EIGHTY YEARS OF JAROSLAV KURZWEIL

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A prominent Czech scientist, Jaroslav Kurzweil, chief research worker of the Mathematical Institute of the Czechoslovak Academy of Sciences (now the Academy of Sciences of the Czech Republic), Professor of Mathematics at Charles University, Prague, reaches eighty years of age on May 7, 2006.[1] Before proceeding to describe the scientific activities of Professor Kurzweil, let us give a brief survey of the main milestones of his life:

1926—born in Prague on May 7

1945—secondary school leaving examination

1949—graduation from Faculty of Science, Charles University, Prague—Assistant Professor at Department of Mathematics and Descriptive Geometry, Czech Technical University, Prague

1950—receives his Doctor of Natural Science (RNDr.) degree

1951—since July 1 a research student (aspirant) at Central Mathematical Institute, later Mathematical Institute of the Czechoslovak Academy of Sciences

1953—research stay in Poland

1954—since January 1 employed in the Mathematical Institute, Czechoslovak Academy of Sciences, Prague

1955—receives the degree of Candidate of Science (CSc.) and is appointed Head of Department of Ordinary Differential Equations of the Mathematical Institute (till 1984)

1957—research stay in the USSR

1958—receives the degree of Doctor of Science (DrSc.)

1964—appointed member of the Scientific Board for Mathematics of the Academy; awarded the State Prize

[1] Twenty and ten years ago J. Kurzweil's jubilee was mentioned among other in the Czechoslovak Mathematical Journal (36 (1986), pp. 147–166, 46 (1996), pp. 375–386), in Časopis pro pěstování matematiky (111 (1986), pp. 91–111 (in Czech)) and in Mathematica Bohemica (121 (1996), pp. 215–222). These articles are extensively used in our present text.

1966—appointed full professor of mathematics

1968—elected corresponding member of the Czechoslovak Academy of Sciences; in the academic year 1968–1969 a visiting professor at Dynamic Centre, Warwick, UK

1978—elected honorary foreign member of the Royal Society of Edinburgh

1981—awarded the Bernard Bolzano silver medal of the Czechoslovak Academy of Sciences "For achievements in mathematical sciences"; elected honourable member of the Union of Czechoslovak Mathematicians and Physicists

1984—appointed Head of Division of Mathematical Analysis in the Mathematical Institute and Head of Department for Didactics of Mathematics

1989—elected regular member of the Czechoslovak Academy of Sciences

1990—elected and appointed Director of the Mathematical Institute, Czechoslovak Academy of Sciences in Prague; he served in this position till 1996

1996—elected foreign member of the Belgian Royal Academy of Sciences; awarded the honorary medal "DE SCIENTIA ET HUMANITATE OPTIME MERITIS" of the Academy of Sciences of the Czech Republic; President of the Union of Czech Mathematicians and Physicists (till 2002); retired but still partially employed in the Mathematical Institute

1997—awarded the State Decoration of the Czech Republic "Medal of Merit (First Grade)" for meritorious service to the state.

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J. Kurzweil started his scientific career as a student of Professor Vojtěch Jarník in the metrical theory of diophantine approximations. The influence of V. Jarník can be seen ever since in Kurzweil's rigorous style and his feeling for fine and ingenious estimates. The very first Kurzweil's paper deals with the properties of Hausdorff measure of the set of real numbers x that admit no $g(q)$ approximation, that is such that there is only a finite number of integers $p, q > 0$ such that $|x - p(q)| < q^{-2}g(q)$, where $g(q)$ is a positive function defined for positive values of q.

The next paper concerning this topic [5] is of great importance. It solves the following Steinhaus problem: if $a < b$ are real numbers, denote

$$I(a,b) = \{(\xi_1, \xi_2) \in \mathbb{R}^2,\ \xi_1 = \cos 2\pi x,\ \xi_2 = \sin 2\pi x,\ x \in [a,b]\},$$

and let B be the set of all nondecreasing sequences (b_k), $k = 1, 2, \ldots$, with positive members satisfying $\sum b_k = +\infty$. Let

$$K = \{(\xi_1, \xi_2) \in \mathbb{R}^2,\ \xi_1^2 + \xi_2^2 = 1\},$$

let μ be the Lebesgue measure on the circumference K and $\alpha(B)$ the set of real numbers $x \in [0,1]$ with the property that for every sequence $(b_k) \in B$, μ-almost all points $y \in K$ belong to infinitely many sets of the form $I \cap (kx - b_k, kx + b_k)$, $k = 1, 2, \ldots$. The set $\alpha(B)$ contains no rational numbers and H. Steinhaus put forward the question whether $\alpha(B)$ contains all irrational numbers from the interval $(0,1)$. Kurzweil characterized the set $\alpha(B)$ by means of the notion of approximability and his considerations implied among other that $\alpha(B) \neq \emptyset$ and that its Lebesgue measure is zero. In this way he answered Steinhaus' question in negative. The paper [5] includes further results, in particular, the problem is modified and solved in the moredimensional case.

In 1953 Kurzweil spent three months in Poznań (Poland) with Prof. Władysław Orlicz. This contact brought new impulses to his work, concerning uniform approximation of a continuous operation by an analytic one. (Basic information on notions involved are found e.g. in the well known monograph E. Hille, R. S. Phillips: *Functional Analysis and Semigroups*, AMS, Providence 1957.)

The paper [3] was directly inspired by Wł. Orlicz. It contains a generalization of the well known theorem of S. N. Bernstein on characterization of real analyticity of a function. Kurzweil proved an assertion of this type for analytic operations defined in a Banach space X with values in a Banach space Y. In the next paper [4] he formulated the following problem: is it possible to uniformly approximate continuous operations from a Banach space X into a real Banach space Y by means of analytic operations?

The answer is given by the following assertion: Let X be a separable real Banach space satisfying the condition

(A) there exists a real polynomial q^* defined on X such that $q^*(0) = 0$ and

$$\inf_{x \in X, \|x\|=1} q^*(x) > 0.$$

Let F be a continuous operation defined on an open set $G \subset X$ with values in an arbitrary Banach space Y. Let φ be a positive continuous functional on G. Then there exists an operation H with values in Y which is analytic in G and satisfies

$$\|F(x) - H(x)\| < \varphi(x).$$

Counterexamples of continuous functionals in $C(0,1)$, l^p and L^p (p odd) which are not uniform limits of analytic functions were presented in the same paper.

The assumption (A) may seem rather surprising. Kurzweil resumed the study of this problem in [11], showing that for a uniformly convex Banach space in which every operation F can be uniformly approximated by analytic functions, the assumption (A) is necessarily fulfilled.

The small excursion into nonlinear functional analysis is remarkable as concerns the depth of the results and only recently has brought its fruits in an apparently distant field dealing with the geometry of Banach spaces.

Functional analysis is the topic also of [25], where Kurzweil, using elementary tools, elegantly proved the known theorem on spectral decomposition of Hermitian operator. Unlike W. P. Eberlein (*A note on the spectral theorem*, Bull. AMS 52 (1946), 328–331) he started with the immediate definition of the so called spectral function.

Also the paper [32] is closely connected with the theory of Hermitian operators. It concerns estimates of eigenvalues of the system of integral equations

$$\int_\Omega K(x,t)u(t)\,\mathrm{d}\mu t = \beta u(x),$$
$$\int_\Omega K(x,t)v(t)\,\mathrm{d}\nu t = \gamma u(x)$$

and its "attached" system

$$\int_\Omega K(x,t)y(t)\,\mathrm{d}\mu t = \alpha z(x),$$
$$\int_\Omega K(x,t)z(t)\,\mathrm{d}\nu t = \alpha y(x).$$

The result obtained by Kurzweil in this direction had been known before only in very special cases.

Theory of stability for ordinary differential equations represents an important field which has been strongly influenced by Kurzweil's research. Although the fundament of this theory had been laid as early as in the last decades of the 19th century (H. Poincaré, A. M. Ljapunov), many problems remained open till the 50's of the last century when this branch again started to flourish.

Given a system of differential equations

$$\dot{x} = f(x,t),\ x \in \mathbb{R},\ t \geqslant 0$$

where $f(0,t) = 0$, $t \geqslant 0$, the solution $x(t) \equiv 0$ is called stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that any solution $y(t)$ of the system with $\|y(0)\| < \delta$ satisfies $\|y(t)\| < \varepsilon$ for all $t \geqslant 0$. A. M. Ljapunov found the following sufficient condition for stability:

If there exist functions $V(x,t)$, $U(x)$ such that $V \in C^1$, U is continuous, $U(x) > 0$ for $x \neq 0$, $V(t,0) \equiv 0$, $V(x,t) \geqslant U(x)$ for $x \in \mathbb{R}$, $t \geqslant 0$, and if

$$W(x,t) := \frac{\partial V}{\partial t} + \sum_{i-1}^{n} \frac{\partial V}{\partial x_i} f_i \leqslant 0,$$

then the solution $x \equiv 0$ is stable.

In 1937 K. P. Persidskij showed that the conditions from this theorem are necessary as well. Persidskij also formulated a sufficient condition for uniform stability in terms of a certain Ljapunov function. The problem whether the conditions of Persidskij's theorem are also necessary was attacked by a number of mathematicians. It was answered in affirmative independently by N. N. Krasovskij and J. Kurzweil under the assumption that the components of the right hand side of the differential equation have continuous partial derivatives. Later, T. Yoshizawa proved the conversion of these theorems for continuous right hand sides. However, the Ljapunov functions constructed by Yoshizawa were not necessarily continuous. This incited the paper [10] where Kurzweil proved that stability or uniform stability can always be characterized by existence of a function satisfying the assumptions of Ljapunov's or Persidskij's theorems. He gave in this work additional (necessary and sufficient) conditions guaranteeing the existence of a smooth Ljapunov function. Analogous problems for the so called second Ljapunov theorem are solved in [9]. Conversion of this theorem, which concerns asymptotic stability, was studied by J. L. Massera for periodic right hand sides of the equation. I. G. Malkin noticed that the assumptions of the second Ljapunov theorem yield results stronger than the original formulation admits. The definitive solution of the problem was given by Kurzweil in [9]. First of all, he showed that the assumptions of the second Ljapunov theorem guarantee even strong stability of the trivial solution $x \equiv 0$. Conversely, if $x \equiv 0$ is a strongly stable solution of the system, he constructed smooth functions satisfying the assumptions of the second Ljapunov theorem. In his constructions Kurzweil developed a method of approximation of Lipschitzian functions, which enabled him to prove that the desired functions are of class C^∞ even if the right hand sides of the equations are merely continuous.

