ORDER CONVERGENCE OF VECTOR MEASURES ON TOPOLOGICAL SPACES

Surjit Singh Khurana, Iowa City

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Abstract. Let X be a completely regular Hausdorff space, E a boundedly complete vector lattice, $C_b(X)$ the space of all, bounded, real-valued continuous functions on X, $\mathcal{F}$ the algebra generated by the zero-sets of X, and $\mu\colon C_b(X) \to E$ a positive linear map. First we give a new proof that μ extends to a unique, finitely additive measure $\mu\colon \mathcal{F} \to E^+$ such that ν is inner regular by zero-sets and outer regular by cozero sets. Then some order-convergence theorems about nets of E^+-valued finitely additive measures on $\mathcal{F}$ are proved, which extend some known results. Also, under certain conditions, the well-known Alexandrov's theorem about the convergent sequences of σ-additive measures is extended to the case of order convergence.

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1. Introduction and notation

All vector spaces are taken over reals. E, in this paper, is always assumed to be a boundedly complete vector lattice (and so, necessarily Archimedean) ([??], [??], [??]). If E is a locally convex space and E' its topological dual, then $\langle \cdot, \cdot \rangle : E \times E' \to \mathbb{R}$ will stand for the bilinear mapping $\langle x, f \rangle = f(x)$. For a completely regular Hausdorff space X, $\mathcal{B}(X)$ and $\mathcal{B}_1(X)$ are the classes of Borel and Baire subsets of X, $C(X)$ and $C_b(X)$ are the spaces of all real-valued, and real-valued and bounded, continuous functions on X and $\widetilde{X}$ is the Stone-Čech compactification of X respectively. For an $f \in C_b(X)$, $\tilde{f}$ is its unique continuous extension to $\widetilde{X}$. The sets $\{f^{-1}(0)\colon f \in C_b(X)\}$ are called the zero-sets of X and their complements the cozero sets of X.

For a compact Hausdorff space X and a boundedly complete vector lattice G, let $\mu\colon \mathcal{B}(X) \to G^+$ be a countably additive (countable additivity in the order convergence of G) Borel measure; then μ is said to be quasi-regular if for any open $V \subset X$, $\mu(V) = \sup\{\mu(C)\colon C \text{ compact}, C \subset V\}$. Integration with respect to these measures is taken in the sense of ([??], [??]). There is a 1-1 correspondence between these quasi-regular, positive, G-valued Borel measures on X and positive linear mappings $\mu\colon C(X) \to G$ ([??], [??], [??]); $M^+_{(o)}(X, G)$ will denote the set of all these measures.

Now suppose that X is a completely regular Hausdorff space. A positive countably additive Borel measure $\mu\colon \mathcal{B}(X) \to G^+$ is said to be tight if for any open $V \subset X$, $\mu(V) = \sup\{\mu(C)\colon C \text{ compact}, C \subset V\}$ ([??], p. 207). This measure gives a positive linear mapping $\tilde{\mu}\colon C(\widetilde{X}) \to G$, $\tilde{\mu}(f) = \mu(f_{|X})$ and the corresponding quasi-regular, positive, G-valued Borel measure on $\widetilde{X}$ is given by $\tilde{\mu}(B) = \mu(X \cap B)$ for every Borel $B \subset \widetilde{X}$. $M^+_{(o,t)}(X, G)$ will denote the set of all tight measures.

If $\mu\colon \mathcal{B}(X) \to G^+$ is a countably additive Borel measure, then μ is said to be τ-smooth if for any increasing net $\{U_\alpha\}$ of open subsets of X, $\mu\big(\bigcup U_\alpha\big) = \sup \mu(U_\alpha)$ (some properties of these measures are given in [??], p. 207). Any such measure gives a positive linear mapping $\tilde{\mu}\colon C(\widetilde{X}) \to G$, $\tilde{\mu}(f) = \mu(f_{|X})$ and the corresponding quasi-regular, positive, G-valued Borel measure on $\widetilde{X}$ is given by $\tilde{\mu}(B) = \mu(X \cap B)$ for every Borel set $B \subset \widetilde{X}$. $M^+_{(o,\tau)}(X, G)$ will denote the set of all τ-smooth measures. If $\mu\colon \mathcal{B}_1(X) \to G^+$ is a countably additive Baire measure then, as in the case of τ-smooth measure, we get $\tilde{\mu}\colon C(\widetilde{X}) \to G$, $\tilde{\mu}(f) = \mu(f_{|X})$; $M^+_{(o,\sigma)}(X, G)$ will denote the set of all these Baire measures.

