ON VAN DOUWEN SPACES AND RETRACTS OF $\beta\mathbb{N}$

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Abstract. Eric van Douwen produced in 1993 a maximal crowded extremally disconnected regular space and showed that its Stone-Čech compactification is an at most two-to-one image of $\beta\mathbb{N}$. We prove that there are non-homeomorphic such images. We also develop some related properties of spaces which are absolute retracts of $\beta\mathbb{N}$ expanding on earlier work of Balcar and Błaszczyk (1990) and Simon (1987).

Keywords: $\beta\mathbb{N}$, retracts, two to one map, Stone-Čech compactification

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1. Introduction

In the paper [3], van Douwen proved the existence of a compact crowded separable extremally disconnected (ED) space which is the at most 2-to-1 continuous image of $\beta\mathbb{N}$. We are interested in the question of whether such an image (henceforth a vD-space) is unique. This is especially interesting now that we have another construction of such a space by a result of Protasov [8, 3.9, p. 235] who constructed a special idempotent of $(\beta\mathbb{N}, +)$ (see Proposition 10). As is the case with any compact separable ED space, each vD-space can be embedded into $\beta\mathbb{N}$ as a retract. There is an unresolved problem of A. Bella, A. Błaszczyk, and A. Szymański, to characterize the so-called absolute retracts of $\beta\mathbb{N}$ (namely a space which is embeddable in $\beta\mathbb{N}$, and every embedding is a retract). Simon [13] presented a very interesting construction of a retract of $\beta\mathbb{N}$ which is not an absolute retract. This had been done earlier by others with special set-theoretic hypotheses (see [14], [15]). Simon's construction and van Douwen's each involve some form of *irresolvability* so we explore Simon's ideas further and see how they might provide information about vD-spaces and/or absolute retracts. We use Kunen's notion of independent matrices to show there are non-homeomorphic vD-spaces (one of which is an absolute retract). Using the set-theoretic assumption, Martin's Axiom for countable posets, we are able to construct a compact vD-space which we can show is not an absolute retract.

Definition 1. A compact space E is a vD-space if E is crowded (has no isolated points), extremally disconnected, and if there is a $\leqslant$2-to-1 map from $\beta\mathbb{N}$ onto E.

Definition 2. A countable space S is a vD-space if S is crowded and there is a 1-to-1 function from $\mathbb{N}$ onto S which extends to a $\leqslant$2-to-1 function from $\beta\mathbb{N}$ onto βS.

In this same spirit, we introduce a new kind of retraction.

Definition 3. A retraction r from $\beta\mathbb{N}$ onto E will be called a 1-to-1 retraction if $r \restriction \mathbb{N}$ is 1-to-1. A space E will be called an absolute 1-to-1 retract of $\beta\mathbb{N}$ if each homeomorphic copy of E in $\beta\mathbb{N}$ is a 1-to-1 retract.

Proposition 4. *Each countable vD-space S is extremally disconnected.*

Proof. Let f denote the 1-to-1 function from $\mathbb{N}$ onto S such that f^β is $\leqslant$2-to-1 from $\beta\mathbb{N}$ onto βS. Since S is crowded, it follows that f^β maps $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ onto βS. By Zorn's Lemma, there is a closed subset $K \subset \mathbb{N}^*$ such that $f \restriction K$ is irreducible. Let S' be the preimage of S in K. Since f^β is $\leqslant$2-to-1, and $f[\mathbb{N}] = S$, it follows that f^β is 1-to-1 on S'. Since K and βS are compact, f^β is a homeomorphism on S'. Now S is extremally disconnected since each countable subset of $\beta\mathbb{N}$ is extremally disconnected. □

Therefore, of course, if S is a countable vD-space, then βS is a compact vD-space. However, it is easily seen that not every countable dense subset of a compact vD-space is itself a vD-space. If E is a compactification of a set S, then $p \in E$ is a *far point* of S (or *far from* S) if it is not in the closure of any countable discrete subset $D \subset S$. We will say that a point $p \in E$ is a *near-point* of S (or *near to* S) if it is the limit of a discrete subset of S.

Proposition 5 [3]. *If $f\colon \beta\mathbb{N} \twoheadrightarrow E$ is $\leqslant$2-to-1 and E is a compact vD-space, then $f \restriction \mathbb{N}$ is 1-to-1, $S = f[\mathbb{N}]$ is a countable vD-space, and for each $p \in E$, $f^{-1}(p) \setminus \mathbb{N}$ is a singleton if and only if p is a far point of $S \setminus \{p\}$. In particular, each point of S is a far point of $S \setminus \{p\}$.*

Proposition 6 [3]. *Let S be a countable regular space, then the following are equivalent:*

(1) *S is a countable vD-space,*

(2) *S is extremally disconnected, crowded and nodec (i.e. nowhere dense sets are closed),*
(3) *for each $A \subset S$, A is open if and only if A is crowded.*

Although, strictly speaking, the following result is presented in van Douwen's paper for the case X is the rationals, the more general statement follows from the same proof.

Theorem 7 [3]**.** *If X is any countable crowded Tychonoff space, there is a stronger topology on X which contains a dense subspace S which is a vD-space.*

Another formulation for being a vD-space is useful in some constructions.

Proposition 8. *A countable space S is a vD-space if it is regular, crowded, and for each $A \subset S$, there is a partition of S, $\{W_n\colon n \in \omega\}$, by clopen sets such that for each n, one of $\{A \cap W_n, W_n \setminus A\}$ is finite.*

P r o o f. It is easy to see that Proposition 6(2) implies that a vD-space S will have this property by using the fact that the boundary of A will be closed discrete in a countable zero-dimensional space. For the converse note simply that each crowded A is open since each W_n will meet it in an open set. $\square$

Definition 9. For $p \in \beta\mathbb{N}$ and $k \in \mathbb{N}$, the ultrafilter $k + p$ is defined as the set $\{A \subset \mathbb{N}\colon (\mathbb{N} \cap (A - k)) \in p\}$. Then, for $q \in \beta\mathbb{N}$, $q + p$ is defined as the image of q by the mapping $\varrho_p\colon \beta\mathbb{N} \to \beta\mathbb{N}$ where $\varrho_p(k) = k + p$ for each $k \in \mathbb{N}$.

The following result is easily deduced (and known) from the comments immediately following Theorem 3.9 of [8].

Proposition 10 [Protasov]**.** *There is an ultrafilter $p \in \mathbb{N}^*$ such that $\varrho_p(p) = p$ and $\varrho_p(q) \neq p$ for all $q \neq p$. It also follows that $\varrho_p \restriction \beta\mathbb{N}$ is $\leqslant$2-to-1, $\varrho_p[\beta\mathbb{N}]$ is a compact vD-space, and $\varrho_p[\mathbb{N}]$ is a vD-space.*

In fact, $\varrho_p[\mathbb{N}] = \mathbb{N}+p$ is a vD-space which is a homogeneous topological semi-group.

2. Retracts

It has been shown (see [2]) that each minimal subset of $\beta\mathbb{N}$ of the form $\varrho_p[\beta\mathbb{N}]$, where $\varrho_p(p) = p$, is homeomorphic to $E(2^{\mathfrak{c}})$. Recall that the topology on $E(2^{\mathfrak{c}})$ is determined by the irreducible map $\varphi\colon E(2^{\mathfrak{c}}) \twoheadrightarrow 2^{\mathfrak{c}}$, where for each open set $U \subset 2^{\mathfrak{c}}$, $\varphi^{-1}(U)$ has clopen closure. Each point $p \in E(2^{\mathfrak{c}})$ can be thought of as the ultrafilter of regular open sets $U \subset 2^{\mathfrak{c}}$ such that p is in the closure of $\varphi^{-1}(U)$.

Proposition 11. *$E(2^{\mathfrak{c}})$ is an absolute retract, in fact, it is an absolute 1-to-1 retract.*

Proof. Assume that $K \subset \beta\mathbb{N}$ is homeomorphic to $E(2^{\mathfrak{c}})$ and let g denote an irreducible map from K onto $2^{\mathfrak{c}}$. For each $\alpha \in \mathfrak{c}$, let $[\langle\alpha, 0\rangle]$ and $[\langle\alpha, 1\rangle]$ denote the canonical basic clopen subsets of $2^{\mathfrak{c}}$. There is subset A_α of $\mathbb{N}$ with the property that $\overline{A_\alpha} \cap K$ is equal to $g^{-1}([\langle\alpha, 0\rangle])$. For each integer $n \in \mathbb{N}$, define the function $z_n \in 2^{\mathfrak{c}}$ by $z_n(\alpha) = 0$ if and only if $n \in A_\alpha$. It is easily checked that $\{z_n \colon n \in \mathbb{N}\}$ is a dense subset of $2^{\mathfrak{c}}$. For each $z \in 2^{\mathfrak{c}}$, it is also easily checked that $g^{-1}(p)$ has cardinality $2^{\mathfrak{c}}$. For each n, let $p_n \in g^{-1}(z_n)$ be chosen, so that $p_n \neq p_m$ for $n \neq m$. Since g is irreducible, $\{p_n \colon n \in \mathbb{N}\}$ is dense in K. We leave as an exercise that $f\colon \mathbb{N} \twoheadrightarrow \{p_n \colon n \in \mathbb{N}\}$ where $f(n) = p_n$, lifts to a retraction from $\beta\mathbb{N}$ onto K. □

Let X be any set and let $\mathcal{F}$ be any filter of subsets of X. A doubly-indexed family $\{A(\alpha, \beta) \colon (\alpha, \beta) \in I \times J\}$ is called an $I \times J$-independent matrix mod $\mathcal{F}$, if

(1) for each finite function ϱ from I into J and each $F \in \mathcal{F}$, the set $A_\varrho = F \cap \bigcap\{A(\alpha, \varrho(\alpha)) \colon \alpha \in \operatorname{dom}(\varrho)\}$ is not empty,

(2) for each $\alpha \in I$ and $\beta \neq \gamma \in J$, $A(\alpha, \beta) \cap A(\alpha, \gamma)$ is disjoint from some member of $\mathcal{F}$.

If no filter $\mathcal{F}$ is mentioned, then it is assumed to be the co-finite filter.

A space X is (strongly) *irresolvable* if (each open subset of) it does not contain disjoint dense subsets. Another interesting property of countable vD-spaces is, that they are strongly irresolvable.

We expand this notion to more general families of dense subsets.

Definition 12. A space X is (ω, κ)-irresolvable if there is a set $Y \subset X$ of cardinality κ so that for any countable $D \subset X \setminus Y$, D is nowhere dense.

Definition 13. A space X is $(\kappa \times \lambda)$-resolvable if there is an $\kappa \times \lambda$-independent matrix which consists of dense subsets of X. A space X is $(\omega, \kappa \times \lambda)$-resolvable, if it has a countable dense subset which is $(\kappa \times \lambda)$-resolvable.

Simon [13] constructs a retract of $\beta\mathbb{N}$ which is not an absolute retract and the notions of independent matrix and some form of irresolvability were shrewdly exploited. In particular, Simon produces an (ω, ω)-irresolvable retract of $\beta\mathbb{N}$.

Proposition 14. *There is a retract G of $\beta\mathbb{N}$ which is (ω, ω)-irresolvable and there is an embedding of G into $\beta\mathbb{N}$ which is not a retract.*

Simon constructs an embedding of G as a non-retract by using a $\mathfrak{c} \times \mathfrak{c}$-independent matrix to simultaneously construct G and the embedding of G. The space G is βG_ω for a now well-known space G_ω. The base set for G_ω is $\omega^{<\omega}$ (the family of functions from an integer into ω). For each $t \in G_\omega$, an ultrafilter $\mathcal{U}_t$ on ω is selected, and a set $U \subset S$ is open if it satisfies that for each $t \in U$, the set $\{n\colon t^\frown n \in U\}$ is in $\mathcal{U}_t$ (see also [4], [16]). The (ω,ω)-irresolvability of such spaces will hold if each $\mathcal{U}_t$ is chosen to be a weak P-point (ensuring that $\beta G_\omega \setminus G_\omega$ is $\aleph_0$-bounded). We explore in more detail what it takes to "kill" all potential retracts. Simon, in effect, shows that examples of *Simon type* (in which each $\mathcal{U}_t$ is a weak P-point), are not $(\omega, 1\times\omega_1)$-resolvable. Although Simon type spaces can be (ω,ω)-irresolvable, they are not irresolvable since, for example, $\bigcup\limits_n \{\omega^n\colon n \in J\}$ is a dense for each infinite subset $J \subset \omega$. The following result is inspired by Simon's construction.