In the fifties, in connection with problems in mechanics, Bogoljubov's averaging method for differential equations became very popular. The method was effective in applications but it was not quite clear how to substantiate it and give it its right place in the framework of the theory of ordinary differential equations. I. I. Gichman in 1952 was the first to notice that the basis of this method is the continuous dependence on a parameter. Gichman's ideas were further developed in 1955 by M. A. Krasnoselskij and S. G. Krejn who pointed out that in order to have continuous dependence on a parameter a certain "integral continuity" of the right hand side of the differential equation is sufficient.

Kurzweil's paper [12] in 1957 then brought the following fundamental result:

Let $f_k\colon\ G \times [0,T] \to \mathbb{R}^n$, $k = 0,1,2,\dots$ *be a sequence of functions,* $G \subset \mathbb{R}$ *an open set. Let* $x_k(t)$ *be a solution of the differential equation*

$$\dot{x} = f_k(x,t),\ x(0) = 0$$

and let $x_0(t)$ be uniquely defined on $[0,T]$. If

$$F_k(x,t) = \int_0^t f_k(x,\tau)\,\mathrm{d}\tau \to \int_0^t f_0(x,\tau)\,\mathrm{d}\tau = F_0(x,t)$$

uniformly with $k \to \infty$ and if the functions $f_k(x,t)$, $k = 0,1,2,\dots$ are equicontinuous in x for fixed t, then for sufficiently large k the solutions $x_k(t)$ are defined on $[0,T]$ and $x_k(t) \to x_0(t)$ with $k \to \infty$ uniformly on $[0,T]$.

The results of [12] discovered the very core of the assertion on continuous dependence for differential equations. When in 1975 Z. Artstein (*Continuous dependence on parameters: on the best possible results.* Journal Diff. Eq. 19, 214–225) studied theorems on continuous dependence from the general viewpoint and introduced topological criteria of comparing them he found that there exist best possible theorems and that the quoted result of [12] is one of them.

However, the results of [12] brought to light also some new problems. For example, direct calculation of the solutions $x_k\colon [0,1] \to \mathbb{R}$ of the sequence of linear differential equations

$$\dot{x} = xk^{1-\alpha}\cos kt + k^{1-\beta}\sin kt,\ x(0) = 0,\ k = 1,2,\dots$$

shows that for $0 < \alpha \leqslant 1$, $0 < \beta \leqslant 1$, $\alpha + \beta > 1$ we have $\lim\limits_{k\to\infty} x_k(t) = 0$ uniformly on $[0,1]$, that is, the solutions converge to the solution of the "limit equation"

$$\dot{x} = 0,\ x(0) = 0.$$

Theorems on continuous dependence on a parameter which could theoretically motivate and justify this convergence phenomenon were not available at the time. Even the above mentioned result from [12] gave a substantiation of the convergence effect in this case only for $\alpha = 1$ and $0 < \beta \leqslant 1$.

Moreover, it was apparent that the knowledge of the function $f(x,t)$ on the right hand side of the differential equation

$$\dot{x} = f(x,t) \tag{1}$$

is in this context needed only to provide the possibility of speaking about the solution of the equation (1). Then all the essential facts can be expressed in terms of the "indefinite integral"

$$F(x,t) = \int_{t_0}^t f(x,\tau)\,\mathrm{d}\tau \tag{2}$$

of the right hand side $f(x,t)$ of the equation (1). A question arose how to describe the notion of a solution of the differential equation (1) in terms of the function F

given by (2). J. Kurzweil answered this question in his work [13] where he introduced the concept of the generalized differential equation. Let us briefly sketch the main points of his theory.

Given a function $F(x,t)\colon\ G\times[0,T]\to\mathbb{R}$, a function $x\colon\ [a,b]\to\mathbb{R}$ is a solution of the generalized differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=DF(x,t) \tag{3}$$

if $(x(t),t)\in G\times[0,T]$ for every $t\in[a,b]$, and for all $s_1,s_2\in[a,b]$ the difference $x(s_2)-x(s_1)$ is approximated with an arbitrary accuracy by the sum

$$\sum_{i=1}^{k}[F(x(\tau_i),\alpha_i)-F(x((\tau_i),\alpha_{i-1})], \tag{4}$$

where $s_1=\alpha_0<\alpha_1<\ldots<\alpha_k=s_2$, $\tau_i\in[\alpha_{i-1},\alpha_i]$ is a sufficiently fine tagged partition of the interval $[s_1,s_2]$. In this way we express in a general form the fact that a solution of the classical equation (1) satisfies the equality

$$x(s_2)-x(s_1)=\int_{s_1}^{s_2}f(x(t),t)\,\mathrm{d}t,\ \ s_1,s_2\in[a,b],$$

and the integral on the right hand side is approximated with an arbitrary accuracy by a sum of the form

$$\sum_{i=1}^{k}\int_{\alpha_{i-1}}^{\alpha_i}f(x((\tau_i),t)\,\mathrm{d}t.$$

Sums of the form (4) are the starting point of Kurzweil's concept of the generalized Perron integral developed in [13]. Here he gave the precise interpretation to the notion of arbitrarily accurate approximation of the difference $x(s_2)-x(s_1)$ by means of (4).

Let $[a,b]\subset\mathbb{R}$ be a compact interval. A finite system of real numbers

$$D=\{\alpha_0,\tau_1,\alpha_1,\ldots,\alpha_{k-1},\tau_k,\alpha_k\}$$

will be called a partition of $[a,b]$ if

$$a=\alpha_0<\alpha_1<\ldots<\alpha_k=b\quad\text{and}\quad\tau_i\in[\alpha_{i-1},\alpha_i],\quad i=1,2,\ldots,k. \tag{5}$$

Given a function $\delta\colon\ [a,b]\to(0,+\infty)$ (a so called gauge), we say that a partition D is δ-fine if

$$[\alpha_{i-1},\alpha_i]\subset[\tau_i-\delta(\tau_i),\tau_i+\delta(\tau_i)],\ \ i=1,2,\ldots,k. \tag{6}$$

With a function $U\colon [a,b] \times [a,b] \to \mathbb{R}^n$ and a partition D we associate the sum

$$S(U,D) = \sum_{i=1}^{k} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})].$$

Definition. We say that $I \in \mathbb{R}^n$ is the *generalized Perron integral* of the function U over $[a,b]$ if for every $\varepsilon > 0$ there is a gauge δ such that for every δ-fine partition D of $[a,b]$, the inequality

$$|S(U,D) - I| < \varepsilon$$

holds. The value I is denoted by the (inseparable) symbol $\int_a^b DU(\tau,t)$.

This definition enables us to give a precise meaning to the notion of solution of the generalized differential equation (3):
a function $x\colon [a,b] \to \mathbb{R}^n$ is a *solution of* (3) if $(x(t),t) \in G \times [0,T]$ and

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau),t)$$

holds for all $s_1, s_2 \in [a,b]$.

The generalized differential equations (3) were thoroughly studied in [13], [14], [15], [17], [20], [29], [34], in which Kurzweil obtained important new results concerning continuous dependence on a parameter for differential equations and substantiated convergence phenomena that had lacked theoretical explanation, including for example the convergence effects for a sequence of ordinary differential equations

$$\dot{x} = f(x,t) + g(x)\varphi_k(x), \ \ k = 1, 2, \ldots,$$

with the sequence (φ_k) converging in the usual way to the Dirac function (see [14], [16]). In [14] Kurzweil showed that generalized differential equations admit discontinuous functions as solutions. This was a quite new phenomenon in the theory of differential equations. Of course, its occurrence was a consequence of the class of right hand sides considered.

The methods of generalized differential equations were extended by Kurzweil also to the case of differential equations in a Banach space. Here he obtained new results concerning partial differential equations and some types of boundary value problems (e.g. in [27], [28], [29], [33], [34]). His contributions in this direction inspired many mathematicians working in the theory of partial differential equations.

For the series of papers on generalized differential equations J. Kurzweil was awarded the State Prize in 1964.

Let us return to the paper [13] and in particular to the above mentioned definition of integral. Kurzweil gave there two equivalent definitions, one of them in terms of majorant and minorant functions analogously to the classical Perron's definition, and the other via the integral sums as we have mentioned above.

If the function U is of the form $U(\tau,t)=f(\tau)t$ then the corresponding integral sum is $\sum_{i=1}^{k} f(\tau_i)(\alpha_i-\alpha_{i-1})$, thus coinciding with the classical Riemann integral sum. In [13] Kurzweil proved that in this special case

$$\int_a^b D[f(\tau)t] \text{ exists if and only if the Perron integral } \int_a^b f(t)\,\mathrm{d}t \text{ exists,}$$

that is, he proved that the Perron integral can be defined by means of Riemannian sums with the above mentioned modification of the "fineness" of a partition of the interval. In this period he contributed to the theory of integral also by the paper [18] devoted to integration by parts.

Independently of Kurzweil's results and with quite different motives, the same definition of integral was later (cca 1960) introduced by R. Henstock (see e.g. his monograph *Theory of Integration*, Butterworths, London 1963).

