FUZZY SEMI α-IRRESOLUTE FUNCTIONS

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(Received December 22, 2005)

Abstract. A new class of functions called fuzzy semi α-irresolute functions in fuzzy topological spaces are introduced in this paper. Some characterizations of this class and its properties and the relationship with other classes of functions between fuzzy topological spaces are also obtained.

Keywords: fuzzy semi α-irresolute function, fuzzy product, fuzzy irresolute function, fuzzy almost irresolute function, nowhere dense fuzzy set

MSC 2000: 54A40

1. Introduction

The fuzzy concept has invaded almost all branches of mathematics ever since the introduction of fuzzy sets by Zadeh [9]. The theory of fuzzy topological spaces was introduced and developed by Chang [4] and since then various notions in classical topology have been extended to fuzzy topological spaces. The concept of semi α-irresolute functions was introduced in [8]. In this paper we introduce and study several interesting properties of the fuzzy version of semi α-irresolute functions.

2. Preliminaries

By a fuzzy topological space we shall mean a non-empty set X together with fuzzy topology T [4] and we shall denote it by (X,T). A fuzzy point in X with support $x \in X$ and value p $(0 < p \leqslant 1)$ is denoted by x_p. If λ and μ are two fuzzy sets in X and Y, respectively, we define (accordingly [1]) $\lambda \times \mu\colon X \times Y \to I$ as follows: $(\lambda \times \mu)(x,y) = \min\,(\lambda(x), \mu(y))$ for every (x,y) in $X \times Y$. A fuzzy topological space X is product related to a fuzzy topological space Y [1] if for any fuzzy sets γ in X and ξ in Y whenever λ' $(= 1 - \lambda) \ngeq \gamma$ and μ' $(= 1 - \mu) \ngeq \xi$ (in which case

$(\lambda' \times 1) \vee (1 \times \mu') \geqslant (\gamma \times \xi))$ where λ is a fuzzy open set in X and μ is a fuzzy open set in Y, there exists a fuzzy open set λ_1 in X and a fuzzy open set μ_1 in Y such that $\lambda_1' \geqslant \gamma$ or $\mu_1' \geqslant \xi$ and $(\lambda_1' \times 1) \vee (1 \times \mu_1') = (\lambda' \times 1) \vee (1 \times \mu')$. Let f be a mapping from X to Y. Then the graph g of f is a mapping from X to $X \times Y$ sending $x \in X$ to $(x, f(x))$. For two mappings $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$, we define the product $f_1 \times f_2$ of f_1 and f_2 to be the mapping from $X_1 \times X_2$ to $Y_1 \times Y_2$ sending $(x_1, x_2) \in X_1 \times X_2$ to $(f_1(x_1), f_2(x_2))$. For any fuzzy set λ in a fuzzy topological space, it is shown in [1] that (i) $1 - \operatorname{cl}\lambda = \operatorname{int}(1-\lambda)$, (ii) $\operatorname{cl}(1-\lambda) = 1 - \operatorname{int}\lambda$. For concepts not defined in this paper we refer to [1] and [2].

Definition 2.1. Let (X,T) be a fuzzy topological space and let λ be any fuzzy set in X.

λ is called fuzzy α-open set [2] if $\lambda \leqslant \operatorname{int}\operatorname{cl}\operatorname{int}\lambda$.

λ is called fuzzy semi-open set [1] if $\lambda \leqslant \operatorname{cl}\operatorname{int}\lambda$.

λ is called fuzzy pre-open set [2] if $\lambda \leqslant \operatorname{int}\operatorname{cl}\lambda$.

λ is called fuzzy β-open set [5] if $\lambda \leqslant \operatorname{cl}\operatorname{int}\operatorname{cl}\lambda$.

The complement of a fuzzy α-open (fuzzy semi-open, fuzzy β-open, respectively) set is called fuzzy α-closed (fuzzy semi-closed, fuzzy β-closed, respectively).

Remark 2.1. It is clear that every fuzzy open (fuzzy closed) set is a fuzzy α-open (fuzzy α-closed) set. But the converse is not true in general [2]. Also, every fuzzy α-open (fuzzy α-closed) set is a fuzzy pre-open (pre-closed) and a fuzzy semi-open (semi-closed) set. However, the converse is false [2]. The intersection of two fuzzy α-open (fuzzy pre-open, fuzzy semi-open, respectively) sets need not be a fuzzy α-open (fuzzy pre-open, fuzzy semi-open [1], respectively) set.

Motivated by the classical concepts introduced in [8] we now define:

Definition 2.2. A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,S) is said to be *fuzzy irresolute* if $f^{-1}(\lambda)$ is fuzzy semi-open in (X,T) for each fuzzy semi-open set λ in (Y,S).

Definition 2.3. A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,S) is said to be *fuzzy α-irresolute* if $f^{-1}(\lambda)$ is fuzzy α-open in (X,T) for each fuzzy α-open set λ in (Y,S).

Definition 2.4. A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,S) is said to be *fuzzy strongly α-irresolute* if $f^{-1}(\lambda)$ is fuzzy open in (X,T) for each fuzzy α-open set λ in (Y,S).

Definition 2.5. A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,S) is said to be *fuzzy almost irresolute* if $f^{-1}(\lambda)$ is fuzzy β-open in (X,T) for each semi-open set λ in (Y,S).

