CONTINUITY IN THE ALEXIEWICZ NORM

Erik Talvila, Abbotsford

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Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. If f is a Henstock-Kurzweil integrable function on the real line, the Alexiewicz norm of f is $\|f\| = \sup_I |\int_I f|$ where the supremum is taken over all intervals $I \subset \mathbb{R}$. Define the translation τ_x by $\tau_x f(y) = f(y-x)$. Then $\|\tau_x f - f\|$ tends to 0 as x tends to 0, i.e., f is continuous in the Alexiewicz norm. For particular functions, $\|\tau_x f - f\|$ can tend to 0 arbitrarily slowly. In general, $\|\tau_x f - f\| \geqslant \operatorname{osc} f|x|$ as $x \to 0$, where $\operatorname{osc} f$ is the oscillation of f. It is shown that if F is a primitive of f then $\|\tau_x F - F\| \leqslant \|f\||x|$. An example shows that the function $y \mapsto \tau_x F(y) - F(y)$ need not be in L^1. However, if $f \in L^1$ then $\|\tau_x F - F\|_1 \leqslant \|f\|_1|x|$. For a positive weight function w on the real line, necessary and sufficient conditions on w are given so that $\|(\tau_x f - f)w\| \to 0$ as $x \to 0$ whenever fw is Henstock-Kurzweil integrable. Applications are made to the Poisson integral on the disc and half-plane. All of the results also hold with the distributional Denjoy integral, which arises from the completion of the space of Henstock-Kurzweil integrable functions as a subspace of Schwartz distributions.

Keywords: Henstock-Kurzweil integral, Alexiewicz norm, distributional Denjoy integral, Poisson integral

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1. Introduction

For $f\colon \mathbb{R} \to \mathbb{R}$ define the translation by $\tau_x f(y) = f(y-x)$ for $x, y \in \mathbb{R}$. If $f \in L^p$ $(1 \leqslant p < \infty)$ then it is a well known result of Lebesgue integration that f is continuous in the p-norm, i.e., $\lim_{x\to 0} \|\tau_x f - f\|_p = 0$. For example, see [4, Lemma 6.3.5]. In this paper we consider continuity of Henstock-Kurzweil integrable functions in

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Alexiewicz and weighted Alexiewicz norms on the real line. Let $\mathcal{HK}$ be the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ that are Henstock-Kurzweil integrable. The Alexiewicz norm of $f \in \mathcal{HK}$ is defined $\|f\| = \sup_I |\int_I f|$ where the supremum is over all intervals $I \subset \mathbb{R}$. Identifying functions almost everywhere, $\mathcal{HK}$ becomes a normed linear space under $\|\cdot\|$ that is barrelled but not complete. See [1] and [5] for a discussion of the Henstock-Kurzweil integral and the Alexiewicz norm. It is shown below that translations are continuous in norm and that for $f \in \mathcal{HK}$ we have $\|\tau_x f - f\| \geqslant \operatorname{osc} f\,|x|$ where $\operatorname{osc} f$ is the oscillation of f. For particular $f \in \mathcal{HK}$ the quantity $\|\tau_x f - f\|$ can tend to 0 arbitrarily slowly. If F is a primitive of f then $\|\tau_x F - F\| \leqslant \|f\||x|$. An example shows that if $f \in \mathcal{HK}$ then the function defined by $y \mapsto \tau_x F(y) - F(y)$ need not be in L^1 but if $f \in L^1$ then $\|\tau_x F - F\|_1 \leqslant \|f\|_1|x|$. For a positive weight function w on the real line, necessary and sufficient conditions on w are given so that $\|(\tau_x f - f)w\| \to 0$ as $x \to 0$ whenever fw is Henstock-Kurzweil integrable. The necessary and sufficient conditions involve properties of the function $g_x(y) = w(y+x)/w(y)$. Sufficient conditions are given on w for $\|(\tau_x f - f)w\| \to 0$. Applications to the Dirichlet problem in the disc and half-plane are given.