This theory of integral, besides its usefulness for the theory of differential equations, is of considerable interest by itself. It is an illustrative summation definition of a general, nonabsolutely convergent integral, which is also of non-negligible didactical value.[2]

Kurzweil's ideas on integration from 1957 are still alive and fruitful. Kurzweil himself returned to his theory of integral in 1973 by papers dealing with the change

[2] This fact was exploited for example by the Belgian mathematician J. Mawhin in his lecture notes *Introduction a l'Analyse*, Louvain 1979 and in all subsequent editions of this book, e.g. *Analyse. Fondements-techniques-évolution*, De Boeck Université, Paris-Bruxelles, 1997. Other monographs devoted to Kurzweil's integral are R. M. McLeod: *The Generalized Riemann Integral*, Carus Math. Monographs 20, MAA, 1980, J. DePree, Ch. Swartz: *Introduction to Real Analysis*, Wiley, New York, 1987, E. J. McShane: *Unified Integration*, Academic Press, 1983, R. Henstock: *Lectures on the Theory of Integration*, World Scientific, Singapore, 1988, P.-Y. Lee: *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, 1989, R. Henstock: *The General Theory of Integration*, Clarendon Press, Oxford, 1991, W. F. Pfeffer: *The Riemann Approach to Integration: Local Geometric Theory*, Cambridge University Press, 1993, R. A. Gordon: *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, American Mathematical Society, 1994, Š. Schwabik: *Integration in* $\mathbb{R}$ *(Kurzweil's theory)* (in Czech), Karolinum, Prague, 1999, P.-Y. Lee and R. Výborný: *The Integral: An Easy Approach after Kurzweil and Henstock*, Cambridge Univ. Press, Cambridge, 2000, R. G. Bartle: *A Modern Theory of Integration*, Amer. Math. Soc., Providence, 2001, S. Leader S: *The Kurzweil-Henstock Integral and its Differentials*, Marcel Dekker, Inc., New York, 2001, R.G. Bartle and R. R. Sherbert: *Introduction to Real Analysis*, Wiley, New York, 2000, W. F. Pfeffer: *Derivation and Integration*, Cambridge University Press, 2001, Ch. Swartz: *Introduction to Gauge Integrals*, World Scientific, Singapore, 2001, D. S. Kurtz and Ch. Swartz: *Theories of Integration*, World Scientific, Singapore, 2004.

of order of two integrations [57] and an interesting problem of multipliers for the Perron integral [58], and published an appendix [B6] to the monograph on measure and integral of K. Jacobs. In 1980 he published a small monograph [B5] summarizing his results and embodying his concept of integral in the framework of the theory of integral.

The survey paper [74] then represents a brief exposition of Kurzweil's approach to the theory of integral based on his works from 1957. We will come back to this topic at the end of our survey.

The years 1957–1959 were the period when principal contributions to the mathematical theory of optimal control appeared. In particular, in 1959 a group of Soviet mathematicians led by L. S. Pontrjagin published the now well known monograph on this subject. J. Kurzweil reacted very soon to this situation and inspired research in this field in Czechoslovakia. In [23] and [31] Kurzweil studied the linear control problem and for this case obtained results concerning especially the geometric properties of accessible sets. The paper [26] is devoted to the linear autonomous problem with a quadratic functional. He proved the existence theorem for the optimal solution approaching zero when $t \to \infty$, and solved also the so called converse problem.

The problems of the optimal control theory form the background of later Kurzweil's papers concerning differential relations (inclusions).

The averaging method did not cease to attract the attention of Prof. Kurzweil. He focused his interest on the application of this method in the case of more general spaces. In [27] he proved a theorem on averaging for differential equations in a Banach space and applied the result to the case of oscillations of a weakly nonlinear string. In particular, he discussed the weak nonlinearity of van der Pol's type. Problems of this type were much more extensively studied in [34]–[44] and [49], in which Kurzweil dealt also with problems concerning integral manifolds for systems of differential equations in a Banach space. He took much care to establish results applicable to the theory of partial differential equations and functional differential equations.

Let us roughly sketch Kurzweil's assertion on the existence of an integral manifold (cf. [41]) for a system of ordinary differential equations in a Banach space $X = X_1 \times X_2$, where X_1, X_2 are also Banach spaces. Let $f = (f_1, f_2)\colon\ G \times \mathbb{R} \to X_1 \times X_2 = X$, where for instance,

$$G = \{(x_1, x_2) \in X;\ x_1 \in X_1, |x_1| < 2, x_2 \in X_2\}.$$

For $x = (x_1, x_2) \in X$ put $|x| = |x_1| + |x_2|$, where $|x|, |x_1|, |x_2|$ are norms of the elements x, x_1, x_2 in the spaces X, X_1, X_2, respectively. Consider the system

$$\dot{x} = f(x, t), \quad \text{i.e.} \quad \dot{x}_1 = f_1(x_1, x_2, t),\ \dot{x}_2 = f_2(x_1, x_2, t) \tag{7}$$

provided the function $f\colon G\times\mathbb{R}\to X$ is continuous, bounded and has a bounded differential $\partial f/\partial x$ uniformly continuous with respect to x and t.

Let $f_1(0,x_2,t)=0$ for $x_2\in X_2$, $t\in\mathbb{R}$, that is, the function $x_1(t)=0$ is a solution of the first equation in (7) on the whole $\mathbb{R}$ and the set

$$M=\{(0,x_2,t)\ x_2\in X_2, t\in\mathbb{R}\}\subset X\times\mathbb{R}$$

is an integral manifold of the system (7). Further, for $\tilde{x}_1\in X_1$, $|\tilde{x}_1|\leqslant\sigma$, $x_2\in X_2$, $\tilde{t}\in\mathbb{R}$ let there exist such a solution (x_1,x_2) of the system (7) defined on $(\tilde{t},+\infty)$ that $x_1(\tilde{t})=\tilde{x}_1$, $x_2(\tilde{t})=\tilde{x}_2$ and

$$|x_1(t)|\leqslant\kappa\mathrm{e}^{-\nu(t-\tilde{t})}|x_1|$$

for $t\geqslant\tilde{t}$.

If $(x_1,x_2),(y_1,y_2)$ are solutions of the system (7) defined for $t\in\mathbb{R}$ and lying in M, then let there exists μ, $\mu<\nu$, such that

$$|x_2(t_2)-y_2(t_2)|\geqslant\frac{1}{\kappa}\mathrm{e}^{-\mu(t_2-t_1)}|x_2(t_1)-y_2(t_1)| \tag{8}$$

holds for $t_2\geqslant t_1$.

If for $x\in G$, $t\in\mathbb{R}$ and $0\leqslant\lambda\leqslant1$ the integral

$$\Bigl|\int_t^{t+\lambda}(f(x,s)-g(x,s))\,\mathrm{d}s\Bigr|$$

is sufficiently small, then there exists such a mapping $p\colon X_2\times\mathbb{R}\to X_1$ that the set

$$\widetilde{M}=\{(x_1,x_2,t);\ x_1=p(x_2,t),x_2\in X_2,t\in\mathbb{R}\}\subset X\times\mathbb{R}$$

is an integral manifold for the system

$$\dot{x}=g(x,t). \tag{9}$$

In other words: if $\tilde{x}_2\in X_2$, $\tilde{t}\in\mathbb{R}$, $\tilde{x}_1=p(\tilde{x}_2,\tilde{t})$ then there exists such a solution (x_1,x_2) of the system (9) defined for $t\in\mathbb{R}$ that $x_1(\tilde{t})=\tilde{x}_1$, $x_2(\tilde{t})=\tilde{x}_2$ and $x_1(t)=p(x_2(t),t)$ for $t\in\mathbb{R}$.

Moreover, the integral manifold $\widetilde{M}$ of the system (9) maintains some properties of the manifold M of the system (7). For example, the mapping p is bounded and Lipschitzian in the variable x_2. Any solution of (9) starting in a neighbourhood of the manifold $\widetilde{M}$ exponentially tends for $t\to\infty$ to a solution of (9) which lies in $\widetilde{M}$, and every couple of solutions of (9) lying in $\widetilde{M}$ satisfies an estimate of the same type as (8).

In order not to complicate the situation too much we do not give a detailed formulation of the results, in which an important role is played by the interrelations of constants characterizing the systems (7) and (9) and their solutions. Of course, these are essential for the result and carry important information as well.

We have already mentioned Kurzweil's efforts to make his results widely applicable. They led him to general formulations as well as to the use of general methods of elaboration. In connection with his investigation of integral manifolds he used the notion of a flow as the basis of his conception. A flow is a certain family of mappings satisfying conditions of axiomatic character, which are motivated by the essential properties possessed by the whole system of solutions of a differential equation. The axioms cover all features of the differential equation which are crucial for the proof of existence of an integral manifold. Kurzweil chose this approach already in the paper [34], and continued in this way in [35]. The whole set of 122 printed pages of these two essays contains numerous applications of the abstract results with proper illustration by pertinent examples. The abstract approach to the problems of existence of invariant manifolds reached its top in Kurzweil's paper [42] where the results are formulated for flows in a metric space. One section of [42] is devoted to functional differential equations in a Banach space. Kurzweil proved that if a functional differential equation is close enough to an ordinary differential equation satisfying certain boundedness conditions, then all solutions defined on the whole $\mathbb{R}$ (the so called global solutions) generate an exponentially stable integral manifold. However, the boundedness condition excluded linear equations from the class for which the result was valid. Therefore Kurzweil published in two notes [45] and [48] analogous results for equations on manifolds, which already covered the case of linear functional differential equations.

Together with A. Halanay, Kurzweil in [40] studied flows in Banach spaces formed by functions defined on the whole real axis or, as the case may be, on a certain halfline. The theory from [42] was modified so that it provided an abstract basis also for functional differential systems (see e.g. [39]).

The modern theory of dynamic systems has very clearly marked connections with modern differential geometry, whose methods Kurzweil has frequently used in his investigations.

As an illustration, let us present his result from [49]: let M be a submanifold of a manifold N and let $f\colon U \to N$, where f is a $C^{(1)}$ mapping from a neighbourhood U of the manifold M, such that the partial mapping $f|_M\colon M \to M$ is a diffeomorphism on M. Under certain additional assumptions, for every $g\colon U \to N$ where g is a $C^{(1)}$

mapping close to f there exists a submanifold M_g in N such that $g|_{M_g}\colon M_g \to M_g$ is a diffeomorphism on M_g.

This result is useful especially in the theory of differential equations with delayed argument.

The research in invariant manifolds was followed by a series of papers from the years 1970–1975, which dealt with global solutions of functional differential equations and, in particular, differential equations with delayed argument [45], [47], [50], [51], [52], [59].