Definition 2.6. A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,S) is said to be *fuzzy semi-continuous* if $f^{-1}(\lambda)$ is fuzzy semi-open in (X,T) for each fuzzy open set λ in (Y,S).

Definition 2.7. A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,S) is said to be *fuzzy strongly semi-continuous* if $f^{-1}(\lambda)$ is fuzzy open in (X,T) for every fuzzy semi-open set λ in (Y,S).

Definition 2.8. A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,S) is said to be *fuzzy α-continuous* if $f^{-1}(\lambda)$ is fuzzy α-open in (X,T) for every fuzzy open set λ in (Y,S).

Definition 2.9. A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,S) is said to be *fuzzy strongly α-continuous* if $f^{-1}(\lambda)$ is fuzzy α-open set in (X,T) for each fuzzy semi-open λ in (Y,S).

3. Fuzzy semi α-irresolute functions

Definition 3.1. A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,S) is said to be *fuzzy semi α-irresolute* if $f^{-1}(\lambda)$ is fuzzy semi-open in (X,T) for each fuzzy α-open set λ in (Y,S).

From the definitions we obtain the following diagram:

$$
\begin{array}{ccc}
\text{fuzzy strongly } \alpha\text{-irresolute} & \overset{\Leftarrow}{\underset{\not\Rightarrow}{}} & \text{fuzzy strongly semi-continuous} \\
\Downarrow \ \not\Uparrow & & \Downarrow \ \not\Uparrow \\
\text{fuzzy } \alpha\text{-irresolute} & \overset{\Leftarrow}{\underset{\not\Rightarrow}{}} & \text{fuzzy strongly } \alpha\text{-continuous} \\
\Downarrow \ \not\Uparrow & & \Downarrow \ \not\Uparrow \\
\text{fuzzy semi } \alpha\text{-irresolute} & \overset{\Leftarrow}{\underset{\not\Rightarrow}{}} & \text{fuzzy irresolute} \\
\Downarrow \ \not\Uparrow & & \Downarrow \ \not\Uparrow \\
\text{fuzzy semi-continuous} & \overset{\Rightarrow}{\underset{\not\Leftarrow}{}} & \text{fuzzy almost irresolute}
\end{array}
$$

The examples given below show that the converses of these implications are not true in general.

Example 3.1. Let μ_1, μ_2 and μ_3 be fuzzy sets on $I=[0,1]$ defined by

$$
\mu_1(x)=\begin{cases} 0, & 0\leqslant x\leqslant \frac{1}{2}, \\ 2x-1, & \frac{1}{2}\leqslant x\leqslant 1; \end{cases}
$$

$$\mu_2(x) = \begin{cases} 1, & 0 \leqslant x \leqslant \frac{1}{4}, \\ -4x+2, & \frac{1}{4} \leqslant x \leqslant \frac{1}{2}, \\ 0, & \frac{1}{2} \leqslant x \leqslant 1; \end{cases}$$

$$\mu_3(x) = \begin{cases} 0, & 0 \leqslant x \leqslant \frac{1}{4}, \\ \frac{1}{3}(4x-1), & \frac{1}{4} \leqslant x \leqslant 1. \end{cases}$$

Let $T_1 = \{0, \mu_1, \mu_2, \mu_1 \vee \mu_2, 1\}$ and $T_2 = \{0, \mu_3, 1\}$. Then T_1 and T_2 are fuzzy topologies on I. Let $f\colon (I, T_1) \to (I, T_2)$ be defined by $f(x) = x$ for each $x \in I$. Then $f^{-1}(0) = 0$; $f^{-1}(1) = 1$; $f^{-1}(\mu_3) = \mu_3$. Since μ_3 fuzzy semi-open in (I, T_1), f is fuzzy irresolute and hence f is fuzzy semi α-irresolute. Since $\operatorname{int}\mu_3 = \mu_1$, $\operatorname{cl}\mu_1 = \mu_2'$, $\operatorname{int}\mu_2' = \mu_1$, we have $\mu_3 \nleqslant \operatorname{int}\operatorname{cl}\operatorname{int}\mu_3 = \mu_1$ and therefore μ_3 is not fuzzy α-open in (I, T_1). Hence f is not fuzzy α-irresolute and it is not fuzzy strongly α-continuous either.

E x a m p l e 3.2. Define $f\colon I \to I$ by $f(x) = x/2$. Let μ_1, μ_2 and μ_3 be fuzzy sets in I described in Example 3.1. Let $T_1 = \{0, \mu_1, \mu_2, \mu_1 \vee \mu_2, 1\}$ and $T_3 = \{0, \mu_3', 1\}$. Then T_1 and T_3 are fuzzy topologies on I. Consider the mapping $f\colon (I, T_3) \to (I, T_1)$. Since $f^{-1}(0) = 0$; $f^{-1}(1) = 1$; $f^{-1}(\mu_1) = 0$; $f^{-1}(\mu_2) = \mu_1' = f^{-1}(\mu_1 \vee \mu_2)$, we conclude that μ_1' is fuzzy α-open in (I, T_3). Then f is fuzzy strongly α-continuous from (I, T_3) to (I, T_1) and it is also fuzzy α-irresolute. Since μ_1' is not fuzzy open in (I, T_3), hence f is not fuzzy strongly α-irresolute and it is not fuzzy strongly semi-continuous either.