In ([??], [??]) some interesting results are derived about the weak order convergence of nets of positive lattice-valued measures. In this paper we extend some of those results to a more general setting. Before doing that we need Alexandrov's Theorem.

2. Alexandrov's theorem

Suppose X is a completely regular Hausdorff space and $\mathcal{F}$ is the algebra generated by the zero-sets of X. Assume that $\mu\colon C_b(X) \to \mathbb{R}$ is a positive linear mapping. By well-known Alexandrov's Theorem, there exists a unique finitely additive measure ν, $\nu\colon \mathcal{F} \to \mathbb{R}^+$, such that

(i) ν is inner regular by zero-sets and outer regular by cozero sets; (ii) $\int f \,\mathrm{d}\nu = \mu(f)$ for all $f \in C_b(X)$ ([??], Theorem 5, p. 165; [??]) (Note that $C_b(X)$ is contained in the uniform closure of $\mathcal{F}$-simple functions on X in the space of all bounded functions on X and so each $f \in C_b(X)$ is ν-integrable; ν is also generally denoted by μ.) This theorem has been extended to the vector case (e.g. [??], p. 353). The proof is quite sophisticated. We give a quite different proof based on the regularity properties of the corresponding quasi-regular Borel measure on $\widetilde{X}$; this also provides a relation between this finitely additive measure and the corresponding quasi-regular Borel measure on $\widetilde{X}$. We start with a lemma.

Lemma 1. *If Z_1 and Z_2 are zero-sets in X then $\overline{(Z_1 \cap Z_2)} = \overline{Z_1} \cap \overline{Z_2}$ (for a subset $A \subset X$, $\bar{A}$ denotes the closure of A in $\widetilde{X}$).*

P r o o f. Suppose this is not true. Take a point $a \in \overline{Z_1} \cap \overline{Z_2} \setminus \overline{(Z_1 \cap Z_2)}$ (note that $Z_1 \cap Z_2$ can be empty). Take an $f \in C_b(X)$, $0 \leqslant f \leqslant 1$, such that $\tilde{f}(a) = 1$ and $f = 0$ on $(Z_1 \cap Z_2)$. For $i = 1, 2$, take $h_i \in C_b(X)$ such that $0 \leqslant h_i \leqslant 1$ and $Z_i = h_i^{-1}(0)$. Define $f_i(x) = f(x)h_i(x)/(h_1(x) + h_2(x))$ for $x \notin (Z_1 \cap Z_2)$ and 0 otherwise. These functions are continuous and $f = f_1 + f_2$. Thus $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$. Since $f_i = 0$ on Z_i, $\tilde{f}_i = 0$ on $\overline{Z_i}$ and so $\tilde{f}_1 + \tilde{f}_2 = 0$ on $\overline{Z_1} \cap \overline{Z_2}$. This means $\tilde{f}(a) = 0$, a contradiction. □

Now we come to the main theorem.

Theorem 2 ([??], p. 353)**.** *Suppose X is a completely regular Hausdorff space, E is a boundedly order-complete vector lattice and $\mu\colon C_b(X) \to E$ is a positive linear mapping. Then there exist a unique finitely additive measure $\nu\colon \mathcal{F} \to E^+$ such that, in terms of order convergence,*

(i) *ν is inner regular by zero-sets and outer regular by cozero sets;*

(ii) *$\int f \,\mathrm{d}\nu = \mu(f)$ for all $f \in C_b(X)$;*

(iii) *For any zero-set $Z \subset X$ we have $\nu(Z) = \tilde{\mu}(\overline{Z})$, $\overline{Z}$ being the closure of Z in $\widetilde{X}$.*

P r o o f. There is no loss of generality if we assume that $E = C(S)$, S being a Stonian space. The given mapping gives a positive linear mapping $\tilde{\mu}\colon C(\widetilde{X}) \to E$; by ([??], [??]) we get an E-valued, positive quasi-regular Borel measure $\tilde{\mu}\colon \mathcal{B}(\widetilde{X}) \to E^+$. If A is a subset of X or $\widetilde{X}$, $\bar{A}$ will denote the closure of A in $\widetilde{X}$. We prove this theorem in several steps.