Let us mention in more detail only the result from [59], where Kurzweil substantially deepened the results of Yu. A. Ryabov. If $x\colon [t-\tau, t] \to \mathbb{R}^n$, $\tau > 0$, then denote by $x_t\colon [-\tau, 0] \to \mathbb{R}^n$ the function defined by the relation $x_t(\sigma) = x(t+\sigma)$ for $\sigma \in [-\tau, 0]$. Consider a functional differential equation

$$\dot{x} = F(t, x_t), \tag{10}$$

where F is continuous in both variables and Lipschitzian in the latter with a constant L independent of t. Ryabov had shown that if the "delay" τ is not too large (precisely, if $L\tau < \mathrm{e}^{-1}$), then for every point (t_0, x_0) there is a unique "special" solution $\tilde{x}(t) = \tilde{x}(t_0, x_0; t)$ of the equation (10) passing through (t_0, x_0), which is defined on $\mathbb{R}$ and exponentially bounded for $t \to -\infty$. Strengthening further the condition on τ he had shown that for every solution x there is a (not necessarily unique) special solution $\bar{x}$ such that $x(t) - \bar{x}(t) \to 0$ for $t \to +\infty$.

In [59] Kurzweil showed that the validity of the original inequality $L\tau < \mathrm{e}^{-1}$ is sufficient even for a substantially stronger assertion:

if x is a solution of (10) *and we put* $x_0 = \lim\limits_{s\to\infty} \bar{x}(s, x(s); t_0)$ *then*

$$\sup\left\{\exp\left(\frac{t}{\tau}\right)|x(t) - \bar{x}(t_0, x_0; t)|;\ t \geqslant t_0\right\} < \infty.$$

Obviously, $\bar{x}(t_0, x_0; t)$ is the only special solution satisfying this inequality.

Another field in which Kurzweil started to engage himself in the 70's of the last century, is the theory of differential relations (inclusions) and the problems of multifunctions connected with it.

A differential relation is a generalization of the differential equation of the form

$$\dot{x} \in F(t, x). \tag{11}$$

The right hand side of this relation is a so called multifunction, that is, a mapping defined on $G \subset \mathbb{R} \times \mathbb{R}^n$ whose values are subsets of the space $\mathbb{R}^n$. As solutions of a differential relation we usually consider locally absolutely continuous functions u

defined on an interval I, which satisfy the relation $\dot{u}(t) \in F(t, u(t))$ for almost all $t \in I$.

The beginnings of the theory of differential relations, which go back to the thirties of the 20th century, are connected with the names of A. Marchaud and S. Zaremba. Their development in the last 30–40 years has been caused by their relations to the optimal control theory, to the study of differential equations with discontinuous right hand sides etc. It was these relations and in particular Filippov's paper (*Differential equations with discontinuous right hand side*, Mat. Sbornik 51 (93) (1960), 99–128 (Russian); English transl. AMS Translat. II, Ser. 92 (1964), 199–231) that incited Kurzweil's still lasting interest in differential relations.

When studying differential relations, Carathéodory-type conditions are often assumed:

(i) $F(t, \cdot)$ is upper semicontinuous for almost every t;

(ii) $F(\cdot, x)$ is measurable for every x;

(iii) F satisfies an "integrable boundedness" condition.

Moreover, the sets $F(t, x)$ are usually assumed to be nonempty, compact and convex subsets of $\mathbb{R}$.

In this connection a question arises whether the validity of (i), (ii) suffices to guarantee "reasonable" behaviour of the multifunction F in both variables. The following condition, evidently implying (i), (ii), seems to be plausible:

(iv) for every $\varepsilon > 0$ there is a set $A_\varepsilon \subset \mathbb{R}$ such that the measure $m(\mathbb{R} \setminus A_\varepsilon) < \varepsilon$ and the restriction $F|_{(A_\varepsilon \times \mathbb{R}^n) \cap G}$ is upper semicontinuous (with respect to the pair of variables (t, x)).

However, the converse implication, that is, (i), (ii) $\Longrightarrow$ (iv), does not hold. In [64] it is proved that in spite of this fact we can restrict the study of differential relations to right hand sides satisfying (iv). Namely, the following theorem holds:

Let $\mathbb{K}^n$ be the system of all nonempty compact convex subsets of $\mathbb{R}^n$. Let $F\colon\ G \to \mathbb{K}$ satisfy (i). Then there is a function $\widetilde{F}\colon\ G \to \mathbb{K}^n \cup \{0\}$ satisfying (iv),

(v) $\widetilde{F}(t, x) \subset F(t, x)$ for all $(t, x) \in G$;

(vi) every solution of (11) is also a solution of the differential relation $\dot{x} \in \widetilde{F}(t, x)$.

Kurzweil gave (iv) the name of the Scorza-Dragoni property, after the Italian mathematician who had studied analogous problems for ordinary differential equations.

The assertion of the above mentioned theorem makes the study of properties of solutions of differential relations easier, as is seen for example in [65]. Here the result analogous to the following well known theorem from the theory of ordinary differential equations was proved:

For a differential equation $\dot{x} = f(t, x)$ there is a set $E \subset \mathbb{R}$ of zero measure such that for every solution $x(t)$ the derivative $\dot{x}(t)$ exists and satisfies the equation for all $t \notin E$.

(For differential relations the term "derivative" must be replaced by that of "contingent derivative".)

In [68] it was proved that the set of solutions of the differential relation (11) is closed with respect to a certain limiting process, which can be roughly described as follows:

Let W be the set of functions $w\colon I_w \to \mathbb{R}^n$, $I_w = \bigcup_{i=1}^{k} [\tau_{i-1}, \tau_i)$, $\tau_0 < \tau_1 < \ldots < \tau_k$, for which there exist such solutions u_i of the differential relation (11) that $w(t) = u_i(t)$ for $t \in [\tau_{i-1}, \tau_i)$. Denote by J_w the jump function of w (that is, $w - J_w$ is continuous, $J_w(t) = 0$ for $t \in [\tau_0, \tau_1)$). Then every function q which is the uniform limit of a sequence of functions $w_j \in W$ satisfying $J_{w_j} \to 0$ uniformly, is a solution of (11).

Conversely, every set of "reasonable" functions closed with respect to the limiting process described is the set of (all) solutions of a certain differential relation. This makes it possible to construct, for a given set of functions, the "minimal" relation for which all the given functions are solutions.

Also Kurzweil's papers [61], [63], [66], [67], [69] and [70] were devoted to differential relations. Let us mention just the paper [70] in which a new summation definition of the integral of a multifunction was given and a theorem on equivalence of the differential and integral relations was proved.

The last paper indicated Kurzweil's comeback to the theory of summation integrals, and he has indeed devoted much effort to this theory during the last two decades. However, the principal impulse was Mawhin's paper (*Generalized multiple Perron integrals and the Green-Goursat Theorem for differentiable vector fields*, Czechoslovak Math. Journal 31 (106), (1981), 614–632) in which the author gave a generalization of the Perron integral in $\mathbb{R}^n$, which guarantees validity of the divergence theorem (Stokes theorem) for all differentiable vector fields without any further assumptions.

Mawhin's definition is based on the above mentioned Riemann-type definition due to Kurzweil, Henstock and McShane, but restricts the class of admissible partitions (of an n-dimensional interval) taking into account only such intervals in which the ratio of the longest and shortest edges is not too big. Nevertheless, Mawhin himself pointed out that it is not clear whether his integral has some natural properties, in particular the following type of additivity: if J, K, $J \cup K$ are intervals and if f is integrable over both J and K, then it is also integrable over $J \cup K$.

In [73] an example was found that Mawhin's integral really lacks this property, and a modified version of Mawhin's definition was proposed: instead of the ratio of the longest and shortest edges, the subintervals J forming a partition of an n-dimensional interval are characterized by the quantity $\sigma(J) = \operatorname{diam} J \cdot m(\partial J)$ (the product of the diameter of the interval and the $(n-1)$-dimensional Lebesgue measure of its boundary).

Define a P-partition (Perron partition) of an interval $I \subset \mathbb{R}^n$ as a finite system Π of pairs (x^j, I^j), $j = 1, 2, \ldots, k$, where I^j are nonoverlapping compact intervals whose union is I, and $x^j \in I^j$. If $\delta\colon I \to (0, +\infty)$ (a gauge) then a given P-partition is called δ-fine if I^j, $j = 1, 2, \ldots, k$ lies in a ball with centre x^j and radius $\delta(x^j)$. For a function $f\colon I \to \mathbb{R}$ put

$$S(f, \Pi) = \sum_{j=1}^{k} f(x^j) m(I^j)$$

(m is the Lebesgue measure) and define: a number $\gamma \in \mathbb{R}$ is the M-integral of the function f if for every $\varepsilon > 0$ and $C > 0$ there is a gauge δ such that $|\gamma - S(f, \Pi)| < \varepsilon$ holds for every δ-fine P-partition Π of I satisfying

$$\sum_{j=1}^{k} \sigma(I^j) \leqslant C. \tag{12}$$

Since the condition (12) is evidently less restrictive then Mawhin's original one, this definition admits a wider class of partitions and hence a narrower class of integrable functions. In [73] the properties of the new notion of integral were studied in detail. It turned out that it preserves those which had led Mawhin to the new definition (in particular, the divergence theorem or the integrability of every derivative). On the other hand, the integral has the additivity property in the above sense and, moreover, the limit theorems on monotone and dominated convergence hold.

A drawback of both Mawhin's and Kurzweil's n-dimensional integral is that we can integrate only over intervals. The intervals are linked with the coordinate system and do not allow even relatively simple transformations.

Therefore, the next step of Kurzweil's research was to find a definition that would remove this drawback. This was successfully achieved in [76]. Here, instead of the above mentioned type of partitions, partitions based on partition of unity were used.

Let us recall the definition. If f is a function with compact support $\operatorname{supp} f$, then any finite system Δ of pairs (x^j, θ_j), $j = 1, 2, \ldots, k$ is called a PU-partition provided θ_j are functions of class C^1 with compact supports, $0 \leqslant \theta_j(x) \leqslant 1$, $\operatorname{Int}\{x \in \mathbb{R}^n;$

$\sum_{j=1}^{k} \theta_j(x) = 1\} \supset \operatorname{supp} f$. We define the integral sum corresponding to f, Δ by

$$S(f,\Delta) = \sum_{j=1}^{k} f(x^j) \int \theta_j(x)\,\mathrm{d}x \tag{13}$$

and introduce the PU-integral of f as the number q such that for every $\varepsilon > 0$ there is a gauge δ such that

$$|q - S(f,\Delta)| < \varepsilon$$

for every δ-fine PU-partition Δ. (Here a gauge is any positive function on $\operatorname{supp} f$ and a PU-partition is δ-fine if $\operatorname{supp}\theta_j \subset B(x^j, \delta(x^j))$ for $j = 1, 2, \ldots, k$.) It is easy to show that the number q is uniquely determined provided it exists.