E x a m p l e 3.3. Let μ_1, μ_2 and μ_3 be fuzzy sets in I described in Example 3.1. Clearly, $T_1 = \{0, \mu_1, \mu_2, \mu_1 \vee \mu_2, 1\}$ is a fuzzy topology on I. Let $f\colon (I, T_1) \to (I, T_1)$ be defined by $f(x) = x$ for each $x \in I$. Since $f^{-1}(0) = 0$; $f^{-1}(1) = 1$; $f^{-1}(\mu_1) = \mu_1$; $f^{-1}(\mu_2) = \mu_2$; $f^{-1}(\mu_1 \vee \mu_2) = \mu_1 \vee \mu_2$, we see that f is fuzzy strongly α-irresolute from (I, T_1) to (I, T_1) and hence f is fuzzy α-irresolute. Now μ_3 is fuzzy semi-open in (I, T_1). Consequently, $f^{-1}(\mu_3) = \mu_3$ is not fuzzy open in (I, T_1). Therefore f is not fuzzy strongly semi-continuous. Further, $f^{-1}(\mu_3) = \mu_3$ is not fuzzy α-open in (I, T_1). Hence f is not fuzzy strongly α-continuous.

E x a m p l e 3.4. Let μ_1, μ_2 and μ_3 be the fuzzy sets in I described in Example 3.1. Clearly $T_1 = \{0, \mu_1, \mu_2, \mu_1 \vee \mu_2, 1\}$ is a fuzzy topology on I. Let $f\colon (I, T_1) \to (I, T_1)$ be defined by $f(x) = x/2$ for each $x \in I$. Then $f^{-1}(0) = 0$, $f^{-1}(1) = 1$, $f^{-1}(\mu_1) = 0$, $f^{-1}(\mu_2) = \mu_1' = f^{-1}(\mu_1 \vee \mu_2)$, hence μ_1' is fuzzy semi-open in (I, T_1). Then f is fuzzy semi α-irresolute and hence f is fuzzy semi-continuous. It can be easily seen that $\operatorname{cl}\mu_1 = \mu_2'$; $\operatorname{cl}\mu_2 = \mu_1'$; $\operatorname{cl}(\mu_1 \vee \mu_2) = 1$; $\operatorname{int}\mu_1' = \mu_2$; $\operatorname{int}\mu_2' = \mu_1$; $\operatorname{int}(\mu_1 \vee \mu_2) = 0$ and $\operatorname{int}\mu_3 = \mu_1$. Since $\mu_1 \leqslant \mu_3 \leqslant \operatorname{cl}\mu_1, \mu_3$ is a fuzzy semi-open set which is not a fuzzy open set (not a fuzzy α-open set). Now, μ_1' is fuzzy β-open in (I, T_1) since

$\operatorname{cl}\mu_1' = \mu_1'$. Let

$$\delta(x) = f^{-1}(\mu_3)(x) = \mu_3 f(x) = \mu_3(x/2) = \begin{cases} 0, & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{3}(2x-1), & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

for each $x \in I$. Then $\operatorname{cl}\operatorname{int}\delta(x) = 0$, and $\operatorname{cl}\operatorname{int}\operatorname{cl}\delta(x) = \mu_1'$ or μ_2'. Since $\delta(x) \leqslant \operatorname{cl}\operatorname{int}\operatorname{cl}\delta(x) = \mu_1'$ or μ_2', $f^{-1}(\mu_3)$ is fuzzy β-open set in I. Hence f is fuzzy almost irresolute. Since $\operatorname{cl}\operatorname{int}\delta(x) = 0$, we have $\delta(x) \nleq \operatorname{cl}\operatorname{int}\delta(x)$. Therefore $f^{-1}(\mu_3)$ is not a fuzzy semi-open set in I. Hence f is not fuzzy irresolute.

Example 3.5. Let μ_1, μ_2 and μ_3 be fuzzy sets in I described in Example 3.1. Clearly $T_4 = \{0, \mu_1, 1\}$ is a fuzzy topology on I. Let $f\colon (I, T_4) \to (I, T_4)$ be defined by $f(x) = x/2$ for each $x \in I$. We have $f^{-1}(0) = 0$; $f^{-1}(1) = 1$ and $f^{-1}(\mu_1) = 0$. Therefore f is fuzzy semi-continuous. It can be easily seen that $\operatorname{int}\mu_3 = \mu_1$; $\operatorname{cl}\mu_1 = 1$. Then μ_3 is fuzzy α-open set in (I, T_4) but not fuzzy open in (I, T_4). Let

$$\delta(x) = f^{-1}(\mu_3)(x) = \mu_3 f(x) = \mu_3(x/2) = \begin{cases} 0, & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{3}(2x-1), & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

for each $x \in I$. Then cl int $\delta(x) = 0$. Therefore $\delta(x) \nleq \operatorname{cl}\operatorname{int}\delta(x)$. Therefore $f^{-1}(\mu_3)$ is not a fuzzy semi-open set in I. Hence f is not fuzzy semi α-irresolute.