I. Let $\overline{\mathcal{Z}} = \{\bar{A}\colon\ A$ a zero-set in $X\}$. Then for every $Q \in \overline{\mathcal{Z}}$, $\inf\{\tilde{\mu}((\widetilde{X} \setminus Q) \setminus W)\colon W \in \overline{\mathcal{Z}}\} = 0$.

P r o o f. Using the quasi-regularity of $\tilde{\mu}$, take an increasing net $\{C_\alpha\}$ of compact subsets of $(\widetilde{X} \setminus Q)$ such that $\inf(\tilde{\mu}((\widetilde{X} \setminus Q) \setminus C_\alpha)) = 0$. Fix α and take a $g \in C(\widetilde{X})$, $0 \leqslant g \leqslant 1$, such that $g = 1$ on C_α and $g = 0$ outside $\widetilde{X} \setminus Q$. Let $V = \{x \in \widetilde{X}\colon\ g(x) > \frac{1}{2}\}$, $Z = \{x \in \widetilde{X}\colon\ g(x) \geqslant \frac{1}{3}\}$. We have $Z \supset \overline{(Z \cap X)} \supset \overline{(V \cap X)} \supset V \supset C_\alpha$ (note that X is dense in $\widetilde{X}$). Now $Z \cap X$ is a zero-set in X and taking $W = \overline{(Z \cap X)}$, we have $C_\alpha \subset W \subset (\widetilde{X} \setminus Q)$. Since $\overline{\mathcal{Z}}$ is closed under finite unions, the result follows.

II. Let $\mathcal{A}$ be the algebra in $\widetilde{X}$ generated by $\overline{\mathcal{Z}}$ and denote by $\mathcal{A}_0$ the elements of $\mathcal{A}$ which have the property that these elements and their complements are inner regular by the elements of $\overline{\mathcal{Z}}$. Then $\mathcal{A}_0 = \mathcal{A}$.

P r o o f. We use I and Lemma 1 to prove it. By I, $\mathcal{A}_0 \supset \overline{\mathcal{Z}}$. By definition, $\mathcal{A}_0$ is closed under complements. Using Lemma 1, it is a routine verification that if A and B are in $\mathcal{A}_0$ then $A \cup B$ and $A \cap B$ are also in $\mathcal{A}_0$. This proves the result.

III. Let $\mathcal{F}$ be the algebra in X generated by zero-sets in X. Then it is a simple verification that $\mathcal{A} \cap X \supset \mathcal{F}$. Also, if $A \in \mathcal{A}$ and $A \cap X = \emptyset$, then $\tilde{\mu}(A) = 0$. To prove this, take any $\overline{Z} \in \overline{\mathcal{Z}}$, Z being a zero-set in X, such that $\overline{Z} \subset A$. This means Z is empty and so $\tilde{\mu}(A) = 0$. Now we can define a $\nu\colon \mathcal{F} \to E$, $\nu(B) = \tilde{\mu}(A)$, A being any element in $\mathcal{A}$ with $B = A \cap X$; it is a trivial verification that ν is well-defined, finitely additive and it is inner regular by zero-sets in X and outer regular by positive-sets in X. We also have $\nu(Z) = \tilde{\mu}(\overline{Z})$ for any zero-set $Z \subset X$.

IV. For any $f \in C_b(X)$, $\mu(f) = \int f \, \mathrm{d}\nu$.

P r o o f. Let $\mathcal{M}$ be the vector space of all $\mathcal{F}$-simple functions on X. With the norm topology $\|\cdot\|$ on $C(S)$, the mapping $\tilde{\mu}\colon \mathcal{M} \to C(S)$, $f \to \int f \, \mathrm{d}\mu$ is positive and continuous and $C_b(X)$ lies in the norm completion of $\mathcal{M}$; this implies that every $f \in C_b(X)$ is ν-integrable. Put $\mu(1) = e \in C(S)$.