Theorem 15. *If E is an absolute* 1*-to-*1 *retract of $\beta\mathbb{N}$, then either E is not $(\omega,\mathfrak{c})$-irresolvable or E is $(\omega,\mathfrak{c}\times\mathfrak{c})$-resolvable.*

P r o o f. If E is not $(\omega,\mathfrak{c})$-irresolvable then there is nothing to prove, so we fix a set $S \subset E$ of cardinality $\mathfrak{c}$ so that D is nowhere dense in E for all countable $D \subset E \setminus S$. Let $\mathfrak{c}+1$ denote the one-point compactification of $\mathfrak{c}$ with the discrete topology. It is shown, in [5], that there is a mapping f from $\mathbb{N}^*$ onto $E \times (\mathfrak{c}+1)^{\mathfrak{c}}$. For each finite function ϱ from a subset of $\mathfrak{c}$ into $\mathfrak{c}$, we let $[\varrho]$ denote the clopen subset of $(\mathfrak{c}+1)^{\mathfrak{c}}$ consisting of all functions that extend ϱ. The mapping f is a mapping version of independent matrices as explored in [5]. For each $(\alpha,\beta) \in \mathfrak{c}\times\mathfrak{c}$, we fix an infinite $B(\alpha,\beta)$ so that $(B(\alpha,\beta))^* = f^{-1}(E \times [\langle(\alpha,\beta)\rangle])$, then the family $\{B(\alpha,\beta)\colon (\alpha,\beta) \in \mathfrak{c}\times\mathfrak{c}\}$ will be a $\mathfrak{c}\times\mathfrak{c}$-independent matrix. For each function ϱ with finite domain contained in $\mathfrak{c}$ and range contained in $\mathfrak{c}$, let $B_\varrho = \bigcap\limits_{\alpha\in\operatorname{dom}(\varrho)} B(\alpha,\varrho(\alpha))$. Similar to the constructions in §3 of [5] we construct a descending chain $\{K_\alpha\colon \alpha < \mathfrak{c}\}$ of closed subsets of $K_0 = \mathbb{N}^*$, sets $I_\alpha \subset \mathfrak{c}$ and maintain the hypothesis that the function f_α, defined as $f \restriction K_\alpha$ composed with the projection map π_{I_α} from $E\times(\mathfrak{c}+1)^{\mathfrak{c}}$ onto $E\times(\mathfrak{c}+1)^{I_\alpha}$, maps K_α onto $E\times(\mathfrak{c}+1)^{I_\alpha}$. This induction will result in a copy $K_{\mathfrak{c}}$ of E since the mapping $f_{\mathfrak{c}}$ will be made to be irreducible (see inductive condition 2). Let $\{r_\alpha\colon \alpha \in \mathfrak{c}\}$ enumerate all the 1-to-1 functions from $\mathbb{N}$ into S. The plan is to select $I_{\alpha+1}$ and $K_{\alpha+1}$ in such a way that r_α will not correspond to a retract from $\mathbb{N}$ into the copy of S sitting in $K_{\mathfrak{c}}$. If we are not able to do this it will because r_α induces an $(I_\alpha \times c)$-independent matrix of dense subsets of its range (this is inductive condition 4). Let $\{A_\alpha\colon \alpha \in \mathfrak{c}\}$ be an enumeration of the infinite subsets of $\mathbb{N}$.

The inductive assumptions on $\beta < \alpha < \mathfrak{c}$ are:

(1) $K_\beta \supset K_\alpha$, $I_\beta \supset I_\alpha$, and $f_\alpha = \pi_{I_\alpha} \circ f$ maps K_α onto $E \times (\mathfrak{c}+1)^{I_\alpha}$,
(2) if $\pi_\emptyset \circ f_\beta$ maps $A^*_\beta \cap K_\alpha$ onto E, then $K_\alpha \subset A^*_\beta$,
(3) the set $\mathfrak{c}\setminus I_\alpha$ has cardinality at most $\omega\cdot|\alpha|$,
(4) either $r_\beta[A]$ is not dense in E for some $A \subset \mathbb{N}$ with $A^* \supset K_\alpha$, or $r_\beta[B_\varrho]$ is dense in E for each function ϱ into $\mathfrak{c}$ with $\operatorname{dom}(\varrho) \in [I_\alpha]^{<\omega}$.

If α is a limit, we may set $K_\alpha = \bigcap\{K_\beta\colon \beta < \alpha\}$ and $I_\alpha = \bigcap\{I_\beta\colon \beta < \alpha\}$. Compactness ensures that f_α maps K_α onto $E \times (\mathfrak{c}+1)^{I_\alpha}$.

Now suppose we have K_α and I_α and we show how to construct $K_{\alpha+1}$ and $I_{\alpha+1}$. If f_α maps $K_\alpha \cap A^*_\alpha$ onto $E \times (\mathfrak{c}+1)^{I_\alpha}$, then replace K_α by $K_\alpha \cap A^*_\alpha$. In this case, for notational convenience, let ϱ_0 denote the empty function. Otherwise, there is a function ϱ_0 into $\mathfrak{c}$ with $\mathrm{dom}(\varrho_0) \in [I_\alpha]^{<\omega}$ and a clopen $W_0 \subset E$, such that the image of $K_\alpha \cap A^*_\alpha$ is disjoint from $W_0 \cap [\varrho_0]$. We will ensure that $K_{\alpha+1}$ is contained in $B^*_{\varrho_0}$ that will guarantee that $\pi_\emptyset \circ f_{\alpha+1}$ will not be onto. Now we turn our attention to r_α. If there is some $\varrho \supset \varrho_0$ such that $r_\alpha[B_\varrho]$ is not dense in E, then choose such a ϱ. If no such ϱ exists, then let $\varrho = \varrho_0$. Define $K_{\alpha+1} = K_\alpha \cap B^*_\varrho$ and $I_{\alpha+1} = I_\alpha \setminus \mathrm{dom}(\varrho)$. Condition 4 will hold since either $r_\alpha[B_\varrho]$ is not dense in E, or $r_\alpha[B_\sigma]$ is dense in E for all suitable σ with $\mathrm{dom}(\sigma) \subset I_{\alpha+1}$ since $B_\sigma \supset B_{\varrho_0 \cup \sigma}$ and the latter is assumed to have dense image.

Assume now that $r\colon \mathbb{N} \to K_\mathfrak{c}$ is a 1-to-1 map that lifts to a retraction r^β from $\beta\mathbb{N}$ onto $K_\mathfrak{c}$. Notice that $f_\mathfrak{c} \circ r$ is a 1-to-1 map from $\mathbb{N}$ into E. Let A be the preimage of S under this map and fix any $\gamma \in \mathfrak{c}$ such that $r_\gamma \restriction A$ equal $f_c \circ r \restriction A$. We first check that $(N \setminus A)^*$ is disjoint from $K_\mathfrak{c}$, hence there is an $\alpha > \gamma$ such that K_α is contained in A^*. Since r is a retraction, r^β maps the clopen subset $K_\mathfrak{c} \cap (N \setminus A)^*$ to itself, and, by continuity, into the closure of $r[(N \setminus A)]$. However, $r[N \setminus A]$ is nowhere dense in K_c since $(f_\mathfrak{c} \circ r)[N \setminus A]$ is a countable subset of $E \setminus S$. Similarly, it follows that $r[\widetilde{A}]$ is a dense subset of $K_\mathfrak{c}$ (hence $r_\gamma[\widetilde{A}]$ is a dense subset of E) for all $\widetilde{A} \subset A$ such that $K_\mathfrak{c} \subset \widetilde{A}^*$. Therefore by inductive condition 4, E is $(\omega, \mathfrak{c} \times \mathfrak{c})$-resolvable. □

The set-theoretic principle $\mathrm{MA}_{\mathrm{ctble}}$ is a very weak form of Martin's Axiom. It is simply the usual statement for Martin's Axiom but restricted to countable posets.

Theorem 16. *If $\mathrm{MA}_{\mathrm{ctble}}$ holds, then any absolute retract of $\beta\mathbb{N}$, that is $(\omega, \mathfrak{c})$-irresolvable, is also $(\omega, \mathfrak{c} \times \mathfrak{c})$-resolvable.*

Before proving the result we record the following (almost) folklore result.

Proposition 17. *If $\mathrm{MA}_{\mathrm{ctble}}$ holds, then every crowded separable Hausdorff space of π-weight less than $\mathfrak{c}$ is $\mathfrak{c} \times \mathfrak{c}$-resolvable.*

Proof. Let S be a countable space, $\lambda < \mathfrak{c}$, and let $\{U_\alpha\colon \alpha < \lambda\}$ be a π-base for S. By a routine application of $\mathrm{MA}_{\mathrm{ctble}}$, we may choose countably many pairwise disjoint dense subsets, $\{S_n\colon n \in \mathbb{N}\}$, of S. Reindex the family by $T = \bigcup\limits_n \mathcal{P}(n)^{\mathcal{P}(n)}$, i.e. $\{S_t\colon t \in T\}$. Recall Simon's construction of a $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$-independent matrix, where $A(X,Y) = \{t \in T\colon t(X \cap n_t) = Y \cap n_t$, and $\mathrm{dom}(t) = n_t\}$. Let $\{X_\gamma\colon \gamma < \mathfrak{c}\}$ be an indexing of $\mathcal{P}(\mathbb{N})$. Construct our new $\mathfrak{c} \times \mathfrak{c}$-independent matrix $\{S(\beta,\gamma)\colon \beta, \gamma \in \mathfrak{c}\}$

by induction on $\beta+\gamma$ so that $S(\beta,\gamma)\cap S_t=\emptyset$ for $t\notin A(X_\beta,X_\gamma)$. What we have gained is that $S(\beta,\gamma)\cap S(\beta,\delta)$ is finite for $\gamma\neq\delta$. We use $\mathrm{MA}_{\mathrm{ctble}}$ to ensure that $S(\beta,\gamma)$ is dense and meets all finite intersections of the, fewer than $\mathfrak{c}$, previously chosen dense sets $S(\zeta,\delta)$. In fact, the necessary inductive assumption is that for $\zeta_0<\zeta_1<\ldots<\zeta_k$, and ξ_i such that $S(\zeta_i,\xi_i)$ have already been defined, there is an infinite set of n such that $\bigcap\limits_{i<k} S(\zeta_i,\xi_i)$ meets S_t for each $t\in\mathcal{P}(n)^{\mathcal{P}(n)}\cap\bigcap\limits_{i<k} A(X_{\zeta_i},Y_{\xi_i})$. We leave the details to the reader. □

2.1. Proof of Theorem 16. Let $S\subset E$ be the dense set of size $\mathfrak{c}$ that witnesses that E is $(\omega,\mathfrak{c})$-irresolvable, and let $\{s_n\colon n\in\mathbb{N}\}$ be a countable dense subset. Let $\{r_\alpha\colon \alpha\in\mathfrak{c}\}$ enumerate all the functions, r, from $\beta\mathbb{N}$ into E such that $r[\mathbb{N}]\subset S$ and let $\{A_\alpha\colon \alpha\in\mathfrak{c}\}$ enumerate all the infinite subsets of $\mathbb{N}$. For notational convenience assume that $A_n=\mathbb{N}$ and r_n is constant for each $n\in\omega$. We construct by induction on $\alpha<\mathfrak{c}$, closed sets $K_\alpha\subset\mathbb{N}^*$ that are $G_{|\alpha|}$-sets, and points $\{s(n,\alpha)\colon n\in\mathbb{N}\}\subset 2^\alpha$ dense in the compact set $E_\alpha\subset 2^\alpha$, and mapping f_α so that for $\beta<\alpha<\mathfrak{c}$:

(1) K_α is a subset of K_β and is equal to the intersection of a family of $|\alpha|$ many clopen subsets of $\beta\mathbb{N}$,

(2) the mapping f_α maps K_α onto E_α and $f_\alpha(x)\restriction\beta=f_\beta(x)$ for each $x\in K_\alpha$,

(3) there is a continuous map from E onto E_α that sends s_n to $s(n,\alpha)$ for each n, and if the closure of $\{s_n\colon n\in A_\beta\}$ in E is disjoint from the closure of $\{s_n\colon n\in\mathbb{N}\setminus A_\beta\}$ then so is their image by this map.