Example 3.6. Let $X = \{a, b, c\}$. Define $T = \{0, 1, \lambda\}$ and $S = \{0, 1, \mu\}$ where $\lambda, \mu\colon X \to I$ are defined by $\lambda(a) = 1$; $\lambda(b) = 2/3$; $\lambda(c) = 1/2$ and $\mu(a) = 1$; $\mu(b) = 0$; $\mu(c) = 0$. Clearly T and S are fuzzy topologies on X. Consider $f\colon (X, T) \to (X, S)$ defined by $f(x) = x$ for each $x \in X$. Since $f^{-1}(0) = 0$; $f^{-1}(1) = 1$; $f^{-1}(\mu) = \mu$, we conclude the μ is a fuzzy β-open in (X, T) (since $\operatorname{cl}\mu = 1$). Therefore f is fuzzy almost irresolute. Then μ is fuzzy open in (X, S) but it is not fuzzy semi-open in (X, T) (since $\operatorname{int}\mu = 0$). Hence f is not fuzzy semi-continuous.

Remark 3.1. From Examples 3.1 to 3.6 and the diagram given after Definition 3.1, we have the following table of implications.

$\Rightarrow$	a	b	c	d	e	f	g	h
a	1	1	1	1	0	0	0	1
b	0	1	1	1	0	0	0	1
c	0	0	1	1	0	0	0	1
d	0	0	0	1	0	0	0	1
e	1	1	1	1	1	1	1	1
f	0	1	1	1	0	1	1	1
g	0	0	1	1	0	0	1	1
h	0	0	0	0	0	0	1	1

1 represent “implies” and 0 represent “does not imply”

In the above table,
a—fuzzy stongly α-irresolute,
b—fuzzy α-irresolute,
c—fuzzy semi α-irresolute,
d—fuzzy semi-continuous,
e—fuzzy strongly semi-continuous,
f—fuzzy stongly α-continuous,
g—fuzzy irresolute,
h—fuzzy almost irresolute.

Theorem 3.1. *If f is a function from a fuzzy topological space (X,T) to another fuzzy topological space (Y,S), then the following assertions are equivalent.*
(a) *f is fuzzy semi α-irresolute.*
(b) *For each fuzzy point $x_p \in X$ and each fuzzy α-open set λ in Y such that $f(x_p) \leqslant \lambda$, there exists a fuzzy semi-open μ such that $x_p \leqslant \mu$ and $f(\mu) < \lambda$.*
(c) *$f^{-1}(\lambda) < \mathrm{cl}(\mathrm{int}(f^{-1}(\lambda)))$ for every fuzzy α-open set λ in Y.*
(d) *$f^{-1}(\eta)$ is fuzzy semi-closed in X for every fuzzy α-closed set η in Y.*
(e) *$\mathrm{int}(\mathrm{cl}(f^{-1}(\delta))) < f^{-1}(\alpha\text{-}\mathrm{cl}(\delta))$ for every fuzzy set δ in Y.*
(f) *$f(\mathrm{int}(\mathrm{cl}(\varrho))) < \alpha\text{-}\mathrm{cl}(f(\varrho))$ for every fuzzy set ϱ in X.*

Proof. (a) $\Rightarrow$ (b) Let us assume that f is fuzzy semi α -irresolute. Suppose x_p is a fuzzy point X and λ is fuzzy α-open in Y such that $f(x_p) \leqslant \lambda$. Then $x_p \in f^{-1}f(x_p) \leqslant f^{-1}(\lambda)$. Let $\mu = f^{-1}(\lambda)$, which is a fuzzy semi-open set in (X,T) (by (a)) such that $x_p \leqslant \mu$. Now $f(\mu) = f(f^{-1}(\mu)) \leqslant \lambda$. Hence (a) $\Rightarrow$ (b) is proved.

(b) $\Rightarrow$ (c) Let λ be any fuzzy α-open set in Y. Let x_p be any fuzzy point in X such that $f(x_p) \leqslant \lambda$. Then $x_p \in f^{-1}(\lambda)$. By (b), there exists a fuzzy semi-open set μ of X such that $x_p \leqslant \mu$ and $f(\mu) < \lambda$. We obtain $x_p \in \mu \leqslant f^{-1}f(\mu) \leqslant f^{-1}(\lambda)$, $x_p \in \mu \leqslant f^{-1}(\lambda)$. We have $x_p \in \mu \leqslant \mathrm{cl}(\mathrm{int}\,\mu) \leqslant \mathrm{cl}(\mathrm{int}\, f^{-1}(\lambda))$. Since $x_p \in f^{-1}(\lambda)$ and $x_p \in \mathrm{cl}(\mathrm{int}\, f^{-1}(\lambda))$, we have $f^{-1}(\lambda) \leqslant \mathrm{cl}(\mathrm{int}\, f^{-1}(\lambda))$. Hence (b) $\Rightarrow$ (c) is proved.

(c) $\Rightarrow$ (d) Let η be a fuzzy α-closed set in Y. Then $1-\eta$ is a fuzzy α-open set in Y. By (c), we get $f^{-1}(1-\eta) \leqslant \mathrm{cl}(\mathrm{int}(f^{-1}(1-\eta)))$. On the other hand, $1 - f^{-1}(\eta) \leqslant \mathrm{cl}(\mathrm{int}(1 - f^{-1}(\eta))) = \mathrm{cl}(1 - \mathrm{cl}\, f^{-1}(\eta)) = 1 - \mathrm{int}\,\mathrm{cl}\, f^{-1}(\eta)$. We obtain $1 - f^{-1}(\eta) \leqslant 1 - \mathrm{int}\,\mathrm{cl}\, f^{-1}(\eta)$. So we have $\mathrm{int}\,\mathrm{cl}(f^{-1}(\eta)) \leqslant f^{-1}(\eta)$. Therefore $f^{-1}(\eta)$ is fuzzy semi-closed in X. Hence (c) $\Rightarrow$ (d) is proved.