Note that the intervals I^j in the definition of δ-fineness of a partition are replaced by the sets $\operatorname{supp}\theta_j$ and the definition of the PU-integral (PU for partition of unity) is introduced formally in the same way as that of the M-integral.

In order to obtain a suitable concept of integral, the family of admissible partitions is reduced by imposing a certain analogue of the regularity condition for intervals (the ratio of the shortest and the longest edge has to be separated from zero).

Instead of the regularity condition (12) let us introduce

$$\sum_{j=1}^{k} f(x^j) \int \|x - x^j\| \sum_{i=1}^{n} \Big|\frac{\partial\theta_j}{\partial x_i}(x)\Big|\,\mathrm{d}x \leqslant C \tag{14}$$

(in both cases, i.e. in (13) and (14), we actually integrate only over certain compact sets).

Roughly speaking, this condition ensures that the Stokes Theorem is valid for differentiable vector fields without further restrictions. This makes it possible to use the integral for integration on manifolds. Technically, it is required that the tag x^j be located so that the corresponding function θ_j is not too small in its neighbourhood. The properties of each individual concept of the integral depend on the character of the regularity condition introduced. In particular, the condition used in [76] guarantees the validity of the Stokes Theorem for vector fields with discontinuities or even singularities. The condition introduced in [90] makes it possible to prove that $C^{(1)}$ functions ψ with

$$\|\psi\|_1 = \sup\{|\psi(x)| + \|\mathrm{D}\psi(x)\|;\ x \in G\} < \infty$$

(G open bounded, $\operatorname{supp} f \subset G$) are multipliers, that is, if $\int f$ exists and ψ is a function as above then $\int f\psi$ also exists. Moreover, there exists $C = C(f) > 0$ such that

$$\Big|(\mathrm{PU})\int f\psi\,\mathrm{d}x\Big| \leqslant C\|\psi\|_1.$$

(This makes it possible to use this integral as a starting point for developing a theory of distributions.)

For the PU-integral the usual transformation theorem and also the Stokes theorem for differentiable functions (or forms on manifolds) hold without any additional assumptions. It is easy to see that among PU-integrable functions there are also some nonabsolutely integrable ones so that the PU-integral is a proper extension of the Lebesgue integral. It is not, however, a generalization of the Perron integral (though there exist PU-integrable functions which do not possess the Perron integral).

In the paper [78] it is shown that a suitable modification of the condition (14) leads to an integral for which the Stokes Theorem can be proved for functions for which the differentiability condition (or even the condition of continuity or boundedness) is violated at some points.

The papers [78], [87] and [90] elaborate the basic idea from [76], namely the idea of an integral defined via partitions of unity (hence the name PU-integral). Two papers, [80] and [89] deal with one-dimensional generalized Perron integrals introduced via Riemann-type sums in which the partitions are subjected to a certain symmetry condition. Namely, the tag is required to be "not too far" from the centre of the interval in question. These integrals have some interesting properties (similar to the "valeur principale") and in some cases allow to establish a standard transformation theorem, which is not possible for the classical Perron integral.

The largest number of papers since 1986 is devoted to a thorough study of summation integrals in $\mathbb{R}^n$ over a compact interval. To obtain generalized Riemann integrals, the partitions of the integration interval are required to consist of regular intervals, i.e. intervals whose regularity (ratio of the shortest and the longest edge) is greater than a certain value ϱ (a constant or, more generally, a function of the tag and/or the diameter of the interval of partition). The main problems considered are

(i) convergence theorems,

(ii) properties of the primitive function.

Besides general results they include a number of examples or rather counterexamples which clarify the relations between individual concepts of regular integrals.

The beginnings of this line of research go back to the paper [73] from 1983, which was inspired by J. Mawhin's paper mentioned above, and its results have appeared in [94]–[101].

In the paper [94] Kurzweil goes back to the original Mawhin's definition, showing that the notion of the α-regular integral really depends on the bound for the regularity: if $\alpha < \beta$, then there exists a function f that is β-integrable but not α-integrable. On the other hand, the notion of α-regular differentiability is independent of the value of α, i.e. a function f α-regularly differentiable at a point is γ-regularly differentiable at the point for any $\gamma > 0$ (which of course does not mean that the regularity condition can be omitted).

Paper [95] offers a comparison of various "regular integrals", among other those of W. F. Pfeffer (*A Riemann type integration and the fundamental theorem of calculus.* Rendiconti Circ. Mat. Palermo, Ser. II, *36* (1987), 482–506), and shows that the "dangerous" points are those on the boundary of the integration interval. This led to the introduction of the so called extensive integral:

Let $I \subset \operatorname{Int} L \subset \mathbb{R}^n$, L a compact interval. For $f\colon I \to \mathbb{R}$ define $f_{\mathrm{ex}}\colon L \to \mathbb{R}$ by extending f from I to L by $f_{\mathrm{ex}}(x) = 0$ for $x \in L \setminus I$. The function $f\colon I \to \mathbb{R}$ is called *extensively integrable* if there is $L, I \subset \operatorname{Int} L$ such that f_{ex} is Mawhin integrable (on L).

In [96] the regularity condition is generalized in the sense that instead of measuring the regularity of the intervals of a partitions "uniformly", i.e. by a constant, it is measured by a function which may depend on the position of the tag t of a pair (t, J) and/or on the diameter of J.

The paper [98] summarizes the properties of the regular integrals; in particular, it contains the descriptive definition and convergence theorems. Finally, the paper [100] deals with the problem whether the regular integral can be introduced via the Bochner approach, i.e. by extending the elementary integration of stepfunctions (piecewise constant functions) using a suitable limiting process. This required further modification of the concept of integral leading to the concept of strong integration.

In [101] the strong integral is further modified by using the L-partitions which differ from the partitions used before by omitting the condition $t \in J$ for any pair (t, J). (The partitions with $t \in J$ are more specifically called P-partitions as was mentioned above. The letters L, P stand for Lebesgue, Perron, respectively, since the respective concepts of integral are connected with the classical Lebesgue and Perron integrals.)

In the common paper with J. Mawhin and W. Pfeffer [92] Kurzweil's idea of PU-partitions is combined with Pfeffer's one of the BV integration (BV for bounded variation in the sense of DeGiorgi). This makes it possible to avoid the shortcomings and accentuate the advantages of both the approaches.

The paper [93] is devoted mainly to the study of convergence theorems for generalized Perron integrals. The simplest assumption which (in addition to the pointwise convergence of the sequence of functions f_k to a limit function f) is that of *equiintegrability* of the sequence, which of course means that the gauge in the definition of the integral can be chosen independently of k.

Lee Peng Yee (*Lanzhou Lectures on Henstock Integration*, Series in Real Analysis, Vol. 2, World Scientific, Singapore, 1989) introduced the notion of controlled convergence which involves a certain kind of generalized absolute continuity of the primitives F_k. For the more dimensional case analogous results are obtained by relaxing the notion of absolute continuity required in [93]. In order to obtain results in a more general setting, an axiomatic approach is chosen which allows to treat simultaneously various kinds of integrals.

The above mentioned "equiintegrability" convergence theorem reads as follows:

Assume that $[a,b] \subset \mathbb{R}$ *is a compact interval and* f_k, $k \in \mathbb{N}$ *is a sequence of Kurzweil integrable functions* $f_k\colon [a,b] \to \mathbb{R}$. *Assume*

(A) $f_i(t) \to f(t)$ *for* $i \to \infty$, $t \in I$,

(B) *for every* $\varepsilon > 0$ *there is a gauge* δ *such that the inequality*

$$\left| \sum_{i=1}^{k} f_k(\tau_i)[\alpha_i - \alpha_{i-1}] - \int_a^b f_k \right| < \varepsilon$$

holds for every δ*-fine partition* $D = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$ *of* $[a,b]$ *and* $k \in \mathbb{N}$.

Then $f\colon [a,b] \to \mathbb{R}$ *is integrable and*

$$\lim_{k\to\infty} \int_a^b f_k = \int_a^b f.$$

This convergence theorem for a pointwise convergent sequence (f_k) of functions on an interval based on the concept of equiintegrability of the sequence (condition (B)) is interesting because, among other, it can be easily proved using basic tools and can be presented at a very early stage of undergraduate studies.

The simple proof works equally for the Kurzweil integral based on P-partitions as well as for the McShane integral based on L-partitions. The last integral is known to be equivalent to the Lebesgue integral.

For a long time it was not precisely known what the position of the "equiintegrability" convergence theorem is in the system of convergence results known for the Lebesgue integral. This problem was solved in [107] where it was shown that the above mentioned theorem is equivalent to the general Vitali convergence theorem based on uniform absolute continuity of the primitives. In [108] it is then shown that this holds also for Banach space valued functions and the McShane integral which is equivalent to the Bochner integral.

In [100] and then later in [102], [103] and [104] the problem of existence of reasonable topologies on the space P of primitives of Kurzweil integrable functions (this is a subspace of all continuous functions on the interval $[a,b]$) was studied. These topologies are connected with the "equiintegrability" convergence theorem mentioned above.

The problem was then clearly formulated in [B8] as follows:

Let $F_i \in P$ for $i \in \mathbb{N}$, $F \in P$.

A sequence F_i, $i \in \mathbb{N}$ is called E-convergent to F, shortly $F_i \xrightarrow{E} F$, if there exist $f_i\colon I \to \mathbb{R}$ for $i \in \mathbb{N}$, $f\colon I \to \mathbb{R}$ such that

(C) F_i is the primitive of f_i for $i \in \mathbb{N}$ and (A) and (B) are valid.

By the "equiintegrability" convergence theorem F is the primitive of f and $F_i \to F$.

The question is:

Does there exist a topology τ on P such that

(D) $F_i \xrightarrow{E} F$ implies that $F_i \to F$ in (P, τ),

(E) (P, τ) is complete,

(F) (P, τ) is a topological vector space?

The main result stated in [B8] says that the answer to the problem is affirmative.

If (F) is strengthened to

(G) (P, τ) is a locally convex vector space,

then the answer is negative.

In the slim and delicate book [B8] many other technical results are presented. They are interesting by themselves.