All of the results also hold when we use the distributional Denjoy integral. Define $\mathcal{A}$ to be the completion of $\mathcal{HK}$ with respect to $\|\cdot\|$. Then $\mathcal{A}$ is a subspace of the space of Schwartz distributions. Distribution f is in $\mathcal{A}$ if there is function F continuous on the extended real line such that $F' = f$ as a distributional derivative. For details on this integral see [6].

First we prove continuity in the Alexiewicz norm.

Theorem 1. *Let $f \in \mathcal{HK}$. For $x, y \in \mathbb{R}$ define $\tau_x f(y) = f(y - x)$. Then $\|\tau_x f - f\| \to 0$ as $x \to 0$.*

Proof. Let $x, \alpha, \beta \in \mathbb{R}$. Then $\int_\alpha^\beta (\tau_x f - f) = \int_{\alpha-x}^{\beta-x} f - \int_\alpha^\beta f$. Write $F(x) = \int_{-\infty}^x f$. Taking the supremum over α and β,

$$\begin{aligned}\|\tau_x f - f\| &\leqslant \sup_{\beta\in\mathbb{R}} |F(\beta - x) - F(\beta)| + \sup_{\alpha\in\mathbb{R}} |F(\alpha - x) - F(\alpha)| \\ &\to 0 \text{ as } x \to 0 \text{ since } F \text{ is uniformly continuous on } \mathbb{R}.\end{aligned}$$

□

Notice that for each $x \in \mathbb{R}$, the translation τ_x is an isometry on $\mathcal{HK}$, i.e., it is a homeomorphism such that $\|\tau_x f\| = \|f\|$. It is also clear that we have continuity at each point: for each $x_0 \in \mathbb{R}$, $\|\tau_x f - \tau_{x_0} f\| \to 0$ as $x \to x_0$.

The theorem also applies on any interval $I \subset \mathbb{R}$. Restrict α and β to lie in I and extend f to be 0 outside I. Or, one could use a periodic extension. The same results also hold for the equivalent norm $\|f\| = \sup_{x\in\mathbb{R}} |\int_{-\infty}^x f|$.

Under the Alexiewicz norm, the space of Henstock-Kurzweil integrable functions is not complete. Its completion with respect to the norm $\|f\| = \sup_{x\in\mathbb{R}} |\int_{-\infty}^{x} f|$ is the subspace of distributions that are the distributional derivative of a function in $\widetilde{C} := \{F\colon \mathbb{R} \to \mathbb{R};\ F \in C^0(\mathbb{R}),\ \lim_{x\to-\infty} F(x) = 0,\ \lim_{x\to\infty} F(x) \in \mathbb{R}\}$, i.e., they are distributions of order 1. See [6], where the completion is denoted $\mathcal{A}$. Thus, if $f \in \mathcal{A}$ then $f \in \mathcal{D}'$ (Schwartz distributions) and there is a function $F \in \widetilde{C}$ such that $\langle F', \varphi\rangle = -\langle F, \varphi'\rangle = -\int_{-\infty}^{\infty} F\varphi' = \langle f, \varphi\rangle$ for all test functions $\varphi \in \mathcal{D} = C_c^\infty(\mathbb{R})$. The distributional integral of f is then $\int_a^b f = F(b) - F(a)$ for all $-\infty \leqslant a \leqslant b \leqslant \infty$. We can compute the Alexiewicz norm of f via $\|f\| = \sup_{x\in\mathbb{R}} |F(x)| = \|F\|_\infty$. If $f \in \mathcal{D}'$ then $\tau_x f$ is defined by $\langle \tau_x f, \varphi\rangle := \langle f, \tau_{-x}\varphi\rangle = \langle F', \tau_{-x}\varphi\rangle = -\langle F, (\tau_{-x}\varphi)'\rangle = -\langle F, \tau_{-x}\varphi'\rangle = -\langle \tau_x F, \varphi'\rangle = \langle (\tau_x F)', \varphi\rangle$. Of course we have $L^1 \subset \mathcal{HK} \subset \mathcal{A}$ and each inclusion is strict.