(d) $\Rightarrow$ (e) Let δ be a fuzzy set in Y. Then $\alpha\text{-}\mathrm{cl}(\delta)$ is a fuzzy α-closed in Y. By (d), since $f^{-1}(\alpha\text{-}\mathrm{cl}(\delta))$ is fuzzy semi-closed in X, we have $\mathrm{int}\ \mathrm{cl}(f^{-1}(\alpha\text{-}\mathrm{cl}(\delta))) \leqslant f^{-1}(\alpha\text{-}\mathrm{cl}(\delta))$. Thus, we have $\mathrm{int}(\mathrm{cl}(f^{-1}(\delta))) \leqslant f^{-1}(\alpha\text{-}\mathrm{cl}(\delta))$. Hence (d) $\Rightarrow$ (e) is proved.

(e) $\Rightarrow$ (f) Let ϱ be a fuzzy set in X. By (e), we have $\operatorname{int}(\operatorname{cl}(\varrho)) \leqslant \operatorname{int}(\operatorname{cl}(f^{-1}(f(\varrho)))) \leqslant \operatorname{int}(\operatorname{cl}(f^{-1}(\alpha\text{-cl}\, f(\varrho)))) \leqslant f^{-1}(\alpha\text{-cl}\, f(\varrho))$. Then $\operatorname{int}(\operatorname{cl}(\varrho)) \leqslant f^{-1}(\alpha\text{-cl}\, f(\varrho))$, and we get $f(\operatorname{int}(\operatorname{cl}(\varrho)) \leqslant \alpha\text{-cl}\, f(\varrho)$. Hence (e) $\Rightarrow$ (f) is proved.

(f) $\Rightarrow$ (a) Let λ be a fuzzy α-open set in Y. Since $f^{-1}(1-\lambda) = 1 - f^{-1}(\lambda)$ is a fuzzy set in X, by (f) we obtain $f(\operatorname{int}\operatorname{cl}(f^{-1}(1-\lambda)) \leqslant \alpha\text{-cl}\, f(f^{-1}(1-\lambda)) \leqslant \alpha\text{-cl}(1-\lambda) = 1 - \alpha\text{-int}\,\lambda = 1 - \lambda$. Therefore

$$f(\operatorname{int}\operatorname{cl} f^{-1}(1-\lambda)) \leqslant 1 - \lambda \tag{1}$$

Now

$$\begin{aligned} 1 - \operatorname{cl}(\operatorname{int}(f^{-1}(\lambda))) &= \operatorname{int}(1 - \operatorname{int} f^{-1}(\lambda)) = \operatorname{int}(\operatorname{cl}(1 - f^{-1}(\lambda)) \\ &\leqslant f^{-1}(f(\operatorname{int}\operatorname{cl}(f^{-1}(1-\lambda)))) \end{aligned} \tag{2}$$

Using (1) in (2) we get $1 - \operatorname{cl}\operatorname{int} f^{-1}(\lambda) \leqslant f^{-1}(1-\lambda) = 1 - f^{-1}(\lambda)$, which implies that $f^{-1}(\lambda) \leqslant \operatorname{cl}\operatorname{int} f^{-1}(\lambda)$. Therefore $f^{-1}(\lambda)$ is fuzzy semi-open in X. Hence f is fuzzy semi α-irresolute. Hence (f) $\Rightarrow$ (a) is proved. $\square$

The following four lemmas taken from [1] and [2], are given here for convenience of the reader.

Lemma 3.1 [1]. *Let $f\colon X \to Y$ be a mapping and $\{\lambda_\alpha\}$ a family of fuzzy sets in Y. Then* (a) $f^{-1}(\bigcup \lambda_\alpha) = \bigcup f^{-1}(\lambda_\alpha)$ *and* (b) $f^{-1}(\bigcap \lambda_\alpha) = \bigcap f^{-1}(\lambda_\alpha)$.

Lemma 3.2 [1]. *For mappings $f_i\colon X_i \to Y_i$ and fuzzy sets λ_i in Y, $i = 1, 2$ we have $(f_1 \times f_2)^{-1}(\lambda_1 \times \lambda_2) = f_1^{-1}(\lambda_1) \times f_2^{-1}(\lambda_2)$.*

Lemma 3.3 [1]. *Let $g\colon X \to X \times Y$ be the graph of a mapping $f\colon X \to Y$. If λ is a fuzzy set in X and μ is a fuzzy set in Y, then $g^{-1}(\lambda \times \mu) = \lambda \wedge f^{-1}(\mu)$.*

Lemma 3.4 [2]. *Let X and Y be fuzzy topological spaces such that X is product related to Y. Then the product $\lambda \times \mu$ of a fuzzy α-open (pre-open) set λ in X and a fuzzy α-open (pre-open) set μ in Y is a fuzzy α-open (pre-open) set in the fuzzy product space $X \times Y$.*

