 2.10. *If F is a frame of a stratified graph G, then $\overline{F}$ is a frame of $\overline{G}$.*

Proof. Suppose that the order of F is n. Thus for every vertex x of G and every vertex y of F, where x and y are colored the same, there exists an induced stratified subgraph H of F containing y and a color-preserving isomorphism φ from G to H such that $\varphi(x) = y$. Therefore, there exists a set $U \subseteq V(F)$ for which $H = \langle U \rangle_F$. Then $U \subseteq V(\overline{F})$ and $\langle U \rangle_{\overline{F}} = \overline{H}$. Thus for each vertex x of $\overline{G}$ and each vertex y of $\overline{F}$, $\overline{H}$ is an induced stratified subgraph of $\overline{F}$ containing y and φ is a color-preserving isomorphism from $\overline{G}$ to $\overline{H}$ such that $\varphi(x) = y$. Therefore, $\overline{G}$ can be homogeneously embedded in $\overline{F}$, implying that $\mathrm{fr}(\overline{G}) \leqslant \mathrm{fr}(G)$. Then we have $\mathrm{fr}(G) = \mathrm{fr}(\overline{\overline{G}}) \leqslant \mathrm{fr}(\overline{G})$. Therefore, $\mathrm{fr}(\overline{G}) = \mathrm{fr}(G) = n$. Since the order of $\overline{F}$ is $n = \mathrm{fr}(\overline{G})$, it follows that $\overline{F}$ is a frame of $\overline{G}$. □

First, we consider the paths P_4 of order 4.

Proposition 2.11. *If G is a 2-stratification of P_4, then $\mathrm{fr}(G) = 4$ or $\mathrm{fr}(G) = 6$*

Proof. The graph P_4 is self-complementary and has the five 2-stratifications (up to color interchange) shown in Figure 5. Observe that $G_3 \cong \overline{G}_2$ and $G_5 \cong \overline{G}_4$. By Corollary 2.4, the 2-stratification G_1 is a frame of itself and so $\mathrm{fr}(G_1) = 4$. Moreover, by Theorem 2.10, $\mathrm{fr}(G_3) = \mathrm{fr}(G_2)$ and $\mathrm{fr}(G_5) = \mathrm{fr}(G_4)$. Thus, it remains to consider $\mathrm{fr}(G_2)$ and $\mathrm{fr}(G_4)$. Let H be a frame of G_2. Then every red vertex of H is adjacent to two independent blue vertices and is not adjacent to a blue vertex. This implies that H contains at least three blue vertices. Similarly, H contains at least three red vertices. Therefore, the order of H is at least 6. Since G_2 can be homogeneously embedded in the 2-stratified graph H_2 of order 6, it follows that H_2 is a frame of G_2 and $\mathrm{fr}(G_2) = 6$. By Theorem 2.10, $\overline{H}_2$ is a frame of G_3 and $\mathrm{fr}(G_3) = 6$.

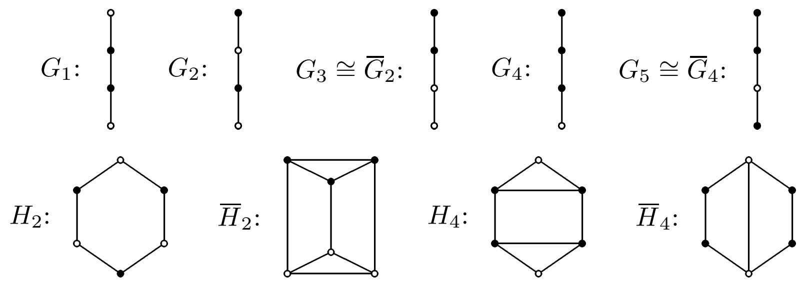

Figure 5. 2-stratifications of P_4 and their frames

Next we consider G_4. Let H be a frame of G_4. Then every red vertex of H is adjacent to two independent red vertices and is not adjacent to a red vertex. This implies that H contains at least four red vertices. Furthermore, every red vertex of H is adjacent to a blue vertex and not adjacent to a blue vertex, implying that H has at least two blue vertices. Hence the order of H is at least 6. Since G_4 can be homogeneously embedded in the 2-stratified graph H_4, it follows that H_4 is a frame of G_4 and $\mathrm{fr}(G_4) = 6$. By Theorem 2.10, $\overline{H}_4$ is a frame of G_5 and $\mathrm{fr}(G_5) = 6$. □

For the graphs $K_4 - e$ and $K_1 + (K_2 \cup K_1)$ of order 4, we only state the framing numbers and give a frame in Figures 6 and 7. For these next two results, H_i is a frame of G_i in each case.