The theorem only depends on uniform continuity of the primitive and not on its pointwise differentiability properties so it also holds in $\mathcal{A}$. The same is true for the other theorems in this paper.

Corollary 2. *Let $f \in \mathcal{A}$. Then $\|\tau_x f - f\| \to 0$ as $x \to 0$.*

The following theorem gives us more precise information on the decay rate of $\|\tau_x f - f\|$.

Theorem 3. (a) *Let $\psi\colon (0,1] \to (0,\infty)$ such that $\lim_{x\to 0} \psi(x) = 0$. Then there is $f \in L^1$ such that $\|\tau_x f - f\| \geqslant \psi(x)$ for all sufficiently small $x > 0$.* (b) *If $f \in \mathcal{HK}$ and $f \neq 0$ a.e. then the most rapid decay is $\|\tau_x f - f\| = O(x)$ as $x \to 0$ and this is the best estimate in the sense that if $\|\tau_x f - f\|/x \to 0$ as $x \to 0$ then $f = 0$ a.e. The implied constant in the order relation is the oscillation of f.*

P r o o f. (a) Given ψ, define $\psi_1(x) = \sup_{0<t\leqslant x} \psi(t)$. Then $\psi_1 \geqslant \psi$ and $\psi_1(x)$ decreases to 0 as x decreases to 0. Define $\psi_2(x) = \psi_1(1/n)$ when $x \in (1/(n+1), 1/n]$ for some $n \in \mathbb{N}$. Then $\psi_2 \geqslant \psi$ and ψ_2 is a step function that decreases to 0 as x decreases to 0. Now let

$$\psi_3(x) = [\psi_2(1/(n-1)) - \psi_2(1/n)]\, n(n+1)\Big(x - \frac{1}{n+1}\Big) + \psi_2(1/n)$$

when $x \in [1/(n+1), 1/n]$ for some $n \geqslant 2$. Define $\psi_3 = \psi_2$ on $(1/2, 1]$. Then $\psi_3 \geqslant \psi$ and ψ_3 is a piecewise linear continuous function that decreases to 0 as x decreases to 0. Define $f(x) = \psi_3'(x)$ for $x \in (0,1]$ and $f(x) = 0$, otherwise. For $0 < x < 1$,

$$\|\tau_x f - f\| \geqslant \left| \int_0^x \Big[f(y - x) - f(y) \Big]\, \mathrm{d}y \right| = \int_0^x f = \psi_3(x) \geqslant \psi(x).$$

Since ψ_3 is absolutely continuous, $f \in L^1$.

(b) Test functions are dense in $\mathcal{HK}$, i.e., for each $f \in \mathcal{HK}$ and $\varepsilon > 0$ there is $\varphi \in \mathcal{D}$ such that $\|f - \varphi\| < \varepsilon$. Let $x \in \mathbb{R}$. Then, since τ_x is a linear isometry, $\|(\tau_x f - f) - (\tau_x\varphi - \varphi)\| = \|\tau_x(f - \varphi) - (f - \varphi)\| < 2\varepsilon$ and $\|\tau_x\varphi - \varphi\| - 2\varepsilon < \|\tau_x f - f\| < \|\tau_x\varphi - \varphi\| + 2\varepsilon$. It therefore suffices to prove the theorem in $\mathcal{D}$. Hence, let $f \in \mathcal{D}$ and let $a, b \in \mathbb{R}$. Write $F(y) = \int_{-\infty}^{y} f$. Then, since $F \in C^2(\mathbb{R})$,

$$\begin{aligned}\int_a^b (\tau_x f - f) &= [F(b - x) - F(b)] - [F(a - x) - F(a)] \\ &= -F'(b)\,x + F''(\xi)\,x^2/2 + F'(a)\,x - F''(\eta)\,x^2/2,\end{aligned}$$

for some ξ, η in the support of f. Now,

$$\|\tau_x f - f\| \geqslant \sup_{a,b\in\mathbb{R}} |f(a) - f(b)||x| - \|f'\|_\infty x^2 = \operatorname{osc} f\,|x| - \|f'\|_\infty x^2.$$