Theorem 3.2. *If $f_i\colon X_i \to Y_i$ $(i = 1, 2)$ are fuzzy semi α-irresolute and X_1 is product related to X_2, then $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ is fuzzy semi α-irresolute.*

P r o o f. Let $A = \bigvee(\lambda_i \times \mu_i)$, where λ_i and μ_i's are fuzzy α-open sets in Y_1 and Y_2, respectively. Since Y_1 is product related to Y_2, we have by Lemma 3.4 that $A = \bigvee(\lambda_i \times \mu_i)$ is fuzzy α-open in $Y_1 \times Y_2$. Using Lemmas 3.1 and 3.2, we obtain $(f_1 \times f_2)^{-1}(A) = (f_1 \times f_2)^{-1}(\bigvee(\lambda_i \times \mu_i)) = \bigvee(f_1^{-1}(\lambda_i) \times f_2^{-1}(\mu_i))$. Since f_1 and f_2 are fuzzy semi α-irresolute, we conclude that $(f_1 \times f_2)^{-1}(A)$ is fuzzy semi-open in $X_1 \times X_2$ and hence $f_1 \times f_2$ is fuzzy semi α-irresolute. $\square$

Theorem 3.3. *Let $f\colon X \to Y$ be a function and assume that X is product related to Y. If the graph $g\colon X \to X \times Y$ of f is fuzzy semi α-irresolute, then so is f.*

P r o o f. Let λ be a fuzzy α-open set in Y. Then, by Lemma 3.3, we have $f^{-1}(\lambda) = 1 \wedge f^{-1}(\lambda) = g^{-1}(1 \times \lambda)$. Now, $1 \times \lambda$ is a fuzzy α-open set in $X \times Y$. Since g is fuzzy semi α-irresolute, $g^{-1}(1 \times \lambda)$ is fuzzy semi-open in X. Hence $f^{-1}(\lambda)$ is fuzzy semi-open in X. Therefore f is fuzzy semi α-irresolute. $\square$

Theorem 3.4. *If a function $f\colon X \to \prod Y_i$ is fuzzy semi α-irresolute, then $P_i \circ f\colon X \to Y_i$ is fuzzy semi α-irresolute, where P_i is the projection of $\prod Y_i$ onto Y_i.*

P r o o f. Let λ_i be any fuzzy α-open set in Y_i. Since P_i is a fuzzy continuous and fuzzy open set, it is a fuzzy α-open set. Now $P_i\colon \prod Y_i \to Y_i$; $P_i^{-1}(\lambda_i)$ is fuzzy α-open in $\prod Y_i$. Therefore, P_i is a fuzzy α-irresolute function. Now $(P_i \circ f)^{-1}(\lambda) = f^{-1}(P_i^{-1}(\lambda_i))$, since f is fuzzy semi α-irresolute. Hence $f^{-1}(P_i^{-1}(\lambda_i))$ is a fuzzy semi-open set, since $P_i^{-1}(\lambda_i)$ is a fuzzy α-open set. Hence $(P_i \circ f)$ is fuzzy semi α-irresolute. $\square$

Lemma 3.5. *λ is a fuzzy semi-open set if and only if* $\operatorname{cl}\lambda = \operatorname{cl}\operatorname{int}\lambda$.

P r o o f. Necessity: Suppose λ is a fuzzy semi-open set. Then $\lambda \leqslant \operatorname{cl}\operatorname{int}\lambda$. Therefore

$$\operatorname{cl}\lambda \leqslant \operatorname{cl}[\operatorname{cl}\operatorname{int}\lambda] = \operatorname{cl}\operatorname{int}\lambda. \tag{1}$$

Also

$$\operatorname{int}\lambda \leqslant \lambda \Rightarrow \operatorname{cl}\operatorname{int}\lambda \leqslant \operatorname{cl}\lambda. \tag{2}$$

From (1) and (2) we have $\operatorname{cl}\lambda = \operatorname{cl}\operatorname{int}\lambda$. Sufficiency: By hypothesis, we have $\operatorname{int}\lambda \leqslant \lambda \leqslant \operatorname{cl}\lambda = \operatorname{cl}\operatorname{int}\lambda$. Therefore $\lambda \leqslant \operatorname{cl}\operatorname{int}\lambda$. Hence λ is a fuzzy semi-open set. $\square$

Lemma 3.6. *If λ is a fuzzy semi-open set and $\lambda \neq 0$, then* $\operatorname{int}\lambda \neq 0$.

P r o o f. Let λ be a fuzzy semi-open set such that $\lambda \neq 0$. Then by Lemma 3.5,

$$\operatorname{cl}\lambda = \operatorname{cl}\operatorname{int}\lambda. \tag{1}$$

If $\operatorname{int}\lambda = 0$, then from (1) we get $\operatorname{cl}\lambda = 0$ and hence $\lambda = 0$. This is a contradiction to our assumption. Therefore $\operatorname{int}\lambda \neq 0$. $\square$

Lemma 3.7. *Let $(X_\alpha, T_\alpha)_{\alpha\in\Gamma}$ be any family of fuzzy topological spaces and λ_α a fuzzy subset of X_α for each $\alpha \in \Gamma$. Then*

(1) $\operatorname{int}\prod \lambda_\alpha = \prod \operatorname{int}\lambda_\alpha$ *if* $\lambda_\alpha = X_\alpha$ *except for a finite number of* $\alpha \in \Gamma$ *and* $\prod \operatorname{int}\lambda_\alpha \neq 0$.