(4) either there is a clopen $W\subset E_\beta$ such that $r_\beta[f_\alpha^{-1}(W)]$ is disjoint from $\{s_n\colon s(n,\alpha)\in W\}$, or there is an $A\subset\mathbb{N}$ such that $A^*\supset K_\alpha$ such that $r_\beta\restriction A$ is 1-to-1 and $r_\beta[A]$ is $\mathfrak{c}\times\mathfrak{c}$-resolvable,

(5) if $f_\alpha[A^*_\beta\cap K_\alpha]=E_\alpha$, then $K_\alpha\subset A^*_\beta$.

We check that if the induction succeeds, then $K_\mathfrak{c}$ is homeomorphic to E and if a retract, then E is $\mathfrak{c}\times\mathfrak{c}$-resolvable. By inductive condition 3, there is a homeomorphism h from $E_\mathfrak{c}$ to E, and by inductive condition 5, $f_\mathfrak{c}$ is an irreducible function from $K_\mathfrak{c}$ onto $E_\mathfrak{c}$. Since E is extremally disconnected, $K_\mathfrak{c}$ is homeorphic to E. Assume $r\colon\beta\mathbb{N}\twoheadrightarrow K_c$ is a retraction. It follows that $h\circ f_\mathfrak{c}\circ r[\mathbb{N}]\cap S$ is a dense subset of S, and let $A=\mathbb{N}\cap(h\circ f_\mathfrak{c}\circ r)^{-1}(S)$. Since $h\circ f_\mathfrak{c}\circ r(\mathbb{N}\setminus A)$ is disjoint from S, it follows that $r[\mathbb{N}\setminus A]$ is nowhere dense in K_c. Since r is a retraction, it follows that $(\mathbb{N}\setminus A)^*\cap K_\mathfrak{c}\subset\overline{[\mathbb{N}\setminus A]}$ is empty. It similarly follows that for each $\widetilde{A}\subset A$ such that $K_\mathfrak{c}\subset\widetilde{A}^*$, $r[\widetilde{A}]$ is dense in $K_\mathfrak{c}$. Fix any $\gamma<\mathfrak{c}$ such that $r_\gamma\supset(h\circ f_\mathfrak{c}\circ r)\restriction A$, and let $\alpha>\gamma$ be large enough so that $(\mathbb{N}\setminus A)^*\setminus K_\alpha$ is empty. By condition 4 there is an $\widetilde{A}\subset A$ such that $r_\gamma\restriction\widetilde{A}$ is 1-to-1 and $r_\gamma[\widetilde{A}]$ is dense in E and is $\mathfrak{c}\times\mathfrak{c}$-resolvable.

Now we carry out the induction. Let h_ω be any map from E onto 2^ω that is 1-to-1 on $\{s_n\colon n\in\mathbb{N}\}$ and let $s(n,\omega)=h_\omega(s_n)$ for each $n\in\mathbb{N}\}$. Let f_ω be the map from

$K_\omega = \beta\mathbb{N}$ onto $E_\omega = 2^\omega$ such that $f_\omega(n) = s(n,\omega)$ for each n. For limit $\alpha > \omega$, let $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$ and, for each n, $s(n,\alpha) = \bigcup\{s(n,\beta)\colon \beta < \alpha\}$. Similarly, for each $x \in K_\alpha$, let $f_\alpha(x) = \bigcup\{f_\beta(x)\colon \beta < \alpha\} \in 2^\alpha$. Since the projection mapping will send E_α onto $E_{\beta+1}$ for $\beta < \alpha$, the last condition is automatically satisfied for all $\alpha > \beta + 1$ if it holds for $\alpha = \beta + 1$. The other induction conditions are routine to verify. Now assume $\alpha = \beta + 1$. For each clopen $W \subset E_\beta$, fix $Y_W \subset \mathbb{N}$ such that $Y_W^* \cap K_\beta = f_\beta^{-1}[W]$. Also choose $\{Y_\xi\colon \xi < \beta\} \subset \mathcal{P}(\mathbb{N})$ so that $K_\beta = \bigcap_{\xi<\beta} Y_\xi^*$. Define a poset P by $p \in P$ if $p \subset \mathbb{N}$ is finite, and $r_\beta \restriction p$ is 1-to-1. For each clopen $W \subset E_\beta$ and $\xi < \beta$, let $D_{W,\xi} = \{p \in P\colon p \cap Y_W \cap Y_\xi \neq \emptyset\}$. Assume there is some W, ξ such that $D_{W,\xi}$ is not dense (P is ordered by $\supset$). Then there is some $p \in P$ such that $q \cap Y_W \cap Y_\xi$ is empty for all $q \supset p$. This means $A_p \supset Y_W \cap Y_\xi$ where $A_p = r_\beta^{-1}(r_\beta[p])$, hence $r_\beta[Y_W \cap Y_\xi]$ is a finite subset of E. It follows then that $r_\beta[(Y_W \cap Y_\xi)^* \cap K_\beta] = r_\beta[f_\beta^{-1}[W]]$ is nowhere dense in E. In this case, let $\widetilde{B}_\beta = \mathbb{N}$ (used below). Therefore to ensure condition 4, we need only worry about the case where $D_{W,\xi}$ is dense for all such W, ξ. Let $G \subset P$ be a filter which, by $\mathrm{MA}_{\mathrm{ctble}}$, can be assumed to meet $D_{W,\xi}$ for each clopen $W \subset E_\beta$ and $\xi < \beta$ and let $B_\beta = \bigcup G \subset \mathbb{N}$. It follows that $r_\beta \restriction B_\beta$ is 1-to-1 and $f_\beta[B_\beta^* \cap K_\beta] = E_\beta$. If there is any $\widetilde{B}_\beta \subset B_\beta$ such that $f_\beta[\widetilde{B}_\beta^* \cap K_\beta] = E_\beta$ and $r_\beta[\widetilde{B}_\beta \cap Y_W \cap Y_\xi] \cap W = \emptyset$, then we will ensure that $K_\alpha \subset \widetilde{B}_\beta^*$. Since we will have $(\widetilde{B}_\beta \cap Y_W \cap Y_\xi)^* \supset f_\alpha^{-1}[W]$, this also would ensure that condition 4 would hold. If there is no such $\widetilde{B}_\beta$, then let $\widetilde{B}_\beta = B_\beta$ and proceed as follows. For each clopen $W \subset E_\beta$ and $\xi < \beta$, let $S_{W,\xi} = \{s(n,\beta)\colon n \in B_\beta \cap Y_W \cap Y_\xi\}$, which by our current assumption will be infinite. Use this family of subsets of $S_\beta = \{s(n,\beta)\colon n \in B_\beta\}$ to generate a Hausdorff topology of weight $< \mathfrak{c}$. Applying Proposition 17, there is a $\mathfrak{c} \times \mathfrak{c}$-independent matrix of dense subsets. For each $D \subset S_\beta$ that is dense in this topology, the set $A_D = \{n\colon s(n,\beta) \in D\}$ will satisfy that $f_\beta[A_D^* \cap K_\beta] = E_\beta$. By our assumption on r_β, (i.e. A_D can not be a $\widetilde{B}_\beta$) $r_\beta[A_D]$ will be dense in E. Since r_β is 1-to-1, E is $(\omega, \mathfrak{c} \times \mathfrak{c})$-resolvable.

Finally we define K_α and $\{s(n,\alpha)\colon n \in \mathbb{N}\}$. If $K_\beta \cap \widetilde{B}_\beta^* \cap A_\beta^*$ maps onto E_β, replace $\widetilde{B}_\beta$ by $\widetilde{B}_\beta \cap A_\beta$. If the closures of $\{s_n\colon n \in A_\alpha\}$ and $\{s_n\colon n \in \mathbb{N} \setminus A_\beta\}$ are disjoint, then let these closure be denoted C_0, C_1 and define $s(n,\alpha) = s(n,\beta)^\frown 0$ for $n \in A_\beta$ and $s(n,\alpha) = s(n,\beta)^\frown 1$ for $n \in \mathbb{N} \setminus A_\beta$; otherwise $C_0 = E_\beta$, $C_1 = \emptyset$, and $s(n,\alpha) = s(n,\beta)^\frown 0$ for all n. Using $\mathrm{MA}_{\mathrm{ctble}}$ (i.e. adding a “Cohen real”) there is a set $Y \subset \widetilde{B}_\beta$ such that each of $Y^* \cap K_\beta$ and $K_\beta \cap (\widetilde{B}_\beta \setminus Y)^*$ map onto E_β by f_β. Since C_0 and C_1 are equal to the intersection of $|\beta|$ many clopen sets in 2^β, each of

$$Y^* \cap K_\beta \cap f_\beta^{-1}[C_0] \quad \text{and} \quad (\widetilde{B}_\beta \setminus Y)^* \cap K_\beta \cap f_\beta^{-1}[C_1]$$

are also equal to the intersection of at most $|\beta|$ many clopen subsets of $\beta\mathbb{N}$. We define K_α to be the union of these two sets. The mapping f_α is easy to define. $\square$

It is a reasonable conjecture (see [1]) to expect that every absolute retract of $\beta\mathbb{N}$ is of the form $E(K)$ where K is some continuous image of $2^{\mathfrak{c}}$. There are structure theorems for such spaces by Shapiro [12] and Koppelberg [9]. A map $f\colon X \twoheadrightarrow Y$ is *semi-open* if the image of each non-empty open set has non-empty interior. This is a notion dual to the important notion of complete or regular embedding in Boolean algebra theory and the language of forcing posets. Borrowing from the forcing notation, let $Y \lessdot_f X$ abbreviate that f is a semi-open mapping from X onto Y. In Boolean algebras one often represents an algebra B as a union of a chain of smaller subalgebras. In addition, the system is usually assumed to be continuous in that at limit levels of the chain, that subalgebra is the union of the earlier algebras. The dual notion in topology is the inverse limit of a continuous system. That is, an inverse system of spaces is an indexed family of spaces and mappings $\{X_\alpha, f_\alpha^\beta\colon\ \alpha < \beta \in \lambda\}$ so that f_α^β is a continuous function from X_β onto X_α and $f_\alpha^\gamma = f_\beta^\gamma \circ f_\alpha^\beta$ for $\alpha < \beta < \gamma$. The inverse limit of the system, $X_\lambda = \varprojlim_\alpha X_\alpha$ for short, is the unique (up to homeomorphism) space for which there is a family of maps $f_\alpha = f_\alpha^\lambda$ extending the system. The notion of a continuous system will mean that for each limit ordinal $\gamma < \lambda$, X_γ will equal $\varprojlim_{\alpha<\gamma} X_\alpha$.

The following result is taken from the book by Heindorf and Shapiro ([7, 5.3.1], see also the remark [7, page 7]).

Proposition 18. *If K is a continuous image of $2^{\mathfrak{c}}$ and $E = E(K)$ is expressed as an inverse limit of a continuous system $\{E_\alpha, f_\alpha^\beta\colon\ \alpha < \beta < \mathfrak{c}\}$ of spaces of weight less than $\mathfrak{c}$ and if f_α denotes the resulting mapping from E onto E_α, then the set of $\lambda < \mathfrak{c}$ such that $E_\lambda \lessdot_{f_\lambda} E$ contains a closed and unbounded (cub) subset of $\mathfrak{c}$.*

Theorem 19. *If an $(\omega, \mathfrak{c})$-irresolvable absolute retract E, of $\beta\mathbb{N}$, is expressed as an inverse limit of a continuous system $\{E_\alpha, g_\alpha^\beta\colon\ \alpha < \beta < \mathfrak{c}\}$ of spaces of weight less than $\mathfrak{c}$ and if g_α denotes the resulting mapping from E onto E_α, then the set $\{\alpha\colon\ E_\alpha \lessdot_{g_\alpha} E\}$ is a stationary subset of $\mathfrak{c}$.*

P r o o f. Let f be a mapping from $\mathbb{N}^*$ onto the compact space $2^{\mathfrak{c}}$ and inductively construct $I_\alpha \subset \mathfrak{c}$, closed sets $J_\alpha \subset 2^{\mathfrak{c}\setminus I_\alpha}$ and $K_\alpha \subset \mathbb{N}^*$ so that $f_\alpha = f \restriction K_\alpha$ maps K_α onto $J_\alpha \times 2^{I_\alpha}$. The plan is to ensure that $J_{\mathfrak{c}} = \bigcap\limits_{\alpha<\mathfrak{c}} J_\alpha \times 2^{I_\alpha} = J_{\mathfrak{c}} \times 2^{\emptyset}$ will be homeomorphic to E and $f_c\colon K_{\mathfrak{c}} \twoheadrightarrow J_c$ will be irreducible. Therefore, $K_{\mathfrak{c}}$ will be a copy of E. Assume that E can be expressed as an inverse limit of a continuous system $\{E_\alpha, g_\beta^\alpha\colon\ \beta < \alpha < \mathfrak{c}\}$ of spaces of weight less than $\mathfrak{c}$ so that there is a cub $C \subset \mathfrak{c}$ such that $E_\lambda \not\lessdot_{g_\lambda} E$ for each $\lambda \in C$. We will ensure that $K_{\mathfrak{c}}$ is not a retract.