The oscillation of $f \in \mathcal{D}$ is positive unless f is constant, but there are no constant functions in $\mathcal{D}$ except 0. The proof is completed by noting that $\|\tau_x f - f\| \leqslant \operatorname{osc} f\,|x| + \|f'\|_\infty x^2$ so that $\|\tau_x f - f\| = O(x)$ as $x \to 0$. $\square$

Part (b) is proven in [3, Proposition 1.2.3] for $f \in L^1$.

It is interesting to note that if $f \in \mathcal{HK}$ and F is its primitive then the function $\tau_x F - F$ is in $\mathcal{HK}$ for each $x \in \mathbb{R}$, even though F need not be in $\mathcal{HK}$.

Theorem 4. *Let $f \in \mathcal{HK}$, let F be one of its primitives and let $x \in \mathbb{R}$. Then the function $y \mapsto \tau_x F(y) - F(y)$ is in $\mathcal{HK}$ even though none of the primitives of f need be in $\mathcal{HK}$. We have the estimate $\|\tau_x F - F\| \leqslant \|f\||x|$. In general, $\tau_x F - F$ need not be in L^1. However, if $f \in L^1$ then $\tau_x F - F \in L^1$ and $\|\tau_x F - F\|_1 \leqslant \|f\|_1|x|$.*

P r o o f. Let $f \in \mathcal{HK}$ and let F be any primitive. Since F is continuous, to prove $\tau_x F - F \in \mathcal{HK}$ we need only show integrability at infinity. Let $a, x \in \mathbb{R}$. Then

$$\int_0^a (\tau_x F - F) = \int_{-x}^{a-x} F - \int_0^a F = \int_{-x}^{0} F - \int_{a-x}^{a} F = \int_{-x}^{0} F - F(\xi)\,x$$

for some ξ between $a - x$ and a, due to continuity of F. So, $\lim\limits_{a\to\pm\infty} \int_0^a (\tau_x F - F) = \int_{-x}^0 F - x \lim\limits_{y\to\pm\infty} F(y)$. Since F has limits at $\pm\infty$, Hake's theorem shows $\tau_x F - F \in \mathcal{HK}$. Now let $a, b \in \mathbb{R}$. Then $\int_a^b(\tau_x F - F) = \int_{a-x}^a F - \int_{b-x}^b F$. Since F is continuous, there are ξ between a and $a - x$ and η between b and $b - x$ such that $\int_a^b(\tau_x F - F) = F(\xi)\,x - F(\eta)\,x = x\int_\eta^\xi f$. It follows that $\|\tau_x F - F\| \leqslant \|f\||x|$.

The example $f = \chi_{[0,1]}$, for which

$$F(y) = \int_{-\infty}^{y} f = \begin{cases} 0, & y \leqslant 0, \\ y, & 0 \leqslant y \leqslant 1, \\ 1, & y \geqslant 1 \end{cases}$$

shows that no primitives need not be in $\mathcal{HK}$. And, if we let $F(y) = \sin(y)/y$, $f = F'$, then for $x \neq 0$,

$$\begin{aligned} \tau_x F(y) - F(y) &= \frac{\sin(y-x)}{y-x} - \frac{\sin(y)}{y} \\ &\sim \frac{[\cos(x) - 1]\sin(y) - \sin(x)\cos(y)}{y} \text{ as } y \to \infty. \end{aligned}$$

Hence, $\tau_x F - F \in \mathcal{HK} \setminus L^1$.