Proposition 2.12. *If G is a 2-stratification of $K_4 - e$, then* $\mathrm{fr}(G) \in \{4, 5, 6\}$.

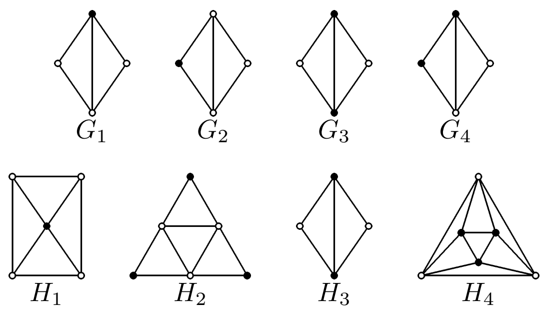

Figure 6. 2-stratifications of $K_4 - e$ and their frames

Proposition 2.13. *If G is a 2-stratification of $K_1 + (K_2 \cup K_1)$, then* $\mathrm{fr}(G) = 5$ *or* $\mathrm{fr}(G) = 6$.

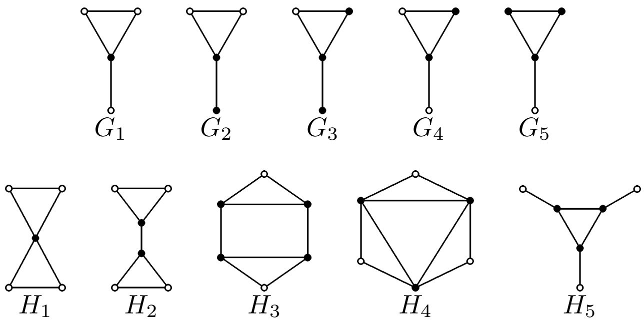

Figure 7. 2-stratifications of $K_1 + (K_2 \cup K_1)$ and their frames

Since we now know the framing number of every 2-stratification of every connected graph of order 4 or less and since the complement of every disconnected graph is connected, it follows by Theorem 2.10 that we know the framing number of every 2-stratification of every graph of order 4 or less.

3. Fences

For a graph G, a 2-stratified graph F of minimum order in which every 2-stratification of G can be homogeneously embedded is called a *fence* of G and the order of F is called the *fencing number* $\mathrm{fe}(G)$ of G. The following observation is useful.

Observation 3.1. *Let G_1 and G_2 be two 2-stratified connected graphs. If the disconnected graph $G_1 \cup G_2$ can be homogeneously embedded in a 2-stratified graph H, so can G_1 and G_2 individually. More generally, if a 2-stratified graph G can be homogeneously embedded in a 2-stratified graph H, then every induced subgraph of G can be homogeneously embedded in H.*

It is a consequence of Theorem 2.1 and Observation 3.1 that every graph has a fence and therefore a fencing number. For example, every 2-stratification of P_3 can be homogeneously embedded in the 2-stratification of Q_3 shown in Figure 8. Thus, $\mathrm{fe}(P_3) \leqslant 8$.

To show that $\mathrm{fe}(P_3) \geqslant 8$, let F be a fence of P_3. We show that F contains at least 4 blue vertices. Since G_3 and G_4 are homogeneously embedded in F, it follows that every blue vertex in F must be adjacent to a blue vertex and not adjacent to a blue vertex. Let u be a blue vertex of F. Suppose that u is adjacent to the blue vertex

v and is not adjacent to the blue vertex w. If v and w are adjacent, then there is a blue vertex x that is not adjacent to v; while if v and w are not adjacent, then there exists a blue vertex x that is adjacent to w. In each case, x is distinct from u, v, and w. Therefore, F contains at least four blue vertices. Similarly, F contains at least four red vertices. Therefore, $\mathrm{fe}(P_3) \geqslant 8$ and so $\mathrm{fe}(P_3) = 8$. Hence the 2-stratification of Q_3 in Figure 8 is a fence of P_3.