(2) $\operatorname{cl}(\prod \lambda_\alpha) = \prod \operatorname{cl}\lambda_\alpha$.

Proof. It is easy to prove. □

Theorem 3.5. *Let $(X_\alpha, T_\alpha)_{\alpha\in T}$ be any family of fuzzy topological spaces. Let $X = \prod\limits_{\alpha\in\Gamma} X_\alpha$, let λ_{α_j} be any fuzzy subset of X_{α_j}, $\alpha_j \in \Gamma$ for each $j = 1, 2, \dots n$. Let $\lambda = \prod\limits_{j=1}^{n} \lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta$ be any fuzzy subset of X. Then λ is a fuzzy semi-open set in $X \Leftrightarrow \lambda_{\alpha_j}$ is a fuzzy semi-open set in X_{α_j} for each $j = 1, 2, \dots n$.*

Proof. Necessity: Suppose λ is fuzzy semi-open in X. Then by Lemma 3.6 we have int $\lambda \neq 0$ and hence $0 \neq \operatorname{int}\lambda = \operatorname{int}\Big[\prod\limits_{j=1}^{n} \lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta\Big] = \prod\limits_{j=1}^{n} \operatorname{int}\lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta$. Therefore int $\lambda_{\alpha_j} \neq 0$ and hence $\lambda_{\alpha_j} \neq 0$. Since λ is fuzzy semi-open in X, by Lemmas 3.5 and 3.7 we obtain $\prod\limits_{j=1}^{n} \operatorname{cl}\operatorname{int}\lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta = \operatorname{cl}\operatorname{int}\Big[\prod\limits_{j=1}^{n} \lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta\Big] = \operatorname{cl}\operatorname{int}\lambda = \operatorname{cl}\lambda = \operatorname{cl}\Big[\prod\limits_{j=1}^{n} \lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta\Big] = \prod\limits_{j=1}^{n} \operatorname{cl}\lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta$. Therefore $\prod\limits_{j=1}^{n} \operatorname{cl}\operatorname{int}\lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta = \prod\limits_{j=1}^{n} \operatorname{cl}\lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta$. Thus we obtain $\operatorname{cl}\operatorname{int}\lambda_{\alpha_j} = \operatorname{cl}\lambda_{\alpha_j}$ for each $j = 1, 2, \dots n$ and hence by Lemma 3.5, λ_{α_j} is a fuzzy semi-open set in X_{α_j} for each $j = 1, 2, \dots n$.

Sufficiency: Suppose λ_{α_j} is fuzzy semi-open in X_{α_j} for each $j = 1, 2, \dots n$. Then $\lambda_{\alpha_j} \neq 0$ for each j ($j = 1, 2, \dots n$) because $\lambda \neq 0$. Therefore, by Lemma 3.6 we have int $(\lambda_{\alpha_j}) \neq 0$. Hence $\prod\limits_{j=1}^{n} \operatorname{int}\lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta \neq 0$. Since λ_{α_j} is a fuzzy semi-open set in X_{α_j} for each j ($j = 1, 2, \dots n$), by Lemmas 3.5 and 3.7 we have $\operatorname{cl}\operatorname{int}\lambda = \operatorname{cl}\operatorname{int}\Big[\prod\limits_{j=1}^{n} \lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta\Big] = \operatorname{cl}\Big[\prod\limits_{j=1}^{n} \operatorname{int}\lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta\Big] = \Big[\prod\limits_{j=1}^{n} \operatorname{cl}\operatorname{int}\lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta\Big] = \Big[\prod\limits_{j=1}^{n} \operatorname{cl}\lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta\Big] = \operatorname{cl}\Big[\prod\limits_{j=1}^{n} \lambda_{\alpha_j} \times \prod\limits_{\beta\neq\alpha_j} X_\beta\Big] = \operatorname{cl}\lambda$. Thus by Lemma 3.5 we obtain that λ is a fuzzy semi-open set in X. Hence the theorem is proved. □

Theorem 3.6. *Let $(X_\alpha, T_\alpha)_{\alpha\in\Gamma}$ be any family of fuzzy topological spaces. Let $X = \prod\limits_{\alpha\in\Gamma} X_\alpha$. Let λ_{α_j} be any fuzzy subset of X_{α_j}, $\alpha_j \in \Gamma$ for each j ($j = 1, 2, \dots n$).*

Let $\lambda = \prod_{j=1}^{n} \lambda_{\alpha_j} \times \prod_{\beta \neq \alpha_j} X_\beta$ be any fuzzy subset of X. Then λ is a fuzzy α-open set in $X \Leftrightarrow \lambda_{\alpha_j}$ is a fuzzy α-open set in X_{α_j} for each j ($j = 1, \ldots, n$).