Take an $f \in C_b(X)$, $0 \leqslant f \leqslant 1$, and fix a large positive integer k. For i, $1 \leqslant i \leqslant k$, let $Z_i = f^{-1}[i/k, 1]$ and $W_i = \tilde{f}^{-1}[i/k, 1]$. On X we get $k^{-1} \sum_{i=1}^{k} \chi_{Z_i} \leqslant f \leqslant k^{-1} + k^{-1} \sum_{i=1}^{k} \chi_{Z_i}$. From this we get $0 \leqslant \nu(f) - k^{-1} \sum_{i=1}^{k} \nu(Z_i) \leqslant k^{-1}e$. On $\widetilde{X}$ we get $\tilde{f} \geqslant k^{-1} \sum_{i=1}^{k} \chi_{W_i} \geqslant k^{-1} \sum_{i=1}^{k} \chi_{\overline{Z_i}}$. Define $h\colon \widetilde{X} \to \mathbb{R}^+$, $h(\tilde{x}) = k^{-1} + k^{-1} \sum_{i=1}^{k} \chi_{\overline{Z_i}}$. Then h is usc (upper semi-continuous). Take an $\tilde{x} \in \widetilde{X}$ and a net $\{x_\alpha\} \subset X$ such that $x_\alpha \to \tilde{x}$. Now $\tilde{f}(\tilde{x}) = \lim f(x_\alpha) \leqslant \overline{\lim}\, h(x_\alpha) \leqslant h(\tilde{x})$ (note that h is usc). Thus $k^{-1} \sum_{i=1}^{k} \chi_{\overline{Z_i}} \leqslant \tilde{f} \leqslant k^{-1} + k^{-1} \sum_{i=1}^{k} \chi_{\overline{Z_i}}$. Integrating relative to $\tilde{\mu}$, we have $0 \leqslant \mu(f) - k^{-1} \sum_{i=1}^{k} \nu(Z_i) \leqslant k^{-1}e$ (note $\tilde{\mu}(\overline{Z_i}) = \nu(Z_i)$). Combining these results, we have $|\mu(f) - \nu(f)| \leqslant k^{-1}e$. Taking the limit over k, we get the result.

V. Uniqueness.

P r o o f. For $i = 1, 2$, let $\nu_i\colon \mathcal{F} \to E^+$ be two finitely additive regular (inner regular by zero-sets and outer regular by positive-sets in X) measures such that $\int f \, \mathrm{d}\nu_1 = \int f \, \mathrm{d}\nu_2$ for all $f \in C_b(X)$. Fix a zero-set $Z \subset X$ and take a decreasing net $\{U_\alpha\}$ of cozero sets in X such that $\nu_i(U_\alpha \setminus Z) \downarrow 0$ for $i = 1, 2$. For each α, take an $f_\alpha \in C_b(X)$ with $0 \leqslant f_\alpha \leqslant 1$, $f_\alpha = 1$ on Z, and $f_\alpha = 0$ outside U_α. For $i = 1, 2$, $\nu_i(U_\alpha) \geqslant \nu_i(f_\alpha) \geqslant \nu_i(Z)$. From this we get, since $\nu_1(f_\alpha) = \nu_2(f_\alpha)$, $\nu_1(U_\alpha) - \nu_2(Z) \geqslant 0 \geqslant \nu_1(Z) - \nu_2(U_\alpha)$. Taking limits we get $\nu_1(Z) = \nu_2(Z)$. By regularity, we have $\nu_1 = \nu_2$. This proves the result. $\square$

We denote by $M^+_{(o)}(X,E)$ the set of all finitely additive $\mu\colon \mathcal{F}\to E^+$ which are inner regular by zero-sets; they are just the positive linear operators $\mu\colon C_b(X)\to E^+$.

3. Order convergence of measures

In this section we consider the order convergence of these measures. A net $\{\mu_\alpha\}\subset M^+_{(o)}(X,E)$ is said to order-converge weakly to a $\mu\in M^+_{(o)}(X,E)$ if $\mu_\alpha(f)\to\mu(f)$ in order-convergence for each $f\in C_b(X)$; this is equivalent to $\tilde\mu_\alpha(f)\to\tilde\mu(f)$ in order-convergence for each $f\in C(\widetilde X)$.