Suppose $f \in L^1$ and $x \geqslant 0$. Then, $|f| \in \mathcal{HK}$ so the theorem gives $\|\tau_x F - F\|_1 \leqslant \int_{-\infty}^{\infty} \int_{y-x}^{y} |f(z)| \, \mathrm{d}z \, \mathrm{d}y \leqslant \| \, |f| \, \| \, x = \|f\|_1 x$. Similarly, if $x < 0$. □

E x a m p l e 5. Let f be 2π-periodic and Henstock-Kurzweil integrable over one period. The Poisson integral of f on the unit circle is

$$u(r\mathrm{e}^{\mathrm{i}\theta}) = u_r(\theta) = \frac{1-r^2}{2\pi} \int_{-\pi}^{\pi} \frac{f(\varphi)\,\mathrm{d}\varphi}{1 - 2r\cos(\varphi - \theta) + r^2}.$$

Differentiating under the integral sign shows that u is harmonic in the disc. And, after interchanging the order of integration, it can be seen that $\|u_r - f\| \to 0$ as $r \to 1^-$. The Poisson integral defines a harmonic function that takes on the boundary values f in the Alexiewicz norm. For details on this Dirichlet problem see [7].

Now we consider continuity in weighted Alexiewicz norms. First we need the following lemma. Lebesgue measure is denoted λ.

Lemma 6. *For each $n \in \mathbb{N}$, suppose $g_n\colon \mathbb{R} \to \mathbb{R}$ and $g_n \chi_E \to g\chi_E$ in measure for some set $E \subset \mathbb{R}$ of positive measure and function g of bounded variation. If $Vg_n \leqslant M$ for all n then g_n is uniformly bounded on $\mathbb{R}$.*

P r o o f. Define $S_n = \{x \in E;\ |g_n(x) - g(x)| > 1\}$. Then $\lambda(S_n) \to 0$ as $n \to \infty$. There is $N \in \mathbb{N}$ such that whenever $n \geqslant N$ we have $\lambda(E \setminus S_n) > 0$. Since $g \in \mathcal{BV}$, g is bounded. Let $n \geqslant N$. There is $x_n \in E \setminus S_n$ such that $|g(x_n)| \leqslant \|g\|_\infty$. Therefore, $|g_n(x_n)| \leqslant 1 + \|g\|_\infty$. Let $x \in \mathbb{R}$. Then $|g_n(x) - g_n(x_n)| \leqslant Vg_n \leqslant M$. So, $|g_n(x)| \leqslant M + 1 + \|g\|_\infty$. Hence, $\{g_n\}$ is uniformly bounded. □

Theorem 7. *Let $w\colon \mathbb{R} \to (0,\infty)$. Define $g_x\colon \mathbb{R} \to (0,\infty)$ by $g_x(y) = w(y+x)/w(y)$ for each $x \in \mathbb{R}$. Then $\|(\tau_x f - f)w\| \to 0$ as $x \to 0$ for all $f\colon \mathbb{R} \to \mathbb{R}$ such that $fw \in \mathcal{HK}$ if and only if g_x is essentially bounded and of essential bounded variation, uniformly as $x \to 0$, and $g_x \to 1$ in measure on compact intervals as $x \to 0$.*

Proof. Let $G(x) = \int_{-\infty}^x fw$. Let $x, \alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}
&\int_\alpha^\beta [f(y-x) - f(y)]\, w(y)\,\mathrm{d}y \\
&\quad = \int_{\alpha-x}^{\beta-x} f(y)w(y)\,\mathrm{d}y - \int_\alpha^\beta f(y)w(y)\,\mathrm{d}y + \int_{\alpha-x}^{\beta-x} f(y)\,[w(y+x) - w(y)]\,\mathrm{d}y \\
&\quad = [G(\beta-x) - G(\beta)] - [G(\alpha-x) - G(\alpha)] + \int_{\alpha-x}^{\beta-x} f(y)w(y)\,[g_x(y) - 1]\,\mathrm{d}y.
\end{aligned}$$