Fix $S \subset E$ of cardinality $\mathfrak{c}$ so that each countable $D \subset E \setminus S$ is nowhere dense and let $\{r_\alpha\colon \alpha < \mathfrak{c}\}$ enumerate all continuous functions from $\beta\mathbb{N}$ into E so that $r[\mathbb{N}] \subset S$. A word of caution, $\mathfrak{c}$ may be singular and the order-type of C may be less than $\mathfrak{c}$. In fact, we may as well assume that the order type of C is the cofinality of $\mathfrak{c}$ and that $\mathfrak{c} \setminus C$ has cardinality $\mathfrak{c}$. At stage $\lambda \in C$, we handle all r_α with $\alpha < \lambda$ and for each $\beta < \mathfrak{c}$, we ensure that J_β is homeomorphic to E_β. Fix an enumeration $\{A_\alpha\colon \alpha \in \mathfrak{c}\}$ of $\mathcal{P}(\mathbb{N})$, and for convenience, we may assume that $A_\lambda = \mathbb{N}$ and $E_{\lambda+1} = E_\lambda$ for each $\lambda \in C$.

To start the induction, fix a countable dense $\{s_n\colon n \in \omega\} \subset S$ and assume without loss of generality that $g_0\colon E \twoheadrightarrow E_0$ is 1-to-1 on $\{s_n\colon n \in \omega\}$. Let μ_0 be the weight of E_0 and choose any embedding of E_0 into 2^{μ_0}, call it J_0. Let $I_0 = \mathfrak{c} \setminus \mu_0$, let $K_0 = f^{-1}(J_0 \times 2^{I_0})$ and $f_0 = f \restriction K_0$. Let h_0 denote the map from E onto J_0 such that $h_0[g_0^{-1}(e)]$ is a singleton for each $e \in E_0$ (and, conversely, $g_0[h_0^{-1}(y)]$ is a singleton for each $y \in J_0$). This condition, in compact spaces, is equivalent to asserting that h_0, g_0 induce a homeomorphism, φ_0, from E_0 to J_0.

Let $\{\lambda_\zeta\colon \zeta \in \kappa\}$ be an increasing enumeration of C. We make the following inductive assumptions on $I_\alpha, J_\alpha, K_\alpha$ and $h_\alpha\colon E \twoheadrightarrow J_\alpha$:

(1) f_α is onto, and for $\beta < \alpha$, $K_\beta \supset K_\alpha$, and the projection $\pi_{\mathfrak{c}\setminus I_\beta}$ maps $J_\alpha \times 2^{I_\alpha}$ onto J_β,

(2) for limit α, $I_\alpha = \bigcap_{\beta<\alpha} I_\beta$, for $\alpha \in C$, $|I_\alpha \setminus I_{\alpha+1}| < \mathfrak{c}$, and for $\alpha \notin C$, $I_\alpha \setminus I_{\alpha+1}$ is finite,

(3) h_α and g_α induce a homeomorphism φ_α from E_α to J_α, and for $\beta < \alpha$ and $e \in E$, $h_\beta(e) = h_\alpha(e) \restriction (I_\alpha \setminus I_\beta)$

(4) if $\lambda < \alpha$ and $\lambda \in C$, then for all $\beta < \lambda$, there is a clopen $W \subset E$ and a clopen $U \subset J_\alpha$ such that either

$$r_\beta^{-1}[W] \cap K_\alpha = \emptyset \quad \text{or} \quad r_\beta[f_\alpha^{-1}(U \times 2^{I_\alpha})] \cap h_\alpha^{-1}(U) = \emptyset$$

(5) if $\beta < \alpha$, then either $K_\alpha \subset A_\beta^*$ or there is a clopen $W \subset J_\alpha$ such that $f_\alpha[A_\beta^* \cap K_\alpha] \cap (W \times 2^{I_\alpha}) = \emptyset$

Item (4) is the condition that guarantees that $K_\mathfrak{c}$ is not a retract of $\beta\mathbb{N}$. By condition (5), f_c is irreducible, hence a homeomorphism from $K_\mathfrak{c}$ onto $J_\mathfrak{c}$. By condition (3), $h_\mathfrak{c}^{-1}$ is a homeomorphism from $J_\mathfrak{c}$ to E. Now assume that r is a retraction mapping onto $K_\mathfrak{c}$ and observe that $H = h_\mathfrak{c}^{-1} \circ f_\mathfrak{c} \circ r$ is a function from $\beta\mathbb{N}$ onto E. Let $A \subset \mathbb{N}$ equal $H^{-1}(S) \cap \mathbb{N}$. Since r is a retraction and $r[\mathbb{N} \setminus A]$ is nowhere dense in $K_\mathfrak{c}$, it follows that $K_\mathfrak{c} \subset A^*$. There is a $\beta < \mathfrak{c}$ such that $H \restriction A = r_\beta \restriction A$ and let $\alpha = \beta + 1$. There can be no W as in (4) since H is onto. So assume $U \subset J_\alpha$ is as in (4). Since $J_\mathfrak{c}$ projects onto J_α, let $p \in K_\mathfrak{c}$ be such that $f_\mathfrak{c}(p) = f_\alpha(p)$ is in $U \times 2^{I_\alpha}$. We obtain a contradiction by showing that

$h_\alpha(r_\beta(p)) \in U$. Since r is a retraction, $r(p) = p$, hence $r_\beta(p) = H(p) = h_{\mathfrak{c}}^{-1}(f_{\mathfrak{c}}(p))$. Therefore, $h_\alpha(r_\beta(p)) = h_\alpha(h_{\mathfrak{c}}^{-1}(f_{\mathfrak{c}}(p))) = f_{\mathfrak{c}}(p) \restriction (\mathfrak{c} \setminus I_\alpha) \in U$.

Assume $\alpha < \mathfrak{c}$ and the induction has succeeded for $\alpha' < \alpha$. If α is a limit, then $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$, $I_\alpha = \bigcap_{\beta<\alpha} I_\beta$, and $\bigcap_{\beta<\alpha} J_\beta \times 2^{I_\beta}$ will be of the form $J_\alpha \times 2^{I_\alpha}$ since $\gamma < \beta < \alpha$ implies $J_\beta \times I_\beta$ projects onto J_γ. It is easily verified that $f_\alpha[K_\alpha] = J_\alpha \times 2^{I_\alpha}$. The function h_α, where for $e \in E$, $h_\alpha(e)$ is the unique point $y \in J_\alpha$ such that $y \restriction \beta = h_\beta(e)$ for each $\beta < \alpha$. It is a routine exercise to verify that $h_\alpha \colon E \twoheadrightarrow E_\alpha$ is a continuous surjection and that h_α, g_α induces a homeomorphism from E_α to J_α (because of the assumption at limits on $\varprojlim_{\beta<\alpha} E_\beta$). Conditions (4) and (5) are immediate because α is a limit.

Now assume $\alpha = \lambda + 1$ for some $\lambda \in C$. Since $E_\lambda \not\leqslant_{g_\lambda} E$, there is a clopen $W \subset E$ such that $g_\lambda[W]$ is nowhere dense in E_λ. Let $K_{\lambda,0} = K_\lambda$. By induction on $\beta < \lambda$, we choose finite functions $\varrho_\beta \subset I_\lambda \times 2$ (pairwise compatible) and $K_{\lambda,\beta} \subset \bigcap_{\gamma<\beta} K_{\lambda,\gamma}$ so that $K_{\lambda,\beta}$ maps onto $J_\lambda \times \{\bigcup_{\gamma<\beta} \varrho_\gamma\} \times 2^{I_{\lambda,\beta}}$ (where $I_{\lambda,\beta} = I_\lambda \setminus \bigcup_{\gamma<\beta} \mathrm{dom}(\varrho_\gamma)$). At stage β, we consider $r_\beta^{-1}[W]$. If there is a non-empty clopen $U \subset J_\lambda$ with $U \cap h_\lambda[W] = \emptyset$ and a finite function $\varrho_\beta \subset I_{\lambda,\beta} \times 2$ such that f_λ maps $K_{\lambda,\beta} \cap r_\beta^{-1}[W]$ onto $U \times \{\bigcup_{\gamma\leqslant\beta} \varrho_\gamma\} \times 2^{I_{\lambda,\beta+1}}$, then let

$$K_{\lambda,\beta+1} = K_{\lambda,\beta} \cap f_\lambda^{-1}\Big[(J_\lambda \setminus U) \times \Big\{\bigcup_{\gamma\leqslant\beta} \varrho_\gamma\Big\} \times 2^{I_{\lambda,\beta+1}}\Big]$$
$$\cup\, r_\beta^{-1}[W] \cap K_{\lambda,\beta} \cap f_\lambda^{-1}\Big[U \times \Big\{\bigcup_{\gamma\leqslant\beta} \varrho_\gamma\Big\} \times 2^{I_{\lambda,\beta+1}}\Big]$$

and notice that $r_\beta[f_\lambda^{-1}(U \times \{\bigcup_{\gamma\leqslant\beta} \varrho_\gamma\} \times 2^{I_{\lambda,\beta+1}})] \subset W$ and so is disjoint from $h_\lambda^{-1}[U]$.

If there is no such ϱ_β, then $f_\lambda[K_{\lambda,\beta} \setminus r_\beta^{-1}[W]]$ maps onto $J_\lambda \times \{\bigcup_{\gamma<\beta} \varrho_\gamma\} \times 2^{I_{\lambda,\beta}}$, so set $\varrho_\beta = \emptyset$, $K_{\lambda,\beta+1} = K_{\lambda,\beta} \setminus r_\beta^{-1}[W]$. In this case we have $r_\beta^{-1}[W] \cap K_{\lambda,\beta+1} = \emptyset$ of course. Let $K_\alpha = K_{\lambda+1} = \bigcap_{\beta<\lambda} K_{\lambda,\beta}$, $I_\alpha = \bigcap_{\beta<\lambda} I_{\lambda,\beta}$, $J_\alpha = J_\lambda \times \{\bigcup_{\beta<\lambda} \varrho_\beta\}$. Since $E_{\lambda+1} = E_\lambda$ and $A_\lambda = \mathbb{N}$, the remaining inductive conditions are immediate by the inductive assumptions.