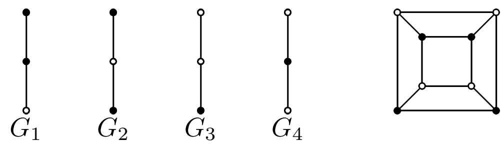

Figure 8. The four 2-stratifications of P_3

First, we determine the fencing numbers of all complete graphs and complete bipartite graphs.

Proposition 3.2. *For each integer $n \geqslant 2$, the fencing number of K_n is $2n-2$.*

P r o o f. First, we show that $\mathrm{fe}(K_n) \leqslant 2n-2$. Let G_0 be the 2-stratification of K_{2n-2} that contains $n-1$ red vertices and $n-1$ blue vertices. Since every 2-stratification of K_n can be homogeneously embedded in G_0, it follows that $\mathrm{fe}(K_n) \leqslant 2n-2$.

Next, we show that $\mathrm{fe}(K_n) \geqslant 2n-2$. Let F be a fence of K_n. We show that F contains at least $n-1$ blue vertices. Let H be the 2-stratification of K_n with exactly one red vertex. Since every blue vertex of H is adjacent to $n-2$ blue vertices in H, it follows that F contains at least $n-1$ blue vertices. Similarly, F contains at least $n-1$ red vertices. Therefore, the order of F is at least $2n-2$ and so $\mathrm{fe}(K_n) \geqslant 2n-2$. □

Proposition 3.3. *For each pair r, t of integers with $1 \leqslant s \leqslant t$, the fencing number of $K_{s,t}$ is $4t$.*

P r o o f. First, let G_0 be the 2-stratification of the complete bipartite graph $K_{2t,2t}$ for which each partite set of G_0 has exactly t red vertices and t blue vertices. Since every 2-stratification of $K_{s,t}$ can be homogeneously embedded in G_0, it follows that $\mathrm{fe}(K_{s,t}) \leqslant 4t$.

Next, we show that $\mathrm{fe}(K_{s,t}) \geqslant 4t$. Let F be a fence of $K_{s,t}$. We show that F contains at least $2t$ blue vertices. Suppose that U and V are the partite sets of $K_{s,t}$ with $|U| = s$ and $|V| = t$. Let H_1 and H_2 be the 2-stratifications of $K_{s,t}$ containing exactly one red vertex, where the red vertex of H_1 is in V and the red vertex of H_2 is in U. In H_1, every blue vertex in U is adjacent to $t-1$ blue vertices in V; while in H_2, every blue vertex in V is not adjacent to $t-1$ blue vertices in V. Since H_1 and H_2 can be homogeneously embedded in F, every blue vertex in F is adjacent to at least $t-1$ blue vertices and not adjacent to at least $t-1$ blue vertices. Thus, F contains at least $2t-1$ blue vertices.

Suppose that F contains exactly $2t-1$ blue vertices. Since every blue vertex in F is adjacent to at least $t-1$ blue vertices, not adjacent to at least $t-1$ blue vertices, and F contains exactly $2t-1$ blue vertices, every blue vertex of F is adjacent to *exactly* $t-1$ blue vertices. Let B be the set of blue vertices of F and let $\langle B\rangle$ be the subgraph of F induced by B. Then $\langle B\rangle$ is $(t-1)$-regular. Let u be the red vertex in H_2. Then $u \in U$ and u is adjacent to the t blue vertices in the independent set V. Since H_2 can be homogeneously embedded in F, every red vertex in F is adjacent to at least t independent blue vertices. This implies that B contains an independent subset B' with $|B'| = t$. Since (1) $\langle B\rangle$ is $(t-1)$-regular, (2) B' is independent, and (3) $B - B'$ contains exactly $t-1$ vertices, each blue vertex in B' must be adjacent to every vertex in $B - B'$. However then, each vertex in $B - B'$ has degree t, contradicting the fact that $\langle B\rangle$ is $(t-1)$-regular. Therefore, as claimed, F contains at least $2t$ blue vertices. Similarly, F contains at least $2t$ red vertices. Therefore, $\mathrm{fe}(K_{s,t}) \geqslant 4t$. □

In the case when $s = t$, then the fencing number of the regular graph $K_{s,t} = K_{s,s}$ is exactly twice of the order of $K_{s,t}$. We now show that the fencing number of every regular graph G that is not complete is at least twice of the order of G.