The book [B9] is closely related to [B8]. It is interesting to observe that if in the definition of the Kurzweil integral point-interval pairs (t, J) with $t \in J$ (P-partitions) are used, we obtain the Perron integral, while not imposing the requirement $t \in J$ (L-partitions) we get the Lebesgue integral.

Scales of integrations ($\mathcal{Y}$-integrations) are studied in [B9] from the point of view of the topological problem presented in [B8]. Roughly speaking, these integrations differ by the partitions involved where some additional conditions are imposed on the point-interval pairs (t, J) in the case $t \notin J$.

A scale of integrations (based on integral sums) connecting the Lebesgue integral and the Perron integral was also introduced by B. Bongiorno (*Un nuovo integrale per il problema delle primitive.* Le Matematiche 51 (1996), 299–313) and B. Bongiorno and W. F. Pfeffer (*A concept of absolute continuity and a Riemann type integral.* Comment. Math. Univ. Carolinae 33 (1992), 189–196). On this scale we have the so called C-integral which integrates every derivative, all Lebesgue integrable functions and is less general than the Perron integral and is in some sense minimal (B. Bongiorno, L. Di Piazza, D. Preiss: *A constructive minimal integral which includes Lebesgue integrable functions and derivatives.* J. London Math. Soc. 62 (2000), 117–126).

A very interesting contribution of J. Kurzweil is concerned with the concept of the multiplicative integral. In the papers [79] and [82] the Kurzweil approach to Perron integration is applied for defining the product integral $\prod_a^b V(t, \mathrm{d}t)$ for an $n \times n$-matrix valued function $V\colon [a, b] \times \mathcal{J} \to L(\mathbb{R}^n)$, where $\mathcal{J}$ is the set of all compact subintervals of the interval $[a,b] \subset \mathbb{R}$ and $L(\mathbb{R}^n)$ denotes the set of all $n\times n$-matrices. The Perron product integral is defined as follows:

Given a positive function $\delta\colon [a,b] \to (0,+\infty)$, called a *gauge*, assume that

$$D = \{(t_i, J_i);\ t_i \in J_i = [x_{i-1}, x_i] \subset \mathcal{J}, i = 1, \dots, k\}$$

is a *tagged partition* of $[a,b]$, i.e.

$$x_0 = a < x_1 < x_2 < \ldots < x_k = b,$$

which is δ-fine, i.e.

$$J_i \subset (t_i - \delta(t_i), t_i + \delta(t_i)).$$

For a given function $V\colon [a,b] \times \mathcal{J} \to L(\mathbb{R}^n)$ and a tagged partition $D = \{(t_i, J_i);$ $i = 1, \dots, k\}$ we denote by

$$P(V,D) = V(t_k, J_k)V(t_{k-1}, J_{k-1})\ldots V(t_1, J_1)$$

the ordered product of matrices $V(t_i, J_i)$.

The function V is called Perron product integrable if there is a regular $Q \in L(\mathbb{R}^n)$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\|P(V,D) - Q\| < \varepsilon$$

for every δ-fine partition D of $[a,b]$. $Q \in L(\mathbb{R}^n)$ is the Perron product integral $\prod_a^b V(t,\,\mathrm{d}t)$.

Typical representatives of functions V are $V(\tau,[\alpha,\beta]) = I + A(\tau)(\beta-\alpha)$ or $V(\tau,[\alpha,\beta]) = \exp(A(\tau)(\beta-\alpha))$ where $A\colon [a,b] \to L(\mathbb{R}^n)$ is a given matrix valued function and $[\alpha,\beta] \subset [a,b]$ is an interval.

The relations to the linear system of ordinary differential equations of the form $\dot{x} = A(t)x$ are studied using the *indefinite* Perron product integral $\prod_a^t V(s,\,\mathrm{d}s)$ which is in fact the fundamental matrix of the system $\dot{x} = A(t)x$ if $V(\tau,[\alpha,\beta]) = I + A(\tau)(\beta-\alpha)$.

Even if the idea of multiplicative integration based on Henstock-Kurzweil δ-fine partitions of an interval can be easily transferred to the case of Banach algebras with unity (e.g. the space of bounded linear operators on a Banach space), the more sophisticated results on the structure of the indefinite integral $\prod_a^t V(s,\,\mathrm{d}s)$ presented in [79] depend strictly on the fact that the function V is matrix valued.

In [83] the system

$$\dot{x} = A(t)x$$

with a continuous $n \times n$-matrix valued function $A(t)$, $t \in \mathbb{R}$ is studied provided $A(t) + A^*(t) = 0$ and A is uniformly quasiperiodic with at most $r+1$ frequencies.

The problem is as follows: Given A and $\eta > 0$, does there exist a matrix valued function C that both C and the matrix solution $X_C(t)$ of

$$\dot{x} = C(t)x, \quad X_C(0) = I$$

are uniformly quasiperiodic with at most $r+1$ frequencies and

$$\|A(t) - C(t)\| \leqslant \eta$$

for $t \in \mathbb{R}$?

The answer to this question is affirmative for such couples (n, r) that the manifold $SO(n)$ of orthonormal $n \times n$-matrices with determinant equal to 1 has the estimation property of homotopies of order $1, 2, \ldots, r$.

The ordinary differential equation

$$\dot{x} = f(x, t)$$

is considered in [88] in the integral form

$$x(t) = x(a) + \int_a^t f(x(s), s)\,\mathrm{d}s$$

with the Perron integral on the right hand side. R. Henstock (*Lectures on the Theory of Integration.* World Scientific, Singapore, 1988) gave an existence result for this equation under some conditions on the right hand side f. In [88] it is shown that Henstock's conditions are satisfied if and only if $f(x,t) = g(t) + h(x,t)$ where g is Perron integrable and h satisfies the well known Carathéodory conditions.

In [86] and [91] the linear difference equation

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{N}_0 = \{0, 1, \ldots\} \tag{15}$$

is studied in the case that $A(n)$ is a $k \times k$ invertible matrix function for $n \in \mathbb{N}$. In the former paper [86] it is shown that if the difference equation has an exponential dichotomy then it is topologically equivalent to the system

$$x_i(n+1) = e_i x_i(n), \quad n \in \mathbb{N} \text{ and } i = 1, 2, \ldots, k \tag{16}$$

where $e_i = \mathrm{e}^{-1}$ or $e_i = \mathrm{e}$, and that the difference equation is structurally stable if and only if it has an exponential dichotomy. In [91] these results are completed by showing that the system (15) is topologically equivalent to the system (16) if and only if the matrix functions $A(n)$ and $A^{-1}(n)$ are bounded on $\mathbb{N}$. Moreover, it is proved in [91] that if two linear difference equations with invertible coefficient matrices are topologically equivalent and one of them has a bounded coefficient matrix whose inverse is bounded as well, then the coefficient matrix of the other system has the same property.

In the papers [81], [84], [85] Kurzweil studied again certain convergence phenomena in ordinary differential equations; these papers amend former results on continuous dependence of solutions of ODE's on a parameter which date back to the fifties, cf. [12]–[14]. A model equation for these results is

$$\dot{x} = \sum_{i=1}^{r} f_i(x) k^{\sigma} \cos(kt + \theta_i).$$

It is shown that for $\sigma = \frac{1}{2}$ the solutions x_k tend to the solution of a "limit equation" which involves the Lie brackets of the functions on the righthand sides. For the above model equation it has the form

$$\dot{x} = \frac{1}{2} \sum_{i<j} [f_i, f_j](x) \sin(\theta_j - \theta_i),$$

where the Lie bracket is given by $[f, g] = Dgf - Dfg = D_f g - D_g f$. The case which leads to a limit equation involving iterated Lie brackets was studied in [84].

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The survey of Kurzweil's results given above represents a choice which is far from being complete. Beside, it is important to point out that Kurzweil's "pure" research activity does not fully cover his contribution to the development of Czech (Czechoslovak) Mathematics.

Prof. Kurzweil has been teaching for many years at Charles University in Prague. At first he delivered special lectures for advanced students in which the students got acquainted with the domains of his own research. Since 1964 he was systematically lecturing the standard course of ordinary differential equations. He created a modern curriculum of this course and prepared the corresponding lecture notes for students ([B1], [B2], [B3]).

His teaching experience was a starting point also for his book [B4] devoted to the classical theory of ordinary differential equations. It is not only a detailed and rigorous textbook in which a complete account of the analytical fundaments of the theory is given, but it also has many features of a monograph, outlining some aspects of the modern theory of differential equations. As an example let us mention the original exposition of the differential relations, which is not to be found in current texts mainly devoted to ordinary differential equations. The book carries the sign of Kurzweil's style consisting in rigorous elaboration of all details. It leads the reader to a thorough study, which in view of the character of the text cannot be superficial. By rearranging and amending some parts of the book [B4], Kurzweil gave rise to its English version [B7]. For instance, the account of boundary value problems in [B7] is really remarkable.

From 1954 prof. Kurzweil led the regular Thursday Seminar in Ordinary Differential Equations in the Mathematical Institute of the Czechoslovak Academy of Sciences. The seminar started in 1952 and is far from being restricted only to the subject of ordinary differential equations, which is a consequence of Kurzweil's extraordinary scope of interest in mathematics. The seminar has received numerous speakers from all parts of the world.

The work of the Department of Ordinary Differential Equations (now Dept. of Real and Probabilistic Analysis) of the Mathematical Institute led by Kurzweil from 1955 till 1984 carries the impress of his scientific personality full of original ideas. The authors of these lines can declare from their own experience that to work with J. Kurzweil is gratifying and extraordinarily stimulative, and that many results of theirs would never come into existence without his advice and support.

Prof. Kurzweil was chief editor of Časopis pro pěstování matematiky (Journal for Cultivation of Mathematics, now Mathematica Bohemica) from 1956 till 1970. In various offices he took part in both the preparation and fulfilment of the National projects of basic research. He was member of the Scientific Board for Mathematics of the Czechoslovak Academy of Sciences, chairman or member of committees for scientific degrees etc.