Now assume $\alpha = \beta + 1$ for $\beta \notin C$. Let the weight of E_α be $\mu_\alpha < \mathfrak{c}$. By inductive condition (2), $\mathfrak{c} \setminus I_\beta$ has cardinality less than $\mathfrak{c}$. Let $I'_\alpha \subset \mathfrak{c} \setminus I_\beta$ have cardinality μ_α and let $J'_\alpha \subset 2^{I'_\alpha}$ be homeomorphic to E_α and let $\varphi'_\alpha \colon E_\alpha \twoheadrightarrow J'_\alpha$ be a homeomorphism. Define the homeomorphism φ^0_α from E_α into $J_\beta \times J'_\alpha$ by $\varphi^0_\alpha(e) = (\varphi_\beta(g_\beta^{\beta+1}(e)), \varphi'_\alpha(e))$ and let J^0_α be the image of E_α. Define $K^0_\alpha = K_\beta \cap f_\beta^{-1}(J^0_\alpha \times 2^{I_\beta \setminus I'_\alpha})$. In order to define $K_\alpha \subset K^0_\alpha$, and $J_\alpha \subset J'_0$ we must now consider A_α. If $f[K^0_\alpha \cap A^*_\alpha] = J^0_\alpha \times 2^{I_\beta \setminus I'_\alpha}$, then $K_\alpha = K^0_\alpha \cap A^*_\alpha$ and $J_\alpha = J^0_\alpha$, $I_\alpha = I_\beta \setminus I'_\alpha$. Otherwise, there is a clopen set $W \subset J^0_\alpha$

and a finite function $\varrho_\alpha \subset (I_\beta \setminus I'_\alpha) \times 2$ such that

$$f[K^0_\alpha \cap A^*_\alpha] \cap W \times \{\varrho_\alpha\} \times 2^{I_\beta \setminus (I'_\alpha \cup \mathrm{dom}(\varrho_\alpha))} = \emptyset.$$

Now define $K_\alpha = K^0_\alpha$, $J_\alpha = J^0_\alpha \times \{\varrho_\alpha\}$, $I_\alpha = I_\beta \setminus (I'_\alpha \cup \mathrm{dom}(\varrho_\alpha))$. In the first case, set $h_\alpha(e) = \varphi^0_\alpha(g_\alpha(e))$ for $e \in E$, in the second, $h_\alpha(e) = \varphi^0_\alpha(g_\alpha(e))^\frown \varrho_\alpha$. $\square$

Corollary 20. *If a Simon type space satisfies that the character of each $\mathcal{U}_t$ is $\mathfrak{c}$, then it is not an absolute retract.*

Proof. Let G_ω be a Simon type space for which each $\mathcal{U}_t$ has character $\mathfrak{c}$. Assume that f is a continuous function from βG_ω onto a space K and assume that $f \restriction G_\omega$ is 1-to-1 and that K has weight less than $\mathfrak{c}$. By Theorem 19, it suffices to show that f is not semi-open. To see this, recall that G_ω is a topology on the base set $\bigcup_n \omega^n$. Let $\emptyset \in T \subset G_\omega$ and for each $t \in G_\omega$, whether in T or not, ensure that $L_t = \{n \in \omega \colon t^\frown n \in T\}$ is not in $\mathcal{U}_t$ while at the same time, $f(t)$ is in the closure of $\{f(t^\frown n) \colon n \in L_t\}$. The reason we can do this is that $f(t)$ is in the closure of $\{f(t^\frown n) \colon n \in \omega\}$ but the neighborhood trace in K will have character less than $\mathfrak{c}$.

It follows that T is nowhere dense in G_ω (see [16]), while at the same time $f[T]$ contains $f[G_\omega]$ in its closure, and so is dense in K. Therefore $\beta G_\omega \setminus \overline{T}$ does not map to a set with non-empty interior, showing that f is not semi-open. $\square$

For a cardinal κ, let $\diamondsuit^*_\kappa$ denote the following (consistent) strengthening of $\diamondsuit_\kappa$: for each $\lambda < \kappa$, there is a family $\mathcal{S}_\lambda \subset \mathcal{P}(\lambda)$ such that $|\mathcal{S}_\lambda| \leqslant |\lambda|$ and for each $X \subset \kappa$, the set $\{\lambda \colon X \cap \lambda \in \mathcal{S}_\lambda\}$ contains a cub subset of κ.

If $\diamondsuit^*_\kappa$ holds and P is a ccc poset of cardinality at most κ, then $\diamondsuit^*_\kappa$ will continue to hold. Therefore $\diamondsuit^*_\mathfrak{c}$ will hold in many models (independent of the continuum hypothesis). The details of the next result are omitted because the proof is so similar to the proof of Theorem 19.

Theorem 21. *$\diamondsuit^*_\mathfrak{c}$ implies that if an absolute retract E, of $\beta\mathbb{N}$, is expressed as an inverse limit of a system $\{E_\alpha, f^\beta_\alpha \colon \alpha < \beta < \mathfrak{c}\}$ of spaces of weight less than $\mathfrak{c}$ and if f_α denotes the resulting mapping from E onto E_α, then the set $\{\alpha \colon E_\alpha \lessdot_{f_\alpha} E\}$ is a stationary subset of $\mathfrak{c}$.*

3. Distinct vD-spaces

In this section we will produce distinct vD-spaces. One interesting fact we will prove is that $E(2^{\mathfrak{c}})$ is itself a compact vD-space (see also [6] for another approach). Our extra effort will yield the additional information that it has quite distinct dense countable vD-spaces. These results will use the technique of independent matrices. We will also show that there is another compact vD-space that is not homeomorphic to $E(2^{\mathfrak{c}})$ by a proof which is suprisingly easy utilizing Simon type spaces, although we can note that a Simon type space is not itself a countable vD-spaces because it is not irresolvable. Before carrying out the constructions we will introduce some other properties that can serve to distinguish these spaces.

3.1. Weak P-sets. A closed subset K of $\mathbb{N}^*$ is a *(discrete) weak P-set* if the closure of each countable (discrete) subset of $\mathbb{N}^* \setminus K$ is disjoint from K. Each $\leqslant$2-to-1 map from $\beta\mathbb{N}$ onto a crowded ED space K gives rise to a canonical embedding of K as a 1-to-1 retract of $\beta\mathbb{N}$.

Proposition 22. *If $f\colon \beta\mathbb{N} \twoheadrightarrow K$ is $\leqslant$2-to-1 and K is a crowded ED space, then $(f \restriction \widetilde{K})^{-1} \circ f$ is a* 1-to-1 *retraction from $\beta\mathbb{N}$ to*

$$\widetilde{K} = \overline{\mathbb{N}^* \cap f^{-1}[\mathbb{N}]}$$

and $\widetilde{K}$ is homeomorphic to K.

If S is a countable vD-space, then any 1-to-1 function f from $\mathbb{N}$ onto S gives rise to an embedding of βS as a 1-to-1 retract, K_S, of $\beta\mathbb{N}$ that is unique up to a permutation on $\mathbb{N}$ (hence a homeomorphism of $\beta\mathbb{N}$). There are internal descriptions for S that correspond to when K_S is a weak P-set or discrete weak P-set of $\mathbb{N}^*$. Recall that a set Y is $\aleph_0$-bounded if the closure of each countable subset of Y is compact.

Proposition 23. *Let S be a countable vD-space and let K_S be the canonical embedding of βS as a* 1-to-1 *retract of $\beta\mathbb{N}$.*

(1) *K_S is a weak P-set of $\mathbb{N}^*$ if and only if the near points of S is $\aleph_0$-bounded.*
(2) *K_S is a discrete weak P-set of $\mathbb{N}^*$ if and only if every countable vD-space contained in the near points of S is nowhere dense in βS.*

P r o o f. Both of the results use the fact that if A is a countable subset of $\mathbb{N}^* \setminus K_S$, then $\overline{A}$ meets K_S if and only if $\overline{A}$ meets S. Also, recall from Proposition 5 that $f[\mathbb{N}^* \setminus K_S]$ is sent onto the near points of S (disjoint from S itself) and that for $A \subset \mathbb{N}^* \setminus K_S$, $f[\overline{A}]$ will be contained in $\overline{f[A]}$. Therefore if A is a countable subset of $\mathbb{N}^* \setminus K_S$, $\overline{A}$ will meet S if and only if $\overline{f(A)}$ meets S. For the second statement, assume that D is a discrete subset of $\mathbb{N}^* \setminus K_S$, then $f[D]$ is a vD-space in the near points of S (since $f \restriction \overline{D} = \beta D$ is a $\leqslant$2-to-1 map onto $\overline{f(D)}$). $\square$

It follows easily that each point of a countable vD-space S is a far-point of its own near points, however it is not at all clear if it is necessarily far from its entire remainder. We show that both situations can happen even with dense subsets of $E(2^{\mathfrak{c}})$.

Theorem 24. *There is a countable vD-space S that is dense in $E(2^{\mathfrak{c}})$ such that there is a countable vD-space Γ dense in the near points of S. Furthermore we can ensure at the same time that S is far from $\beta S\setminus S$. Of course Γ is not far from $\beta\Gamma\setminus\Gamma$.*

Theorem 25. *There is a countable vD-space S such that βS is not homeomorphic to $E(2^{\mathfrak{c}})$.*

Theorem 26. $\mathrm{MA}_{\mathrm{ctble}}$ *implies there is a countable vD-space S such that βS is (ω,ω)-irresolvable and not $(\omega, 1\times 2)$-resolvable. In particular, $\beta S\setminus S$ is $\aleph_0$-bounded.*

Corollary 27. *There are countable vD-spaces S_1, S_2, S_3 such that in $\mathbb{N}^*$,*
(1) *K_{S_1} is not a discrete weak P-set,*
(2) *K_{S_2} is a discrete weak P-set but not a weak P-set, and*
(3) $\mathrm{MA}_{\mathrm{ctble}}$ *implies K_{S_3} is a weak P-set.*

3.2. Proof of theorem 24. We are able to break the proof of Theorem 24 into a series of Lemmas. Lemmas 28 and 30 combine to easily prove the first statement of Theorem 24 using a straightforward induction which we omit. The rest of Theorem 24 is similarly a consequence of Lemmas 35 and 34.

For the next lemma, it might help to imagine starting out with the usual topology $(\mathbb{Q}\times\mathbb{Q})$ and then choosing a function from $\mathbb{Q}$ into the irrationals whose graph, Γ, is dense. Now let $\mathcal{Q}$ denote the set of rational numbers and $(\mathcal{Q}\times\mathcal{Q})\cup\mathcal{Q}$ denote the same set, $(\mathbb{Q}\times\mathbb{Q})\cup\Gamma$, where $\mathcal{Q}$ is also identified with Γ in the obvious way. In Theorem 24, S will be $\mathcal{Q}\times\mathcal{Q}$, while $\mathcal{Q}$ will be the vD-space that is dense in the near points of S.

Lemma 28. *Assume that $X = (\mathcal{Q}\times\mathcal{Q})\cup\mathcal{Q}$ is a space such that X is zero-dimensional, $\mathcal{Q}$ is dense and*
(1) *each $q\in\mathcal{Q}$ is a limit point of $\{q\}\times\mathcal{Q}$,*
(2) *for each $A\subset\mathcal{Q}$, there is an $A_0\subset A$ such that $\overline{A_0}$ is clopen in X, $\mathcal{Q}\cap\overline{A}\setminus A_0$ is closed discrete in $\mathcal{Q}$, and $\overline{A}\setminus((A_0\times\mathcal{Q})\cup\mathcal{Q})$ is closed and discrete in $\mathcal{Q}\times\mathcal{Q}$,*
(3) *if $B\subset\mathcal{Q}\times\mathcal{Q}$ is such that $q\notin\overline{B\cap(\{q\}\times\mathcal{Q})}$ for each $q\in\mathcal{Q}$, then B is closed*

then $\mathcal{Q}$ is vD-space and is C^*-embedded in the extremally disconnected space X. Furthermore $\mathcal{Q}\times\mathcal{Q}$ is a vD-space that contains $\mathcal{Q}$ densely in its near points.

P r o o f. The space X is crowded because it has disjoint dense sets $\mathcal{Q}$ and $\mathcal{Q}\times\mathcal{Q}$. It is immediate by condition 2 that if $A\subset\mathcal{Q}$ is crowded, then A_0 is dense in A. Furthermore, since $\mathcal{Q}\cap\overline{A_0}\setminus A_0$ is closed discrete in $\mathcal{Q}$, A is open in $\mathcal{Q}$ because it is equal to $\mathcal{Q}\cap\overline{A_0}$ with a closed discrete set removed. Therefore, by Proposition 6, $\mathcal{Q}$ is a vD-space. Condition 2 also implies that $\mathcal{Q}$ is extremally disconnected and C^*-embedded in X. For each $q\in\mathcal{Q}$, $\{q\}\times\mathcal{Q}$ is discrete by applying 2 to the set $A=\mathcal{Q}\setminus\{q\}$. Therefore condition 1 implies that $\mathcal{Q}$ is contained in the near points of $\mathcal{Q}\times\mathcal{Q}$. To finish the proof we need only show that $\mathcal{Q}\times\mathcal{Q}$ is a vD-space. Consider any crowded subset B of $\mathcal{Q}\times\mathcal{Q}$. By Proposition 6, we must show that B is open in $\mathcal{Q}\times\mathcal{Q}$. For $q\in\mathcal{Q}$, let $B_q=\{r\in\mathcal{Q}\colon (q,r)\in B\}$ and let $A=\{q\in\mathcal{Q}\colon q\in\overline{\{q\}\times B_q}\}$. Let $A_0\subset A$ be such that A_0 is the interior in $\mathcal{Q}$ of A. Clearly B contains a dense subset of the relatively clopen set $U=(\mathcal{Q}\times\mathcal{Q})\cap\overline{A_0}$ and $U\setminus B$ is closed because it equals the union of the two closed sets $U\setminus(A_0\times\mathcal{Q})$ (by condition 2) and $(A_0\times\mathcal{Q})\setminus B$ (by condition 3). Therefore $B\cap U$ is open.