Since G is uniformly continuous on $\mathbb{R}$, we have $\|(\tau_x f - f)w\| \to 0$ if and only if the supremum of $|\int_a^b f(y)w(y)[g_x(y) - 1]\,\mathrm{d}y|$ over $a, b \in \mathbb{R}$ has limit 0 as $x \to 0$, i.e., $\|fw(g_x - 1)\| \to 0$. Given $h \in \mathcal{HK}$ we can always take $f = h/w$. Hence, the theorem now follows from Lemma 6 (easily modified for the case of essential boundedness and essential variation) and the necessary and sufficient condition for convergence in norm given in [2, Theorem 6]. □

Corollary 8. *Suppose that for each compact interval I there are real numbers $0 < m_I < M_I$ such that $m_I < \|w\|_\infty < M_I$; w is continuous in measure on I; $w \in \mathcal{BV}_{\mathrm{loc}}$. Then for all $f\colon \mathbb{R} \to \mathbb{R}$ such that $fw \in \mathcal{HK}$ we have $\|(\tau_x f - f)w\| \to 0$ as $x \to 0$.*

Proof. Fix $\varepsilon > 0$. Let I be a compact interval for which $0 < m_I < \|w\|_\infty < M_I$. Define

$$\begin{aligned}
S_x &:= \{y \in I;\ |g_x(y) - 1| > \varepsilon\} \\
&= \{y \in I;\ |w(y+x) - w(y)| > \varepsilon\, w(y)\} \\
&\subset \{y \in I;\ |w(y+x) - w(y)| > \varepsilon\, m_I\} \quad \text{except for a null set.}
\end{aligned}$$

Since w is continuous in measure on I we have $\lambda(S_x) \to 0$ as $x \to 0$ and $g_x \to 1$ in measure on I.

Using

$$g_x(s_n) - g_x(t_n) = \frac{w(s_n + x) - w(t_n + x)}{w(t_n)} - \frac{w(s_n + x)[w(s_n) - w(t_n)]}{w(s_n)w(t_n)}$$

we see that $V_I g_x \leqslant V_{I+x} w/m_I + M_I V_I w/m_I^2$ where $I + x = \{y + x;\ y \in I\}$ and $V_I w$ is the variation of w over interval I. Hence, g_x is of uniform bounded variation on I. With Lemma 6 this then gives the hypotheses of the theorem. □

Of course, we are allowing w to be changed on a set of measure 0 so that w is of bounded variation rather than just equivalent to a function of bounded variation. This redundancy can be removed by replacing w with its limit from the right at each point so that w is right continuous.

As pointed out in [2], convergence of g_x to 1 in measure on compact intervals in the theorem can be replaced by convergence in L^1 norm: For each compact interval I, $\|(g_x-1)\chi_I\|_1 \to 0$ as $x \to 0$. In the corollary, we can replace continuity in measure with the condition: As $x \to 0$, $\int_I |\tau_x w - w| \to 0$ for each compact interval I.

The first two conditions in the corollary are necessary. Suppose $\|(\tau_x f - f)w\| \to 0$ whenever $fw \in \mathcal{HK}$. Then the essential infimum of w must be positive on each compact interval. If there is a sequence $a_n \to a \in \mathbb{R}$ for which $w(a_n) \to 0$ then $\operatorname{esssup}_{|x|<\delta} \sup_n g_x(a_n) = \infty$ for each $\delta > 0$ unless $w = 0$ a.e. in a neighbourhood of a. Similarly, the essential supremum of w must be finite on compact intervals. This asserts the existence of m_I and M_I in the corollary. Also, let I be a compact interval on which $0 < m < \|w\|_\infty < M$. Let $\varepsilon > 0$ and define

$$\begin{aligned} T_x &:= \{y \in I;\ |w(y+x) - w(y)| > \varepsilon\} \\ &\subset \{y \in I;\ |g_x(y) - 1| > \varepsilon/M\} \text{ except for a null set.} \end{aligned}$$

Hence, w is continuous in measure.