Theorem 3. *Suppose X is a Hausdorff completely regular space, E is a boundedly order-complete vector-lattice, $\{\mu_\alpha\}$ is a uniformly order-bounded net in $M^+_{(o)}(X,E)$ and $\mu\in M^+_{(o)}(X,E)$. Then, with order convergence, the following statements are equivalent:*

(i) *$\mu_\alpha\to\mu$, pointwise on $C_b(X)$;*

(ii) *$\overline{\lim}_\alpha\,\mu_\alpha(Z)\leqslant\mu(Z)$ for every zero-set Z and $\mu_\alpha(X)\to\mu(X)$;*

(iii) *$\underline{\lim}_\alpha\,\mu_\alpha(U)\geqslant\mu(U)$ for every positive-set U and $\mu_\alpha(X)\to\mu(X)$;*

If μ is τ-smooth, then each of the above statements is also equivalent to

(iv) *$\mu_\alpha\to\mu$ pointwise on $C_{ub}(X)$, where $C_{ub}(X)$ is the set of all uniformly continuous functions on X relative to a uniformity $\mathcal{U}$ on X which gives the original topology on X (if the uniformity $\mathcal{U}$ comes from a single metric, then it is enough to assume that μ is σ-smooth).*

Proof. The positive linear mappings $\mu\colon C_b(X)\to E$ and $\mu_\alpha\colon C_b(X)\to E$ give the positive linear mappings $\tilde\mu\colon C(\widetilde X)\to E$ and $\tilde\mu_\alpha\colon C(\widetilde X)\to E$. Since the net $\{\mu_\alpha\}$ is a uniformly order-bounded, we can assume that $\mu_\alpha(1)\leqslant p$ for all α, for some $p\in E$ $(p>0)$.

(ii) and (iii) are easily seen to be equivalent.

(i) implies (ii). Fix a zero-set $Z\subset X$ and let $\overline{Z}$ be its closure in $\widetilde X$. Take a decreasing net $\{\tilde f_\gamma\}\subset C(\widetilde X)$, $0\leqslant\tilde f_\gamma\leqslant 1$ for every γ such that $\tilde f_\gamma\downarrow\chi_{\overline{Z}}$. This means that, for some $\eta_\gamma\downarrow 0$ in E we have $\mu(Z)=\tilde\mu(\overline{Z})=\tilde\mu(\tilde f_\gamma)-\eta_\gamma\geqslant\tilde\mu(\tilde f_\gamma)-2\eta_\gamma=\lim\limits_\alpha\tilde\mu_\alpha(\tilde f_\gamma)-2\eta_\gamma\geqslant\overline{\lim}_\alpha(Z)-2\eta_\gamma$. Taking the limit over γ, we get the result.

(ii) implies (i). Take an $f\in C_b(X)$, $0\leqslant f\leqslant 1$, and fix a large positive integer k. For i, $1\leqslant i\leqslant k$, put $Z_i=f^{-1}[i/k,1]$. We get $\sum\limits_{i=1}^k\chi_{Z_i}\leqslant f\leqslant\sum\limits_{i=1}^k\chi_{Z_i}+k^{-1}$. From this we get $\mu_\alpha(f)\leqslant\sum\limits_{i=1}^k\mu_\alpha(Z_i)+k^{-1}p$. This means $\overline{\lim}_\alpha(\mu_\alpha(f))\leqslant\overline{\lim}_\alpha\Big(\sum\limits_{i=1}^k\mu_\alpha(Z_i)\Big)+k^{-1}p$. Using (ii), this gives $\overline{\lim}_\alpha(\mu_\alpha(f))\leqslant\Big(\sum\limits_{i=1}^k\mu(Z_i)\Big)+k^{-1}p$. From this it follows that $\overline{\lim}_\alpha(\mu_\alpha(f)) \leqslant \mu(f) + k^{-1}p$. Taking the limit as $k \to \infty$, we get $\overline{\lim}\,\alpha(\mu_\alpha(f)) \leqslant \mu(f)$. The same result holds for $1 - f$ also (note that $\mu_\alpha(X) \to \mu(X)$). Combining these two results, we get the desired implication.

(i) implies (iv) trivially.

(iv) implies (ii). Fix a zero-set $Z \subset X$ and take a decreasing net $\{f_\gamma\} \subset C_{ub}(X)$ such that $f_\gamma \downarrow \chi_Z$ (if the uniformity comes from a single metric then the net $\{f_\gamma\}$ can be taken to be a sequence). Since μ is τ-smooth, $\mu(Z) = \lim\limits_{\gamma} \mu(f_\gamma)$ (in case the uniformity is metrizable, it is enough to assume μ to be σ-smooth). The rest of the proof is identical with that given above in ((i) implies (ii)). □

Remark 4. This generalizes ([??], Theorem 7, p. 4).