To complete the proof that B is open, we show that $B\setminus U$ is actually empty because it is the union of two discrete sets. The first is simply $B\setminus(A\times\mathcal{Q})$ by the last condition. Then $(A\setminus A_0)\times\mathcal{Q}$ is discrete by the second condition applied to $A_0\cup(\mathcal{Q}\setminus A)$, that is crowded since $(A\setminus A_0)$ is discrete. $\square$

To prove Theorem 24 using Lemma 28 we will construct an embedding of the set $X=\mathcal{Q}\times\mathcal{Q}\cup\mathcal{Q}$ into $E(2^{\mathfrak{c}})$. For convenience we will instead use the homeomorphic representation $(\omega+1)^{\mathfrak{c}}$ of $2^{\mathfrak{c}}$ where $\omega+1$ is the compact ordinal space. Recall that the points of $E((\omega+1)^{\mathfrak{c}})$ are ultrafilters of regular open subsets of $(\omega+1)^{\mathfrak{c}}$. The basic structure of the embedding is captured in the next definition.

Definition 29. A collection $\mathcal{X}=\{\mathcal{W}_x\colon x\in X\}$ satisfies E_α (or is an E_α structure) if for each $x\in X$, $\mathcal{W}_x\subset RO((\omega+1)^\alpha)$ is a filter base such that
(1) for each $q\in\mathcal{Q}$ and $W\in\mathcal{W}_q$, $\{r\in\mathcal{Q}\colon W\in\mathcal{W}_{(q,r)}\}$ is infinite,
(2) for each non-empty compact open $W\subset(\omega+1)^\alpha$, $\{q\in\mathcal{Q}\colon W\in\mathcal{W}_q\}$ is infinite,
(3) for each $W\in\mathcal{W}_{\mathcal{X}}=\{\emptyset\}\cup\bigcup\{\mathcal{W}_x\colon x\in X\}$ and each $x\in X$, either $W\in\mathcal{W}_x$ or $W\cap W'=\emptyset$ for some $W'\in\mathcal{W}_x$.

We say that $\mathcal{X}\prec\mathcal{X}'$ if for some $\alpha\leqslant\alpha'$, $\mathcal{X}'$ satisfies $E_{\alpha'}$ and $W\times(\omega+1)^{(\alpha'\setminus\alpha)}\in\mathcal{W}'_x\in\mathcal{X}'$ for each $W\in\mathcal{W}_x\in\mathcal{X}$.

Let $\{b_{(\gamma,n)}\colon (\gamma,n)\in\mathfrak{c}\times\omega\}$ be an $\mathfrak{c}\times\omega$-independent matrix of subsets of $\mathcal{Q}$. As above, for a finite function $\varrho\subset\mathfrak{c}\times\omega$, let b_ϱ denote the intersection $\bigcap\{b_{\gamma,n}\colon (\gamma,n)\in\varrho\}$. We may assume that each b_ϱ is a dense subset of $\mathbb{Q}$ and that $\bigcup\limits_{n\in\omega} b_{(\gamma,n)}=\mathbb{Q}$ for each $\gamma\in\mathfrak{c}$. For a subset I of $\mathfrak{c}$, let $\mathcal{B}_I$ denote the $I\times\omega$-independent submatrix. We say that $\mathcal{B}_I$ is independent mod $\mathcal{X}$, for an E_α structure $\mathcal{X}$, if for each finite function $\varrho\subset I\times\omega$ and each $W\in\mathcal{W}_{\mathcal{X}}\setminus\{\emptyset\}$, $B_\varrho\cap\{q\in\mathcal{Q}\colon W\in\mathcal{W}_q\}$ is infinite. Clearly the topology mentioned above on $(\mathbb{Q}\times\mathbb{Q})\cup\Gamma$ can be embedded densely into $(\omega+1)^\omega$ and by choosing the filter of clopen neighborhoods of each point for $\mathcal{W}_x$, we have $\mathcal{X}_\omega$ as an E_ω structure.

If $\mathcal{X}$ is an E_α structure, then for each $W\in\mathcal{W}_{\mathcal{X}}$, $\widetilde{W}=\{x\in X\colon\ W\in\mathcal{W}_x\}$ is going to be a clopen subset of X. The independent matrix assumption is simply asserting that each b_ϱ is a dense subset of $X\cap\mathcal{Q}$ with this topology. When discussing or constructing an E_α structure $\mathcal{X}$ we will implicitly assume that for $W\subset W'\in\mathcal{W}_{\mathcal{X}}$ and $x\in X$, if $W\in\mathcal{W}_x$, then W' is also in $\mathcal{W}_x$.

Lemma 30. *Assume that $\mathcal{X}$ is*

(1) *an $E_{\mathfrak{c}}$ structure, and*
(2) *for each $A\subset\mathcal{Q}$, there is a pairwise disjoint family $\{W(A,n)\colon\ n\in\omega\}\subset\mathcal{W}_{\mathcal{X}}$ such that for each $x\in X$, there is an $n\in\omega$, $W(A,n)\in\mathcal{W}_x$, and for each $n\in\omega$ $\{q\in\mathcal{Q}\setminus A\colon\ W(A,2n)\in\mathcal{W}_q\}$ and $\{q\in A\colon\ W(A,2n+1)\in\mathcal{W}_q\}$ are finite, and*
(3) *for each $B\subset\mathcal{Q}\times\mathcal{Q}$, there is a pairwise disjoint family $\{W(B,n)\colon\ n\in\omega\}\subset\mathcal{W}_{\mathcal{X}}$ such that for each $x\in X$, there is an $n\in\omega$, $W(B,n)\in\mathcal{W}_x$, and for each $n\in\omega$ $B\supset\{(q,r)\in\mathcal{Q}\times\mathcal{Q}\colon\ W(B,2n)\in\mathcal{W}_{(q,r)}\}$ and $\{(q,r)\in\mathcal{Q}\times\mathcal{Q}\colon\ W(B,2n+1)\in\mathcal{W}_{(q,r)}\}$ is finite, and*
(4) *$RO((\omega+1)^{\mathfrak{c}})=\mathcal{W}_{\mathcal{X}}$,*

then $\mathcal{X}$ is an embedding of X densely into $E((\omega+1)^{\mathfrak{c}})$ and with the inherited topology satisfies Lemma 28.

Proof. Condition 28(1) follows from the fact that $\mathcal{X}$ satisfies $E_{\mathfrak{c}}$, and the embedding of $\mathcal{Q}$ is dense because of the second condition of the definition of $E_{\mathfrak{c}}$. For any $A\subset\mathcal{Q}$, it is easily checked that $A_0=\{a\in A\colon\ (\exists n\in\omega)W(A,2n)\in\mathcal{W}_a\}$ will satisfy the requirements of condition 28(2) with $\overline{A_0}=\{x\in X\colon\ (\exists n\in\omega)W(A,2n)\in\mathcal{W}_x\}$. For example, each $\widetilde{W}(A,n)$ meets $A\setminus A_0$ in a finite set, hence $A\setminus A_0$ is closed discrete. Similarly, condition 28(3), follows easily from hypothesis (3) of this Lemma. □

The proof of Theorem 24 is achieved by a sequence of Lemmas. It is a routine induction using alternatingly, Lemmas 31, 32, and 33 to produce a space satisfying Lemma 30.

Lemma 31. *Assume $\alpha<\mathfrak{c}$, $\mathcal{X}$ is an E_α structure, I is an uncountable set such that $\mathcal{B}_I$ is independent* mod *$\mathcal{X}$, and $A\subset\mathcal{Q}$, then there is an $E_{\alpha+1}$ structure $\mathcal{X}'$ and $I'\subset I$ and a pairwise disjoint family $\{W(A,n)\colon\ n\in\omega\}\subset\mathcal{W}_{\mathcal{X}'}$ such that*

(1) *$\mathcal{X}\prec\mathcal{X}'$ and $I\setminus I'$ is countable, and*
(2) *for each $x\in X$, there is an $n\in\omega$, $W(A,n)\in\mathcal{W}'_x$, and for each $n\in\omega$ $\{q\in\mathcal{Q}\setminus A\colon\ W(A,2n)\in\mathcal{W}'_q\}$ and $\{q\in A\colon\ W(A,2n+1)\in\mathcal{W}'_q\}$ are finite.*

(3) $\mathcal{B}_{I'}$ *is independent* mod $\mathcal{X}'$.

Proof. Let $A \subset \mathcal{Q}$. It follows easily that for each $W \in \mathcal{W}_{\mathcal{X}} \setminus \{\emptyset\}$, that there is a $W^* \subset W$ in $\mathcal{W}_{\mathcal{X}} \setminus \{\emptyset\}$ and a finite function $\varrho_{W^*} \subset I \times \omega$ such that one of the following holds:

$$\{a \in A\colon\ W^* \in \mathcal{W}_a\} \cap b_{\varrho_{W^*}} = \emptyset \tag{3.1}$$

or, for all finite functions τ such that

$$\varrho_{W^*} \subset \tau \subset I \times \omega \quad \{a \in A\colon\ W^* \in \mathcal{W}_a\} \cap b_\tau \text{ is infinite.} \tag{3.2}$$

Therefore, there is a maximal disjoint family $\{W^*_n\colon\ n \in \omega\} \subset \mathcal{W}_{\mathcal{X}}$ and a family, $\{\varrho^*_n\colon\ n \in \omega\}$, of finite functions from I into ω, such that for each n, either 3.1 or 3.2 holds for W^*_n, ϱ^*_n.

By re-indexing the families, we may assume that for each n, W^*_{2n}, ϱ^*_{2n} satisfy condition 3.2, and $W^*_{2n+1}, \varrho^*_{2n+1}$ satisfy 3.1. Therefore, in the space $(\omega+1)^\alpha$,

$$W_A = \operatorname{int}\operatorname{cl}\bigcup_n W^*_{2n} \quad \text{and} \quad W_{-A} = \operatorname{int}\operatorname{cl}\bigcup_n W^*_{2n+1}$$

are complementary members of $RO((\omega+1)^\alpha)$. In defining our new $E_{\alpha+1}$ structure, $\mathcal{X}'$, we will define an intermediary E_α-structure which we will also denote as $\mathcal{X}'$. In particular we will add $\{W_A, W_{-A}\}$ to $\mathcal{W}_{\mathcal{X}'}$ and for each $x \in X$ we will have to select W_x from $\{W_A, W_{-A}\}$ to add to $\mathcal{W}_x$. The most important consideration in making this selection will be the preservation of condition (1) of E_α-structure.

For each $x = (q, r) \in \mathcal{Q} \times \mathcal{Q}$, if $W \cap W_A \neq \emptyset$ for each $W \in \mathcal{W}_{(q,r)}$, then we let $W_x = W_A$. Otherwise, we have that $W \cap W_{-A} \neq \emptyset$ for each $W \in \mathcal{W}_{(q,r)}$ and we set $W_x = W_{-A}$. For each $q \in \mathcal{Q}$, we may put $W_q = W_A$ if for each $W \in \mathcal{W}_q$, we have that $\{r \in \mathcal{Q}\colon\ W_{(q,r)} = W_A \text{ and } W \in \mathcal{W}_{(q,r)}\}$ is infinite. Otherwise, let $W_q = W_{-A}$ and, in either case, let B_q be the infinite set $\{r \in \mathcal{Q}\colon\ W_q = W_{(q,r)}\}$. We have ensured that for each $W \in \mathcal{W}_q$, $\{r \in B_q\colon\ W \in W_{(q,r)}\}$ will be infinite. For each $x \in X$, let $\mathcal{W}'_x$ temporarily denote the filter $\mathcal{W}_x \cup \{W_x \cap W\colon\ W \in \mathcal{W}_x\}$.

Let i_α be any element of $I \setminus \bigcup\{\operatorname{dom}(\varrho^*_n)\colon\ n \in \omega\}$ and $I' = I \setminus (\{i_\alpha\} \cup \bigcup\{\operatorname{dom}(\varrho^*_n)\colon n \in \omega\}$.