It is not known if $\|(\tau_x f - f)w\| \to 0$ for all f such that $fw \in \mathcal{HK}$ implies $w \in \mathcal{BV}_{\text{loc}}$. The example $w(y) = \mathrm{e}^y$ shows W need not be of bounded variation and can have its infimum zero and its supremum infinity. For, note that $g_x(y) = \exp(y+x)/\exp(y) = \mathrm{e}^x$ and so satisfies the conditions of the theorem. And, by the corollary, $w(y) = 1$ for $y < 0$ and $w(y) = 2$ for $y \geqslant 0$ is a valid weight function. Hence, w need not be continuous.

E x a m p l e 9. Let $w(y) = 1/(y^2+1)$. A calculation shows that the variation of $y \mapsto w(y+x)/w(y)$ is $2|x|\sqrt{x^2+1}$ so w is a valid weight for Theorem 7. The half-plane Poisson kernel is $\Phi_y(x) = w(x/y)/(\pi y)$. For $f\colon \mathbb{R} \to \mathbb{R}$ the Poisson integral of f is

$$u_y(x) = (\Phi_y * f)(x) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)\,\mathrm{d}t}{(x-t)^2 + y^2}.$$

Define $\Psi_z(t) = \Phi_y(x-t)/w(t)$ for $z = x + \mathrm{i}y$ in the upper half-plane, i.e., $x \in \mathbb{R}$ and $y > 0$. For fixed z both Ψ_z and $1/\Psi_z$ are of bounded variation on $\mathbb{R}$. Hence, necessary and sufficient for the existence of the Poisson integral on the upper half-plane is $fw \in \mathcal{HK}$.

Define $G(t) = \int_{-\infty}^t fw$. Integrate by parts to get $u_y(x) = y\,G(\infty)/\pi - \int_{-\infty}^{\infty} G(t)\,\Psi_z'(t)\,\mathrm{d}t$. Since G is continuous on the extended real line (with $G(\infty) := \lim_{t\to\infty} G(t)$), dominated convergence now allows differentiation under the integral. This shows $u_y(x)$ is harmonic in the upper half-plane.

Neither f nor u_y need be in $\mathcal{HK}$. For example, the Poisson integral of 1 is 1. But, we have the boundary values taken on in the weighted norm: $\|(u_y - f)w\| \to 0$ as $y \to 0^+$. We sketch out the proof, leaving the technical detail of interchanging repeated integrals for publication elsewhere. For $a, b \in \mathbb{R}$ we then have

$$\begin{aligned}\int_a^b [u_y(t) - f(t)]\, w(t)\, \mathrm{d}t &= \int_a^b \left\{(f * \Phi_y)(t) - f(t) \int_{-\infty}^{\infty} \Phi_y(s)\, \mathrm{d}s\right\} w(t)\, \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \Phi_y(s) \int_a^b [f(t-s) - f(t)]\, w(t)\, \mathrm{d}t\, \mathrm{d}s.\end{aligned}$$

Therefore, $\|(u_y - f)w\| \leqslant \int_{-\infty}^{\infty} \Phi_y(s) \|(\tau_s f - f)w\|\, \mathrm{d}s$. But, $s \mapsto \|(\tau_s f - f)w\|$ is continuous at $s = 0$. By the usual properties of the Poisson kernel (an approximate identity), we have $\|(\tau_s f - f)w\| \to 0$ as $y \to 0^+$.

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Author's address: *Erik Talvila*, Department of Mathematics and Statistics, University College of the Fraser Valley, Abbotsford, BC Canada V2S 7M8, e-mail: `Erik.Talvila@ucfv.ca`.