Suppose X is a uniform space. An $H \subset C_{ub}(X)$ is called ueb if it is uniformly bounded and uniformly equicontinuous. Now we have the following theorem:

Theorem 5. *Suppose X is a topological space having a uniformity $\mathcal{U}$ which gives the same topology on X, E is a boundedly order-complete vector-lattice and $\{\mu_\alpha\}$ is a uniformly order-bounded net in $M_o^+(X,E)$. Suppose there is a $\mu \in M_{(o,t)}^+(X,E)$ such that $\mu_\alpha \to \mu$ pointwise on $C_{ub}(X)$ and H is a ueb set in $C_{ub}(X)$. Then $\mu_\alpha \to \mu$ uniformly on H.*

Proof. Because $\{\mu_\alpha\}$ is uniformly order-bounded, we can take $E = C(S)$ for some Stonian compact Hausdorff space S and we can also assume that $\mu_\alpha(1) \leqslant e$ for every α, e being the unit function in $C(S)$. Also assume H to be absolutely convex and pointwise compact and $\|f\| \leqslant 1$ for all $f \in H$. Take a compact $K \subset X$. By the Arzelà-Ascoli theorem, $H_{|K}$ is norm compact in $C(K)$. Further $d(x,y) = \sup\limits_{f\in H} |f(x) - f(y)|$ is a uniformly continuous pseudometric on X. Fix $c > 0$. Define $h\colon X \to \mathbb{R}$, $h(x) = d(x,K)$; then $h \in C_{ub}(X)$. This means $V = \{x\colon h(x) < c\}$ is a positive set, it is open in X, $V \supset K$, and for an $x \in V$ there is a $y \in K$ such that $d(x,y) < c$. By the Arzelà-Ascoli theorem, there is a finite subset $\{f_i\colon 1 \leqslant i \leqslant n\} \subset H$ such that $H = \bigcup\limits_{i=1}^{n} H_i$ where $H_i = \{f \in H\colon \|f - f_i\|_{|K} < c\}$. Now take an $x \in V$ and $f \in H_i$. There is a $y \in K$ such that $d(x,y) < c$. We get $|f(x) - f_i(x)| \leqslant |f(x) - f(y)| + |f(y) - f_i(y)| + |f_i(y) - f_i(x)| \leqslant 3c$. So $|f - f_i| \leqslant 3c$ on V. From the given hypothesis, $\mu_\alpha \to \mu$ uniformly on finite subsets of $C_{ub}(X)$. Thus there exists a net $\{\eta_\alpha\} \subset E$ such that $\eta_\alpha \downarrow 0$ and $|\mu_\alpha(f_i) - \mu(f_i)| \leqslant \eta_\alpha$ for $1 \leqslant i \leqslant n$. Fix i and take an $f \in H_i$. We have $|\int f\,\mathrm{d}\mu_\alpha - \int f\,\mathrm{d}\mu| \leqslant |\int (f - f_i)\,\mathrm{d}\mu_\alpha - \int (f - f_i)\,\mathrm{d}\mu| + |\int f_i\,\mathrm{d}\mu_\alpha - \int f_i\,\mathrm{d}\mu| \leqslant |\int_V (f - f_i)\,\mathrm{d}\mu_\alpha| + |\int_{X\setminus V} (f - f_i)\,\mathrm{d}\mu_\alpha| + \int_K (f - f_i)\,\mathrm{d}\mu| + \int_{X\setminus K} (f - f_i)\,\mathrm{d}\mu| + \eta_\alpha \leqslant 3ce + 2\mu_\alpha(X \setminus V) + 3ce + 2\mu(X \setminus K) + \eta_\alpha$. Since this is true for each i, $1 \leqslant i \leqslant n$, the above result holds for every $f \in H$. So we get