As an intermediary step it is helpful to note that $\mathcal{X}' = \{\mathcal{W}'_x\colon\ x \in X\}$ is an E_α-structure and we will check that $\mathcal{B}_{I'}$ is independent mod $\mathcal{X}'$. We actually prove a stronger statement: for each $k \in \omega$, finite function $\tau \subset I' \times \omega$, and each $W \in \mathcal{W}_{\mathcal{X}'}$,

$$W \cap W_{-A} \neq \emptyset \ \Rightarrow b_{(i_\alpha,k)} \cap b_\tau \cap \{q \in Q \setminus A\colon\ W \cap W_{-A} \in \mathcal{W}'_q\} \text{ is infinite} \tag{3.3}$$

and

$$(3.4)\qquad W \cap W_A \neq \emptyset \ \Rightarrow\ b_{(i_\alpha,k)} \cap b_\tau \cap \{q \in A\colon\ W \cap W_A \in \mathcal{W}'_q\} \text{ is infinite.}$$

To see this, notice that, for each n, if $W \cap W^*_{2n} \neq \emptyset$ and/or $W \cap W^*_{2n+1} \neq \emptyset$, then

$$b_{(i_\alpha,k)} \cap b_\tau \cap b_{\varrho^*_{2n}} \cap \{q \in A\colon\ W \cap W_A \in \mathcal{W}'_q\} \text{ is infinite}$$

and/or

$$b_{(i_\alpha,k)} \cap b_\tau \cap b_{\varrho^*_{2n+1}} \cap \{q \in \mathcal{Q} \setminus A\colon\ W \cap W_{-A} \in \mathcal{W}'_q\} \text{ is infinite}$$

For each integer k, we define

$$W(A, 2k) = W_A \times \{(\alpha, k)\} \subseteq (\omega + 1)^{\alpha+1}$$

and

$$W(A, 2k+1) = W_{-A} \times \{(\alpha, k\}.$$

Let $A_0 = \{a \in A\colon\ W_A \in \mathcal{W}'_\alpha\} = \{a \in A\colon\ W_a = W_A\}$ and let $A_1 = \{a \in A\colon\ W_A \notin \mathcal{W}'_a\} = \{a \in A\colon\ W_a = W_{-A}\}$. Similarly, set $\mathcal{Q}_0 = \{q \in \mathcal{Q} \setminus A\colon\ W_{-A} \in \mathcal{W}'_q\}$ and $\mathcal{Q}_1 = \{q \in \mathcal{Q} \setminus A\colon\ W_{-A} \notin \mathcal{W}'_q\} = \{q \in \mathcal{Q} \setminus A\colon\ W_A = W_q\}$. For each $q \in \mathcal{Q}$ recall that we have defined $B_q \subset \mathcal{Q}$ above so that for each $W \in \mathcal{W}'_q$, $\{r \in B_q\colon\ W \in \mathcal{W}_{(q,r)}\}$ is infinite. Define a function $f_\alpha\colon\ X \to \omega$ as follows:

(1) for each $q \in A_0 \cup \mathcal{Q}_0$, $f_\alpha(q) = k$ where k is the unique integer such that $q \in b_{(i_\alpha,k)}$,
(2) $f_\alpha\colon\ A_1 \cup \mathcal{Q}_1 \to \omega$ is 1-to-1,
(3) for each $q \in \mathcal{Q}$ and $r \in B_q$, $f_\alpha((q,r)) = f_\alpha(q)$, and
(4) $f_\alpha\colon\ (\mathcal{Q} \times \mathcal{Q}) \setminus \bigcup_{q\in\mathcal{Q}} B_q \to \omega$ is 1-to-1.

For each $x \in X$, we define

$$(3.5)\ \mathcal{W}'_x = \{W \times \{(\alpha, f_\alpha(x))\}\colon\ W \in \mathcal{W}_x\} \cup \{(W_x \cap W) \times \{(\alpha, f_\alpha(x))\}\colon\ W \in \mathcal{W}_x\}$$

that is a filter in $RO((\omega+1)^{\alpha+1})$.

Note that if $W_x = W_A$ and $f_\alpha(x) = n$, then $W(A, 2n) \in \mathcal{W}'_x$ and similarly, $W(A, 2n+1) \in \mathcal{W}'_x$ if $W_x = W_{-A}$. Condition (2) in the statement of the Lemma certainly holds since, for each n,

$$\{q \in Q \setminus A\colon\ W(A, 2n) \in \mathcal{W}'_q\} \cup \{q \in A\colon\ W(A, 2n+1) \in \mathcal{W}'_q\}$$

has at most one point in it because this union is contained in $(A_1 \cup \mathcal{Q}_1) \cap f_\alpha^{-1}(n)$. Properties (1) and (3) of $E_{\alpha+1}$-structure are immediate by the definition of $\mathcal{X}'$. To see that property (2) holds, assume that $W \subset (\omega+1)^\alpha$ is clopen and $k \in \omega$, we must show that the clopen set $W \times \{(\alpha, k)\}$ is in $\mathcal{W}'_q$ for some $q \in \mathcal{Q}$. Assume first that $W \cap W_A \neq \emptyset$ and recall that $b_{(i_\alpha,k)} \cap b_\emptyset \cap \{q \in A\colon W \cap W_A \in \mathcal{W}_q\}$ is infinite, so let q be any member of this set. Since $W_A \in \mathcal{W}'_q$, we have $q \in A_0$ and so $f_\alpha(q) = k$. Therefore, $W \times \{(\alpha, k)\} \in \mathcal{W}'_q$. The argument for the case that $W \cap W_{-A} \neq \emptyset$ is similar.

The fact that $\mathcal{B}_{I'}$ is independent mod $\mathcal{W}'_{\mathcal{X}'}$ follows routinely from 3.3 and 3.4. $\square$

The proofs of the Lemmas 32 and 33 are so similar to the previous proof that they can be omitted.

Lemma 32. *Assume $\alpha < \mathfrak{c}$, $\mathcal{X}$ is an E_α structure, I is an uncountable set such that $\mathcal{B}_I$ is independent* mod *$\mathcal{X}$, and $B \subset \mathcal{Q} \times \mathcal{Q}$, then there is an $E_{\alpha+1}$ structure $\mathcal{X}'$ and $I' \subset I$ and a pairwise disjoint family $\{W(B,n)\colon n \in \omega\} \subset \mathcal{W}_{\mathcal{X}'}$ such that*

(1) *$\mathcal{X} \prec \mathcal{X}'$ and $I \setminus I'$ is countable, and*
(2) *for each $x \in X$, there is an $n \in \omega$, $W(B,n) \in \mathcal{W}'_x$, and for each $n \in \omega$ $B \supset \{(q,r) \in \mathcal{Q} \times \mathcal{Q}\colon W(B,2n) \in \mathcal{W}'_{(q,r)}\}$ and $\{(q,r) \in \mathcal{Q} \times \mathcal{Q}\colon W(B,2n+1) \in \mathcal{W}'_{(q,r)}\}$ is finite,*
(3) *$\mathcal{B}_{I'}$ is independent* mod *$\mathcal{X}'$.*

Lemma 33. *Assume $\alpha < \mathfrak{c}$, $\mathcal{X}$ is an E_α structure, I is an uncountable set such that $\mathcal{B}_I$ is independent* mod *$\mathcal{X}$, and $W \in RO((\omega+1)^\alpha)$, then there is an E_α structure $\mathcal{X}'$ such that*

(1) *$\mathcal{X} \prec \mathcal{X}'$, and*
(2) *$W \in \mathcal{W}_{\mathcal{X}'}$, and*
(3) *$\mathcal{B}_I$ is independent* mod *$\mathcal{X}'$.*

This next lemma introduces the trick to ensure that $\mathcal{Q} \times \mathcal{Q}$ will be far from $\beta(\mathcal{Q} \times \mathcal{Q}) \setminus (\mathcal{Q} \times \mathcal{Q})$ (the final condition in Theorem 24), and then Lemma 35 incorporates the technique into the construction of E_α-structures.

Lemma 34. *Suppose that S is a countable space with the property that for each pairwise disjoint collection $\{W_n\colon n \in \omega\}$ of clopen sets, there is another pairwise disjoint collection $\{W(n,m)\colon m \leqslant n \in \omega\}$ of clopen sets such that for each $s \in S$ either there is an $n \in \omega$, such that $s \in W_n$ or for each selection $0 \leqslant m_n \leqslant n$ ($n \in \omega$), there is a clopen W such that $s \in W$ and $W \cap \bigcup_{n \in \omega} W(n, m_n)$ is empty. Then S is far from $\beta S \setminus S$.*

P r o o f. Assume that A is any countable discrete subset of $\beta S \setminus S$. Choose any collection $\{W_n^*\colon n \in \omega\}$ of clopen subsets of βS such that for each $a \in A$, there is an n satisfying $W_n^* \cap A = \{a\}$. For each n, let $W_n = S \cap W_n^*$, and let $\{W(n,m)\colon m \leqslant n \in \omega\}$ be as in the lemma. For each n, $W_n^* = \bigcup\limits_{m\leqslant n} \overline{W(n,m)}$, hence there is an $m_n \leqslant n$ such that $\overline{W(n,m_n)} \cap A \neq \emptyset$. For each n, clearly no point of W_n is a limit point of A. In addition, it clearly follows that no other point of S is a limit point of A since A is contained in the closure of $\bigcup\limits_n W(n,m_n)$. $\square$

Lemma 35. *Assume $\alpha < \mathfrak{c}$, $\mathcal{X}$ is an E_α structure, I is an uncountable set such that $\mathcal{B}_I$ is independent* mod $\mathcal{X}$, *and $\{W_n\colon n \in \omega\} \subset \mathcal{W}_{\mathcal{X}}$, is a pairwise disjoint family, then there is an $E_{\alpha+1}$ structure $\mathcal{X}'$ and $I' \subset I$ and a pairwise disjoint family $\{W(n,m)\colon m \leqslant n \in \omega\} \subset \mathcal{W}_{\mathcal{X}'}$ such that*

(1) *$\mathcal{X} \prec \mathcal{X}'$ and $I \setminus I'$ is countable, and*

(2) *for each $x \in \mathcal{Q} \times \mathcal{Q}$, either there is an $n \in \omega$, such that $W_n \in \mathcal{W}_x$, or for each selection $0 \leqslant m_n \leqslant n$ ($n \in \omega$), there is a $W \in \mathcal{W}'_x$ such that $W \cap W(n,m_n) = \emptyset$.*

(3) *$\mathcal{B}_{I'}$ is independent* mod *$\mathcal{X}'$.*

P r o o f. Let $\{W_n\colon n \in \omega\} \subset \mathcal{W}_{\mathcal{X}}$ be any pairwise disjoint family. We will be defining an $E_{\alpha+1}$ structure $\mathcal{X}'$. There is no loss of generality in assuming that $\{W_n\colon n \in \omega\}$ is maximal since if we add some set W_0 to the collection and $W_0 \in \mathcal{W}_x$ for some x, then it will be immediate that $W_0 \times (\omega+1)^{\{\alpha\}}$ will already be a set that is disjoint from $W(n,m_n)$ for all $n \neq 0$. Choose any $i_\alpha \in I$ and let $I' = I \setminus \{i_\alpha\}$. For brevity, let b_k denote $b_{(i_\alpha,k)}$ for each k.

For each $0 \leqslant m < n$, let $W(n,m) = W_n \times \{(\alpha,m)\}$ and $W(n,n) = W_n \times (\omega+1 \setminus \{0,\dots,n-1\})^{\{\alpha\}}$. Clearly we have that $W_n \times (\omega+1)^{\{\alpha\}}$ is equal to $\bigcup\limits_{m\leqslant n} W(n,m)$.

Now we are ready to define the family $\{\mathcal{W}'_x\colon x \in X\}$.

For each $q \in \mathcal{Q}$ such that there is an n with $W_n \in \mathcal{W}_q$, let k_q denote the integer k such that $q \in b_k$. Also, let B_q denote the infinite family of $r \in \mathcal{Q}$ such that $W_n \in \mathcal{W}_{(q,r)}$ (by condition (1) of E_α structure). For each $x \in \{q\} \cup (\{q\} \times B_q)$ (where there is an n such that $W_n \in \mathcal{W}_q$), set

$$\mathcal{W}'_x = \{W \times \{(\alpha,k_q)\}\colon W \in \mathcal{W}_x\}.$$

Since the family $\{W_n\colon n \in \omega\}$ is maximal and $\mathcal{B}_I$ is independent mod $\mathcal{X}$, this assignment will already ensure condition (2) of $E_{\alpha+1}$ structure. It will also preserve condition (1) for those q for which $W_n \in \mathcal{W}_q$ for some n, and it is immediate that $\mathcal{B}_{I'}$ is going to be independent mod $\mathcal{X}'$.