The survey of Kurzweil's activity in mathematics would be incomplete without mentioning his deep interest in the problems of mathematical education in our (even elementary) schools. In this field he has been active both in the Institute and in the Union of Czechoslovak Mathematicians and Physicists. Here he has always supported approaches based on the employment of children's natural intellect, experience and skills. Although being confident that it is necessary to educate children and young people in accordance with the present state of science, he is firmly convinced that abstract concepts and schemes which have significantly contributed to the development of mathematics as a branch of science lead the pupils in many cases to formal procedures which are irrational at least to the same extent as the old system of mathematical education. Prof. Kurzweil devoted much time and energy to these questions till 1989 and also later in the position of the President of the Union of Czech Mathematicians and Physicists.

The scientific activity of Jaroslav Kurzweil has been lasting for about 55 years. During this period he has created admirable work of research that has notedly influenced Czechoslovak mathematics and enriched contemporary mathematical knowledge in an exceptionally broad part of its spectrum. He is a specialist and known and respected throughout the world, with friends (both professional and personal) in many countries. The deep trace of Kurzweil's work and personality in Czech and world Mathematics is evident and incontestible to everybody who met him either as a mathematician or simply as a man.

All those who have met Prof. Jaroslav Kurzweil know him as a good and wise man who does not lack the sense of humor, who loves people with all their assets and drawbacks, and they respect and love him in return.

It would be superfluous to dilate upon Kurzweil's role in cultivating, fostering and developing Mathematics. The greater are his merits that his work was done under a totalitarian regime which certainly did not create and ensure adequate conditions for scientific work in spite of its frequent big-mouthed declarations. So obvious was his integrity and natural authority that he was respected by practically everybody, except perhaps the most hardline party bosses, in spite of his rather openly pronouncing critical opinions unwelcome to the regime. It was his merit that the microclimate in the Department as regarded both the scientific work and the human relations remained so exceptional even in the relatively favourable atmosphere of the Institute generally, during all the peripetias of four decades of communist reign. It was also a sense of humour of his own that helped him to get over the absurdities of the period. Let us just recall the opening ceremony of the EQUADIFF 7 Conference in 1989 (still before the "velvet revolution" in Czechoslovakia) at which Kurzweil delivered an opening address. He started quite innocently: "Today we celebrate an extraordinary anniversary." Nevertheless, the audience (at least the Czechs and Slovaks, but many foreigners as well) held their breath: it was August 21, the day of Soviet invasion to the country in 1968. After a well-timed pause, Kurzweil went on: "Exactly two hundred years passed since the birth of one of the greatest mathematicians of all times, Augustin Cauchy..."

Soon after the revolution in 1989 J. Kurzweil was elected Director of the Mathematical Institute, and held this office till 1996. Since 1990 he has been chairman of the Board for Accreditation attached to the government of the Czech Republic which is an advisory body of the government for the scientific and teaching level of all institutions of higher education in the Czech Republic, approving among other their right to grant the academic degrees of Master and Doctor.

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We present in the References a (hopefully) complete chronological list of scientific publications of Jaroslav Kurzweil. Original papers are listed first, books are at the end of the list marked in the form [Bn] where n is the number in the chronological order of the books. Occasional articles are not included.

References

[1] A contribution to the metric theory of diophantine approximations. Czechoslovak Math. J. *1* (1951), 149–178; Russian version: Čehoslovackij matematičeskij žurnal 1 (1951), 173–203.

[2] On the uniqueness of solution of the modified Dirichlet problem. Čas. pěst. mat. *78* (1953), 213–214. (In Czech.)

[3] A characterization of analytic operations in real Banach spaces. Studia Mathematica *14* (1953), 82–83.

[4] On approximation in real Banach spaces. Studia Mathematica *14* (1954), 214–231.

[5] On the metric theory of inhomogeneous diophantine approximations. Studia Mathematica *15* (1955), 84–112.

[6] On the theory of oscillations of an autonomous quasilinear system of ordinary differential equations. Czechoslovak Math. J. *5* (1955), 517–531. (In Russian.)

[7] On periodic and almost periodic solutions of the system of ordinary differential equations. Czechoslovak Math. J. *5* (1955), 362–370. (In Russian.)

[8] On the converse of the first Ljapunov theorem on stability of motion. Czechoslovak Math. J. *5* (1955), 382–398. (In Russian.)

[9] On the converse of the second Ljapunov theorem on stability of motion. Czechoslovak Math. J. *6* (1956), 217–260, 455–485.

[10] On the converse of Ljapunov stability theorem and Persidskij uniform stability theorem. Czechoslovak Math. J. *7* (1957), 254–272 (with I. Vrkoč). (In Russian.)

[11] On approximation in real Banach spaces by analytic operations. Studia Mathematica *16* (1957), 124–129.

[12] On continuous dependence of solutions of differential equations on a parameter. Czechoslovak Math. J. *7* (1957), 568–583 (with Z. Vorel). (In Russian.)

[13] Generalized ordinary differential equations and continuous dependence on a parameter. Czechoslovak Math. J. *7* (1957), 418–449.

[14] Generalized ordinary differential equations. Czechoslovak Math. J. *8* (1958), 360–388.

[15] Unicity of solutions of generalized differential equations. Czechoslovak Math. J. *8* (1958), 502–509.

[16] On Dirac function in nonlinear differential equations. Aplikace matematiky *3* (1958), 348–359 (with V. Doležal, Z. Vorel). (In Czech.)

[17] On generalized ordinary differential equations possessing discontinuous solutions. Prikl. mat. i mech. *22* (1958), 27–45. (In Russian.)

[18] On integration by parts. Czechoslovak Math. J. *8* (1958), 356–359.

[19] On some properties of linear differential equations. Aplikace matematiky *4* (1959), 163–176 (with V. Doležal). (In Czech.)

[20] Addition to my paper "Generalized ordinary differential equations and continuous dependence on a parameter". Czechoslovak Math. J. *9* (1959), 564–573.

[21] Linear differential equations with distributions as coefficients. Bull. de l'Academie Polonaise des Sciences, Ser. des sci. math., astr. et phys. *VII* (1959), 557–560.

[22] Note on oscillatory solutions of the equation $y'' + f(x)y^{2n-1} = 0$. Čas. pěst. mat. *85* (1960), 357–358. (In Russian.)

[23] On linear control systems. Bul. Inst. Politeh. Iasi, N.Ser. *6(10)* (1960), 13–19 (with Z. Vorel).
[24] Contributions to the theory of nonlinear oscillations 5. Princeton (1960), 25–35 (with J. Jarník).
[25] On spectral decomposition of Hermitian operator. Čas. pěst. mat. *86* (1961), 90–92. (In Czech.)
[26] On analytic construction of regulators. Avtomatika i telemechanika *26* (1961), 688–695. (In Russian.)
[27] On the averaging principle in some special cases of boundary value problems for partial differential equations. Čas. pěst. mat. *88* (1963), 444–456. (In Russian.)
[28] The averaging principle and its applications in partial differential equations. Nonlinear Vibration Problems (1963), 142–148.
[29] Problems which lead to a generalization of the concept of an ordinary nonlinear differential equation. Proc. of Conf. Equadiff, Academia, Praha (1963), 65–76.
[30] On a system of the third order describing the motion of a material point accounting for the inner friction. Trudy Mezhdunarod. simp. po nelinejnym kolebanijam. Kiev (1963), 220–222. (In Russian.)
[31] To the linear theory of optimal control. Čas. pěst. mat. *89* (1964), 90–101. (In Russian.)
[32] On an inequality for eigenvalues of integral equations. Čas. pěst. mat. *89* (1964), 205–210. (In Russian.)
[33] Über die asymptotische Methode in der Theorie der nichtlinearen Differentialgleichungen. III. Konferenz über nichtlineare Schwingungen, Abhandlungen der Deutschen Akad. der Wiss., Klasse Math., Phys., Astr. (1965), 1–9.
[34] Exponentially stable integral manifolds, averaging principle and continuous dependence on a parameter. Czechoslovak Math. J. *16* (1966), 380–423, 463–492.
[35] Invariant manifolds for differential systems. Atti VIII Congr. UMI, Trieste (1967), 291–292.
[36] Van der Pol perturbation of the equation for a vibrating string. Czechoslovak Math. J. *17* (1967), 558–608.
[37] Invariant manifolds for flows. Acta Fac. Rerum Nat. Univ. Comenianae, Proc. Equadiff Bratislava, 1966, Mathematica *17* (1967), 89–92.
[38] Invariant manifolds of differential systems. Proc. of Fourth Conf. on Nonlinear Oscillations, Academia, Praha (1968), 41–49.
[39] Invariant manifolds of a class of linear functional differential equations. Revue Roum. Math. Pures et Appl. *13* (1968), 113–1120.
[40] A theory of invariant manifolds for flows. Revue Roum. Math. Pures et Appl. *13* (1968), 1079–1097 (with A. Halanay).
[41] Invariant sets of differential systems. Differencialnye uravnenija *4* (1968), 785–797. (In Russian.)
[42] Invariant manifolds for flows. Differential Equations and Dyn. Systems. Proc. Int. Symp. Mayaguez 1967, Academic Press, 431–468.
[43] Invariant manifolds of differential systems. Z. Angew. Math. Mech. *49* (1969), 11–14.
[44] On invariant sets and invariant manifolds of differential systems. J. Differ. Equations *6* (1969), 247–263 (with J. Jarník).
[45] Global solutions of functional differential equations. Seminar on Diff. Equations and Dynamical Systems 11, Lecture Notes in Math. *144*, Springer (1970), 134–139.
[46] Uniqueness and differentiability of solutions of ordinary differential equations and total differential equations. Math. Inst. Univ. of Warwick, preprint (1969), 1–26.
[47] Global solutions of functional differential equations. Univ. of Maryland, The Inst. for Fluid Dynamics and Applied Math., Technical Note BN629, 1969.