Proof. The proof is similar to that of Theorem 3.5 and is thus omitted. □

Theorem 3.6. *If the fuzzy product function $f\colon \prod X_\alpha \to \prod Y_\alpha$ is fuzzy semi α-irresolute, then $f_\alpha\colon X_\alpha \to Y_\alpha$ is fuzzy semi α-irresolute for each $\alpha \in \Gamma$.*

Proof. Let $\alpha_0 \in \Gamma$ be an arbitrary fixed index and λ_{α_0} any fuzzy α-open set in Y_{α_0}. Then, by Theorem 3.6, $\lambda_{\alpha_0} \times \prod_{\beta \neq \alpha_0} Y_\beta$ is fuzzy α-open in $\prod Y_\alpha$. Since f is fuzzy semi α-irresolute, $f^{-1}\Big(\lambda_{\alpha_0} \times \prod_{\beta \neq \alpha_0} Y_\beta\Big) = f_{a_0}^{-1}(\lambda_{\alpha_0}) \times \prod_{\beta \neq \alpha_0} X_\beta$ is a fuzzy semi-open set in $\prod X_\alpha$ and hence by Theorem 3.5, $f_{a_0}^{-1}(\lambda_{\alpha_0})$ is a fuzzy semi-open set in X_{α_0}. This shows that f_{α_0} is fuzzy semi α-irresolute. Hence the theorem is proved. □

Definition 3.2. Let (X, T) be any fuzzy topological space and let λ be any fuzzy set in X. Then λ is called a dense fuzzy set if $\operatorname{cl}\lambda = 1$ and λ is called a nowhere dense fuzzy set if $\operatorname{int}\operatorname{cl}\lambda = 0$.

Theorem 3.8. *If a function $f\colon (X, T) \to (Y, S)$ is fuzzy semi α-irresolute, then $f^{-1}(\lambda)$ is fuzzy semi-closed in X for any nowhere dense fuzzy set λ of Y.*

Proof. Let λ be nowhere dense fuzzy set in Y. Then $\operatorname{int}\operatorname{cl}\lambda = 0$. Now $1 - \operatorname{int}(\operatorname{cl}(\lambda) = 1 \Rightarrow \operatorname{cl}(1 - \operatorname{cl}(\lambda)) = 1 \Rightarrow \operatorname{cl}(\operatorname{int}(1 - \lambda)) = 1$. Since $\operatorname{int} 1 = 1$, we have $\operatorname{int}(\operatorname{cl}(\operatorname{int}(1 - \lambda))) = \operatorname{int} 1 = 1$. Therefore $1 - \lambda \leqslant \operatorname{int}\operatorname{cl}\operatorname{int}(1 - \lambda) = 1$. Then $1 - \lambda$ is a fuzzy α-open set in Y. Since f is fuzzy semi α-irresolute, $f^{-1}(1 - \lambda)$ is a fuzzy semi-open set in X. Consequently, $f^{-1}(1 - \lambda) = 1 - f^{-1}(\lambda)$ is a fuzzy semi-open set in X. Hence $f^{-1}(\lambda)$ is fuzzy semi-closed set in X. □

Theorem 3.9. *The following assertions hold for functions $f\colon X \to Y$ and $g\colon Y \to Z$:*

(a) *If f is fuzzy irresolute and g is fuzzy semi α-irresolute, then $g \circ f$ is fuzzy semi α-irresolute.*
(b) *If f is fuzzy semi-continuous and g is fuzzy strongly α-irresolute, then $g \circ f$ is fuzzy semi α-irresolute.*
(c) *If f is fuzzy semi α-irresolute and g is fuzzy strongly α-continuous, then $g \circ f$ is fuzzy irresolute.*
(d) *If f is fuzzy strongly α-continuous and g is fuzzy semi α-irresolute, then $g \circ f$ is fuzzy α-irresolute.*

Proof. (a) Let μ be a fuzzy α-open set in Z. Since g is fuzzy semi α-irresolute, $g^{-1}(\mu)$ is fuzzy semi-open in Y. Now $(g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))$, since f is fuzzy irresolute. Then $f^{-1}(g^{-1}(\mu))$ is fuzzy semi-open in X and $g^{-1}(\mu)$ is fuzzy semi-open in Y. Hence $(g \circ f)$ is fuzzy semi α-irresolute.

(b) Let μ be fuzzy α-open in Z. Since g is fuzzy strongly α-irresolute, $g^{-1}(\mu)$ is a fuzzy open set in Y. Now $(g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))$. Since $g^{-1}(\mu)$ is a fuzzy open set in Y and f is fuzzy semi-continuous, we conclude that $f^{-1}(g^{-1}(\mu))$ is fuzzy semi-open in X. Hence $(g \circ f)$ is fuzzy semi α-irresolute.

(c) Let μ be any fuzzy semi-open set in Z. Since g is fuzzy strongly α-continuous, $g^{-1}(\mu)$ is fuzzy α-open in Y. Now $(g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))$, since $g^{-1}(\mu)$ is fuzzy α-open in Y and f is fuzzy semi α-irresolute. Hence $f^{-1}(g^{-1}(\mu))$ is fuzzy semi-open in X. Hence $(g \circ f)$ is fuzzy irresolute.

(d) Let μ be any fuzzy α-open set in Z. Since g is fuzzy semi α-irresolute, $g^{-1}(\mu)$ is fuzzy semi-open in Y. Now $(g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}\ (\mu))$. Since $g^{-1}\ (\mu)$ is fuzzy semi-open in Y and f is fuzzy strongly α-continuous, $f^{-1}(g^{-1}(\mu))$ is a fuzzy α-open set in X. Hence $(g \circ f)$ is fuzzy α-irresolute. □

Acknowledgement. The authors wish to thank the referee for his suggestions and corrections which helped to improve this paper.

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