For each $(q,r) \in \mathcal{Q} \times \mathcal{Q}$ such that there is an n such that $W_n \in \mathcal{W}_{(q,r)}$ and $W_n \notin \mathcal{W}_q$, set

$$\mathcal{W}'_{(q,r)} = \{W \times \{(\alpha, 0)\} \colon W \in \mathcal{W}_{(q,r)}\}.$$

It remains to define $\mathcal{W}'_x$ for all those x such that $W_n \notin \mathcal{W}_x$ for each n. If $x \in \mathcal{Q} \times \mathcal{Q}$ is such an x, then set $\mathcal{W}'_x$ to be the filter generated by including

$$\operatorname{intcl} \bigcup (W \cap W_n) \times (n \setminus H_n)^{\{\alpha\}}$$

for all $W \in \mathcal{W}_x$ and each sequence, $\langle H_n \colon n \in \omega \rangle$, such that $\{|H_n| \colon n \in \omega\} \subset \omega$ is bounded.

For the remaining $q \in \mathcal{Q}$ there are two cases so as to preserve condition (1) of $E_{\alpha+1}$ structure.

If for each $W \in \mathcal{W}_q$, there is an r such that $W \cap W_n \in \mathcal{W}_{(q,r)}$, then set $B_q = \{r \in \mathcal{Q} \colon (\exists n) W_n \in \mathcal{W}_{(q,r)}\}$ and define

$$\mathcal{W}'_q = \{W \times \{(\alpha, 0)\} \colon W \in \mathcal{W}_q\},$$

otherwise, $B_q = \{r \in \mathcal{Q} \colon (\forall n) W_n \notin \mathcal{W}_{(q,r)}\}$, and define $\mathcal{W}'_q$ to be the filter generated by

$$\operatorname{intcl} \bigcup (W \cap W_n) \times (n \setminus H_n)^{\{\alpha\}}$$

for all $W \in \mathcal{W}_q$ and each sequence, $\langle H_n \colon n \in \omega \rangle$, such that $\{|H_n| \colon n \in \omega\} \subset \omega$ is bounded. $\square$

3.3. Proof of Theorem 25. Let X be a countable Simon type space with the property that βX is (ω, ω)-irresolvable (use Proposition 14). Of course X can be chosen so that $\beta X \setminus X$ is $\aleph_0$-bounded. By Theorem 7, there is a vD-space S that maps continuously by a 1-to-1 function into a dense subset of X. Therefore there is a continuous function f from βS onto βX with the property that $f \restriction S$ is 1-to-1. Since S is crowded, it also follows that each fiber, $f^{-1}(x)$ $(x \in \beta X)$, is nowhere dense in βS. The proof that βS is not homeomorphic to $E(2^{\mathfrak{c}})$ follows from the next Lemma.

Lemma 36. *If f is a mapping from $E(2^{\mathfrak{c}})$ onto an (ω, ω)-irresolvable space K, then there is a point $x \in K$ such that $f^{-1}(x)$ has interior.*

Proof. Assume that $Y \subset K$ is a countable dense subset such that every countable $D \subset K \setminus Y$ is nowhere dense in K. Also assume that f maps $E(2^{\mathfrak{c}})$ into K so that $f^{-1}(x)$ is nowhere dense for each $x \in K$. We show that the range of f is nowhere dense in K. Let φ denote the canonical irreducible map from $E(2^{\mathfrak{c}})$ onto $2^{\mathfrak{c}}$. It follows that for each $y \in Y$, there is a dense open subset U_y of $2^{\mathfrak{c}}$ so that $\varphi(f^{-1}(y))$

is disjoint from U_y. It is routine to show that each G_δ subset of $2^{\mathfrak{c}}$ is separable, so let T be any countable dense subset of the dense set $\bigcap_{y\in Y} U_y$. Let $D \subset E(2^{\mathfrak{c}})$ be any countable set such that $\varphi(D) = T$. Since φ is irreducible and $\varphi(D)$ is dense in $2^{\mathfrak{c}}$, it follows that D is dense in $E(2^{\mathfrak{c}})$. Since D is disjoint from $f^{-1}(Y)$, it follows that $f(D)$, and $\overline{f(D)}$, is nowhere dense in K □

3.4. Proof of Theorem 26. Before constructing our vD-space, we prove the following Lemma to provide a condition that will ensure we have an $\aleph_0$-bounded remainder.

Lemma 37. *Assume that S is a Lindelöf zero-dimensional space and that for each countable family $\{W(n,m)\colon n,m \in \omega\}$ of clopen sets such that for each n, the family $\{W(n,m)\colon m \in \omega\}$ is a descending sequence with empty intersection, there is a partition $\{Y_n\colon n \in \omega\}$ of clopen sets such that for each n,k there is an m such that Y_n is disjoint from $W(k,m)$. Then $\beta S \setminus S$ is $\aleph_0$-bounded.*

Proof. Let $\{a_n\colon n \in \omega\}$ be a subset of $\beta S \setminus S$. Since S is zero-dimensional Lindelöf, there is, for each $n \in \omega$, a countable descending sequence $\{W(n,m)\colon m \in \omega\}$ of clopen subsets of S such that $a_n \in \overline{W(n,m)}$ for each n, and $\bigcap_{m\in\omega} W(n,m)$ is empty. Let $\{Y_n\colon n \in \omega\}$ be the partition of S as given. For each $s \in S$, there is an n such that $s \in Y_n$, hence it suffices to prove that $a_k \notin \overline{Y_n}$ for each k,n. By assumption, for each n,k there is an $m_{n,k} \in \omega$ so that $Y_n \cap W(k,m_{n,k})$ is empty. For each k, $a_k \in \overline{W(k,m_{n,k})}$, hence $a_k \notin \overline{Y_n}$. □

Now, assume that $\mathrm{MA}_{\mathrm{ctble}}$ holds. We start with $\mathcal{B}_0$, any countable base of clopen sets for the usual topology on $\mathbb{Q}$. By induction on $\alpha < \mathfrak{c}$, we construct subalgebras $\mathcal{B}_\alpha \subset \mathcal{P}(\mathbb{Q})$ so that $|\mathcal{B}_\alpha| \leqslant |\alpha+\omega|$ and the 0-dimensional topology, X_α, on $\mathbb{Q}$ induced by $\mathcal{B}_\alpha$ is crowded (i.e. each member of $\mathcal{B}_\alpha$ is infinite). For limit α we will simply let $\mathcal{B}_\alpha$ be the union of the $\mathcal{B}_\gamma$ for $\gamma < \alpha$.

Let $\{A_\gamma\colon \gamma \in \mathfrak{c}\}$ be an enumeration of $\mathcal{P}(\mathbb{Q})$ and let $\{\{W(\gamma,n,m)\colon n,m \in \omega\}\colon \gamma \in \mathfrak{c}\}$ be an enumeration of all the doubly-indexed countable families of subsets of $\mathbb{Q}$ so that each is listed cofinally often in $\mathfrak{c}$.

For each $\alpha \in \mathfrak{c}$, we ensure there is a partition $\{Y_n\colon n \in \omega\} \subset \mathcal{B}_{\alpha+1}$ so that

(1) for each n, one of $Y_n \cap A$ or $Y_n \setminus A$ is finite, and

(2) if, for each $k \in \omega$, $\{W(\alpha,k,m)\colon m \in \omega\} \subset \mathcal{B}_\alpha$ is descending and has empty intersection, then for each $n \in \omega$, there is a function $f_n \in \omega^\omega$ such that $Y_n \cap W(k, f_n(k))$ is empty for each k.

Given $\mathcal{B}_\alpha$, once we show how to construct $\{Y_n\colon n \in \omega\}$ we simply let $\mathcal{B}_{\alpha+1}$ be the Boolean algebra generated by $\mathcal{B}_\alpha \cup \{Y_n\colon n \in \omega\}$.

Let $\{q_i\colon\ i \in \omega\}$ enumerate $\mathbb{Q}$. We construct the family $\{Y_n\colon\ n \in \omega\}$ by induction on n so that $\{q_i\colon\ i \leqslant n\} \subset \bigcup_{i \leqslant n} Y_i$, Y_n is crowded in X_α, and $\mathbb{Q} \setminus \bigcup_{i \leqslant n} Y_i$ is dense (in X_α).

Let A denote A_α, and for each n, m, $W(n,m) = W(\alpha, n, m)$. Let q_{l_n} be the point of $\mathbb{Q} \setminus \bigcup_{i \leqslant n} Y_i$ with minimal index (thus $l_n \geqslant n$). We will ensure that $q_{l_n} \in Y_n$ and that $Y_n \setminus \{q_{l_n}\}$ is either contained in A or is disjoint from A.

If $A \setminus \bigcup_{i<n} Y_i$ is dense in some X_α-neighborhood of q_{l_n}, then let $B_n \in \mathcal{B}_\alpha$ be such a neighborhood. Otherwise, let $B_n \in \mathcal{B}_\alpha$ be a neighborhood of q_{l_n} with a dense subset that is disjoint from $A \cup \bigcup_{i<n} Y_i$. By Lemma 17, we can choose a dense subset Z_n of B_n such that $B_n \setminus (Z_n \cup \bigcup_{i<n} Y_i)$ is also dense in B_n and so that, again, either $Z_n \subset A$ or Z_n is disjoint from A. Let $I \subset \omega$ be the indices so that $\{q_j\colon\ j \in I\} = \{q_{l_n}\} \cup Z_n \setminus \{q_i\colon\ i < l_n\}$. We will choose $Y_n \ni q_{l_n}$ to be a subset $Z_n \cup \{q_{l_n}\}$, hence we will preserve that $\mathbb{Q} \setminus \bigcup_{i \leqslant n} Y_i$ is dense and we will have that either $Y_n \cap A$ or $Y_n \setminus A$ is finite. We need only work to ensure that Y_n is crowded and to select our function f_n.

By a simple re-enumeration (and possibly choosing subsequences) we may assume that for each $k \in \omega$ and each $l \in I$, $W(k,l)$ is disjoint from $\{q_0, \ldots, q_l\}$. We define a poset P that is a subposet of the strictly increasing functions from some integer into I ordered by usual extension. A function p will be in P if $p(0) = l_n$ and for each $k \in \mathrm{dom}(p)$, $q_{p(k)}$ is not in $W(i, p(i))$ for each $i < k$. It follows then that $q_{p(k)}$ is not in $W(i, p(i))$ for all $i \in \mathrm{dom}(p)$ since p is an increasing function and $q_{p(k)}$ is not in $W(i, p(k))$ for all i.

Once we have selected a filter $G \subset P$ using $\mathrm{MA}_{\mathrm{ctble}}$, we will have a function $f_n = \bigcup_{p \in G} p$ and we will let $Y_n = \{q_{f_n(k)}\colon\ k \in \omega = \mathrm{dom}(f_n)\}$. It follows immediately that $Y_n \cap W(k, f_n(k))$ is empty for each k. We need only to identify enough dense subsets of P to ensure that $\mathrm{dom}(f_n) = \omega$ and that Y_n is crowded.

To see that $\mathrm{dom}(f_n)$ will be ω, we can note that each $p \in P$ with $i = \mathrm{dom}(p)$ has an extension p' with $i \in \mathrm{dom}(p')$. To see this, note that $\{q_i\colon\ i \in I'\} = Z_n \setminus \bigcup_{j<i} W(j, p(j))$ is crowded since it contains q_{l_n} and each of the $W(j, p(j))$ are clopen. Therefore $p'(i)$ can be chosen to be any member of I' that is greater than $p(i-1)$.

Finally to see that Y_n will be crowded, it suffices to show that for each $k \in \omega$ and $U \in B_\alpha$ the set

$$D(k,U) = \{p \in P\colon\ (k \in \mathrm{dom}(p) \text{ and } p(k) \in U) \Rightarrow (\exists j > k) p(j) \in U\}$$

is dense in P. Assume $p \in P$, $k \in \mathrm{dom}(p)$ and $p(k) \in U \in \mathcal{B}_\alpha$. Let $j = \mathrm{dom}(p)$ and choose $p'(j) \in I$ with $p'(j) > p(j-1)$ so that $q_{p'(j)} \in U \setminus \bigcup_{i<j} W(i, p(i))$, which we may do since Z_n is dense in B_n and $q_{p(k)} \in U \setminus \bigcup_{i<j} W(i, p(i))$.

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