[48] Small delays don't matter. Math. Inst. Univ. of Warwick, preprint (1969), 1–17.
[49] Invariant manifolds I. CMUC *11* (1970), 309–336.
[50] Existence of global solutions of delayed differential equations on compact manifolds. Rend. Ist. di Matem. Univ. Trieste, vol. II, fasc. II (1970), 106–108.
[51] On solutions of nonautonomous linear delayed differential equations which are defined and bounded for $t \to -\infty$. CMUC *12* (1971), 69–72.
[52] On solutions of nonautonomous linear delayed differential equations which are exponentially bounded for $t \to -\infty$. Čas. pěst. mat. *96* (1971), 229–238.
[53] Solutions of linear nonautonomous functional differential equations which are exponentially bounded for $t \to -\infty$. J. Differ. Equations *11* (1972), 376–384.
[54] On a system of operator equations. J. Differ. Equations *11* (1972), 364–375.
[55] On submultiplicative nonnegative functionals on linear maps of linear finitedimensional normed spaces. Czechoslovak Math. J. *22* (1972), 454–461.
[56] On the maximum value of a class of determinants. Matem. časopis *23* (1973), 40–42.
[57] On Fubini theorem for general Perron integral. Czechoslovak Math. J. *29* (1973), 286–297.
[58] On multiplication of Perron-integrable functions. Czechoslovak Math. J. *23* (1973), 542–566.
[59] Ryabov's special solutions of functional differential equations. Boll. U.M.I. *11*; Suppl. Fasc. *3* (1975), 198–208 (with J. Jarník).
[60] On Ryabov's Special solutions of functional differential equations. Colloquia Math. Soc. J. Bolyai I5, Differential Equations, Keszthely, Hungary (1975), 303–307 (with J. Jarník).
[61] Right-hand sides of differential inclusions which cannot be reduced. Problemy asymptotičeskoj teorii nelinejnych kolebanij, Naukova Dumka, Kiev, 1977, 122–132 (with J. Jarník).
[62] Reducing differential inclusions. Abh. Akad. Wiss. DDR, Abt. Mathematik, Naturwissenschaften, Technik No. 3 (1977), 477–479 (with J. Jarník).
[63] On differential relations and on Filippov's concept of differential equations. Proc. Uppsala 1977 Int. Conf. on Diff. Eq., Uppsala (1977), 109–113.
[64] On conditions on right hand sides of differential relations. Čas. pěst. mat. *102* (1977), 334–349 (with J. Jarník).
[65] Extension of a Scorza-Dragoni theorem to differential relations and functional differential relations. Commentat. Math. Spec. Vol. I, dedic. L. Orlicz (1978), 147–158 (with J. Jarník).
[66] On differential relations. Proc. 8. Int. Conf. on Nonlin. Osc., Praha (1978), 39–49.
[67] Kneser's theorem for multivalued differential delay equations. Čas. pěst. mat. *104* (1979), 1–8 (with P. Krbec).
[68] Sets of solutions of differential relations. Čas. pěst. mat. *106* (1981), 554–568 (with J. Jarník).
[69] Nonautonomous Differential relations and their connections with differential equations the right hand side of which is discontinuous with respect to space variables. Proc. of the Second Conference Rousse'81, Rousse, 1982.
[70] Integral of multivalued mappings and its connection with differential relations. Čas. pěst. mat. *108* (1983), 8–28 (with J. Jarník).
[71] On a problem in the theory of linear differential equations with quasi-periodic coefficients. Proc. of the 9th Int. Conf. on Nonlin. Osc., Kiev 1981, Vol. 1, Kiev (1984), 214–217 (with A. Vencovská).

[72] On linear differential equations with almost periodic coefficients and the property that the unit sphere is invariant. Proc. Int. Conf. Equadiff 82, Lecture Notes in Math. 1017, Springer, 1983, 364–368 (with A. Vencovská).

[73] On Mawhin's approach to multiple nonabsolutely convergent integrals. Čas. pěst. mat. *108* (1983), 356–380 (with J. Jarník, Š. Schwabik).

[74] The integral as a limit of integral sums. Jahrbuch Ueberblicke Mathematik 1984. Bibliographisches Inst. AG (1984), 105–136.

[75] A non-absolutely convergent integral which admits C^1-transformations. Čas. pěst. mat. *109* (1984), 157–167 (with J. Jarník).

[76] A non-absolutely convergent integral which admits transformation and can be used for integration on manifold. Czechoslovak Math. J. *35* (1985), 116–139 (with J. Jarník).

[77] On regularization of right hand sides of differential relations. Proc. Royal Soc. Edinburgh *97A* (1984), 151–159 (with J. Jarník).

[78] A new and more powerful concept of the PU-integral. Czechoslovak Math. J. *38* (1988), 8–48 (with J. Jarník).

[79] Perron integral, Perron product integral and ordinary linear differential equations. Equadiff 6., Lecture Notes in Mathematics 1192 (eds. J. Vosmanský, M. Zlámal), Berlin, Springer, 1986, pp. 149–154 (with J. Jarník).

[80] On some extensions of the Perron integral on one dimensional intervals. An approach by integral sums fulfilling a symmetry condition. Functiones et Approximatio Commentarii Mathematici *17* (1987), 49–55 (with J. Jarník).

[81] Limit processes in ordinary differential equations. Zeitschrift für Angewandte Mathematik und Physik *38* (1987), 241–256 (with J. Jarník).

[82] A general form of the product integral and linear ordinary differential equations. Czechoslovak Math. J. *37* (1987), 642–659 (with J. Jarník).

[83] Linear differential equations with quasiperiodic coefficients. Czechoslovak Math. J. *37* (1987), 424–470 (with A. Vencovská).

[84] Iterated Lie brackets in limit processes in ordinary differential equations. Resultate der Mathematik *14* (1988), 125–137 (with J. Jarník).

[85] A convergence effect in ordinary differential equations. Asymptotic Methods of Mathematical Physics. (ed. V. S. Korolyuk) Naukova Dumka, Kiev (1988), 134–144 (with J. Jarník).

[86] Structural stability of linear discrete systems via the exponential dichotomy. Czechoslovak Math. J. *38* (1988), 280–284 (with G. Papaschinopoulos).

[87] The PU-integral and its properties. Real Analysis Exchange *14* (1988–89), 34–43 (with J. Jarník).

[88] Ordinary differential equations the solutions of which are ACG_*-functions. Archivum Mathematicum *26* (1990), 129–136 (with Š. Schwabik).

[89] On a generalization of the Perron integral on one-dimensional intervals. Annales Polonici Mathematici (Zdzislaw Opial in memoriam) *51* (1990), 205–218 (with J. Jarník).

[90] The PU-integral: its definitions and some basic properties. New Integrals, Proceedings of the Henstock Conference held in Coleraine, Northern Ireland, August 9–12, 1988 (eds. P. S. Bullen, P. Y. Lee, J. Mawhin, P. Muldowney, W. F. Pfeffer) Lecture Notes in Math. 1419, Springer, Berlin (1990), 66–81 (with J. Jarník).

[91] Topological equivalence and structural stability for linear difference equations. J. Differ. Equations *89* (1991), 89–94 (with G. Papaschinopoulos).

[92] An integral defined by approximating BV partitions of unity. Czechoslovak Math. J. *41* (1991), 695–712 (with J. Mawhin and W. Pfeffer).

[93] Equiintegrability and controlled convergence of Perron-type integrable functions. Real Analysis Exchange *17* (1991–92), 110-139 (with J. Jarník).

[94] Differentiability and integrability in n dimensions with respect to α-regular intervals. Results in Mathematics *21* (1992), 138–151 (with J. Jarník).
[95] Equivalent definitions of regular generalized Perron integral. Czechoslovak Math. J. *42* (1992), 365–378 (with J. Jarník).
[96] Generalized multidimensional Perron integral involving a new regularity condition. Results in Mathematics *23* (1993), 363–373 (with J. Jarník).
[97] Pfeffer integrability does not imply M_1-integrability. Czechoslovak Math. J. *44* (1994), 47–56 (with J. Jarník).
[98] Perron-type integration on n-dimensional intervals and its properties. Czechoslovak Math. J. *45* (1995), 79–106 (with J. Jarník).
[99] Nonabsolutely convergent integrals via integral sums in $\mathbb{R}^n$. Tatra Mountains Math. Publications *8* (1996), 191–201.
[100] Perron type integration on n-dimensional intervals as an extension of integration of stepfunctions by strong equiconvergence. Czechoslovak Math. J. 46 *121* (1996), 1–20 (with J. Jarník).
[101] Another Perron type integration in n-dimensions as an extension of integration of stepfunctions. Czechoslovak Math. J. *47* (1997), 557–575 (with J. Jarník).
[102] A convergence theorem for Henstock-Kurzweil integral and its relation to topology. Bull. Acad. Royal de Belgique, Classe de Sci. *7–12* (1997), 217–230 (with J. Jarník).
[103] Convergence cannot be replaced by a locally convex topology. Bulletin de la Classe des Sciences. Acad. Royale Belgique *9* (1998), 331–347 (with J. Jarník).
[104] The set of primitives of Henstock-Kurzweil integrable functions as a convergence vector space. Tatra Mount. Math. Publ. *14* (1998), 29–37.
[105] A nonexistence result for the Kurzweil integral. Math. Bohem. *127* (2002), 571–580 (with P. Krejčí).
[106] The revival of the Riemannian approach to integration. Banach Center Publications, Vol. 64 (Orlicz Centenary Volume), Warszawa (2004), 147–158.
[107] McShane equi-integrability and Vitali's convergence theorem. Math. Bohem. *129* (2004), 141–157 (with Š. Schwabik).
[108] On McShane integrability of Banach space-valued functions. Real Analysis Exchange *29* (2003/2004), 763–780 (with Š. Schwabik).
[B1] Integral Equations. Lecture Notes, SPN, Praha, 1963. (In Czech.)
[B2] Differential Equations (Ordinary differential equations in the real domain). Lecture Notes SPN, Praha, 1968. (In Czech.)
[B3] Ordinary Differential Equations in the Real Domain. Lecture Notes SPN, Praha, 1970. (In Czech.)
[B4] Ordinary Differential Equations. TKI, SNTL Praha, 1978. (In Czech.)
[B5] Nichtabsolut konvergente Integrale. Teubner Texte zur Mathematik, Bd. 26. Teubner, Leipzig, 1980.
[B6] Appendix A: The Perron-Ward Integral and Related Concepts. In the monograph K. Jacobs: Measure and Integral, Academic Press, 1978, pp. 515–533.
[B7] Ordinary Differential Equations. SNTL Praha-Elsevier Publ., Amsterdam, 1986.
[B8] Henstock-Kurzweil Integration: Its Relation to Topological Vector Spaces. World Scientific, Singapore, 2000.
[B9] Integration between the Lebesgue Integral and the Henstock-Kurzweil Integral. Its Relation to Local Convex Vector Spaces. World Scientific, Singapore, 2002.