Proposition 3.4. *If G is a regular graph of order n that is not a complete graph, then*

$$\mathrm{fe}(G) \geqslant 2n.$$

Proof. Suppose that F is a fence of an r-regular graph G of order n such that G is not complete. We show that F contains at least n blue vertices. Let $v \in V(G)$. Let H_1 be the 2-stratification of G in which every vertex in $N[v]$ is blue and the remaining $n-(r+1) \geqslant 1$ vertices are red, and let H_2 be the 2-stratification of G in which every vertex in $N(v)$ is red and the remaining vertices are blue. Thus, in H_1 the blue vertex v is adjacent to r blue vertices; while in H_2, the blue vertex v is not adjacent to $n-(r+1)$ blue vertices. Since H_1 and H_2 are homogeneously embedded in F, it follows that each blue vertex in F is adjacent to at least r blue vertices and not adjacent to at least $n-(r+1)$ blue vertices. This implies that F has at least n blue vertices. Similarly, F has at least n red vertices. Therefore, the order of F is at least $2n$. □

For a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, the *reflection graph* $\mathrm{Ref}(G)$ of G is constructed from G by taking another copy G' of G with $V(G') = \{v'_1, v'_2, \ldots, v'_n\}$, where v_i' corresponds to v_i for $1 \leqslant i \leqslant n$, and (1) joining each vertex v_i in G to the neighbors of v_i' in G' and (2) assigning the color red to every vertex in G and the color blue to every vertex in G'.

Theorem 3.5. *If G is a vertex-transitive graph of order n that is not a complete graph, then*

$$\mathrm{fe}(G) = 2n.$$

Proof. Since every vertex-transitive graph is regular, it follows by Proposition 3.4 that $\mathrm{fe}(G) \geqslant 2n$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. We show that every 2-stratification of G can be homogeneously embedded in $\mathrm{Ref}(G)$, which has order $2n$. Let H be a 2-stratification of G and $v \in V(H)$. Assume, without loss of generality, that v is blue. Let y be a blue vertex in $\mathrm{Ref}(G)$. Then $v = v_i$ for some i $(1 \leqslant i \leqslant n)$ and $y = v_j'$ for some j $(1 \leqslant j \leqslant n)$. Since G is vertex-transitive, there exists an automorphism α of G such that $\alpha(v_i) = v_j$. Let F be the 2-stratified subgraph of $\mathrm{Ref}(G)$ with $V(F) = \{v_1^*, v_2^*, \ldots, v_n^*\}$, where

$$v_i^* = \begin{cases} \alpha(v_i) & \text{if} \quad v_i \text{ is red} \\ \alpha(v_i)' & \text{if} \quad v_i \text{ is blue.} \end{cases}$$

Then $\langle V(F) \rangle_{\mathrm{Ref}(G)} \cong G$. □

Corollary 3.6. *For each integer $n \geqslant 4$, the fencing number of C_n is $2n$.*

The following observation is useful.

Observation 3.7. *If H is an induced subgraph of a graph G, then*

$$\mathrm{fe}(H) \leqslant \mathrm{fe}(G).$$

Proposition 3.8. *For each integer $n \geqslant 4$, the fencing number of P_n is $2(n+1)$.*

Proof. By Observation 3.7 and Corollary 3.6, $\mathrm{fe}(P_n) \leqslant \mathrm{fe}(C_{n+1}) = 2(n+1)$. To show that $\mathrm{fe}(P_n) \geqslant 2(n+1)$, let F be a fence of P_n. We show that F contains at least $n+1$ blue vertices. Let $P_n\colon v_1, v_2, \ldots, v_n$, let H_1 be the 2-stratification in which v_1 is the only red vertex, and let H_2 be the 2-stratification in which v_2 is the only red vertex. In H_1, the blue vertex v_3 is adjacent to blue vertices v_2 and v_4, while in H_2, the blue vertex v_1 is not adjacent to $n-2$ blue vertices $v_3, v_4, \ldots, v_n$. Since H_1 and H_2 are homogeneously embedded in F, each blue vertex in F is adjacent to at least two blue vertices and not adjacent to at least $n-2$ blue vertices. This implies that F has at least $n+1$ blue vertices. Similarly, F has at least $n+1$ red vertices. Therefore, the order of F is at least $2(n+1)$. □

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