$\sup\limits_{f\in H} |\int f\,\mathrm{d}\mu_\alpha - \int f\,\mathrm{d}\mu| \leqslant 6ce + 2\mu_\alpha(X\setminus V) + 2\mu(X\setminus K) + \eta_\alpha$. Taking limit superior, we get $\overline{\lim}_\alpha(\sup\limits_{f\in H} |\int f\,\mathrm{d}\mu_\alpha - \int f\,\mathrm{d}\mu|) \leqslant 2\mu(X\setminus V) + 2\mu(X\setminus K) + 6ce$. Letting $c\downarrow 0$, we get $\overline{\lim}_\alpha(\sup\limits_{f\in H} |\int f\,\mathrm{d}\mu_\alpha - \int f\,\mathrm{d}\mu|) \leqslant 4\mu(X\setminus K)$. Since $\mu \in M^+_{(o,t)}(X,E)$, the result follows. □

Corollary 6. *Suppose X is a Hausdorff completely regular space, E is a boundedly order-complete vector-lattice and $\{\mu_\alpha\}$ is a uniformly order-bounded net in $M^+_o(X,E)$. Suppose there is a $\mu \in M^+_{(o,t)}(X,E)$ such that $\mu_\alpha \to \mu$ pointwise on $C_b(X)$ and H is a uniformly bounded and pointwise equicontinuous subset of $C_b(X)$. Then $\mu_\alpha \to \mu$ uniformly on H.*

P r o o f. Consider X to be a uniform space with uniformity determined by all continuous pseudo-metrics on X. In this uniformity, H is a ueb set and so the result follows from Theorem 5. □

4. Alexandrov's theorem for a σ-additive case

In this case we take E to be a boundedly complete vector lattice and, E^* and E^*_n to be its order dual and order continuous dual. E^*_n is a band in E^* and we assume that E^*_n separates the points of E. Take a sequence $\{\mu_n\} \subset M^+_{(o,\sigma)}(X,E)$ and assume that, in order convergence, $\mu(g) = \lim \mu_n(g)$ exists for every $g \in C_b(X)$. If $E = \mathbb{R}$, the well-known Alexandrov's theorem says that $\mu \in M^+_\sigma(X)$ ([??], p. 195); in ([??], Theorem 2, p. 73), this result is extended to the case when E is a topological vector space. In the next theorem we extend the result to the case when E is a boundedly complete vector lattice.

Theorem 7. *Suppose X is a Hausdorff completely regular space, E is a boundedly order-complete vector lattice and E^*_n its order dual. Assume that E is weakly σ-distributive ([??]) and E^*_n separates the points of E. Let $\{\mu_n\} \subset M^+_{(o,\sigma)}(X,E)$ be a sequence such that, in order convergence, $\mu(g) = \lim \mu_n(g)$ exists for every $g \in C_b(X)$. Then the positive $\mu\colon C_b(X) \to E$ is generated by the E^+-valued Baire measure on X.*

P r o o f. E^*_n is a band in E^* and so the order intervals of E^*_n are $\sigma(E^*_n, E)$-compact and convex. Now the topology on E of uniform convergence on the order intervals of E^*_n is a locally convex topology for which lattice operations are continuous and so, in this topology, the positive cone E_+ of E is closed and convex. Since this topology is compatible with the duality $\langle E, E^*_n\rangle$, E_+ is also closed in $\sigma(E, E^*_n)$. Now we consider E to be a locally convex space with the topology $\sigma(E, E_n^*)$. By given hypothesis, $\mu_n\colon C_b(X) \to E$ are countably additive measures (note that E_n^* is the order continuous dual of E) and $\mu(g) = \lim \mu_n(g)$ exists for every $g \in C_b(X)$. By ([??], Theorem 2, p. 73), if $g_m \downarrow 0$ in $C_b(X)$, then $\mu_n(g_m) \to 0$ uniformly in n. So we get $\mu(g_m) \to 0$ in E. We claim that in order convergence in E, $\mu(g_m) \to 0$. This will be proved if we prove that $\inf_m \mu(g_m) = 0$ (note that $\mu(g_m) \downarrow$). Let $\inf_m \mu(g_m) = a > 0$. Take a positive element $f \in E_n^*$ such that $f(a) > 0$ (note that E_n^* separates the points of E). This implies that $\lim \langle f, \mu(g_m)\rangle = f(a) > 0$. This contradicts $\mu(g_m) \to 0$ in $(E, \sigma(E, E_n^*)$ and so the claim is proved. We get a positive linear mapping $\tilde{\mu}\colon C(\widetilde{X}) \to E$, $\tilde{\mu}(f) = \mu(f_{|X})$. For any zero-set $Z \subset \widetilde{X} \setminus X$, take a sequence $\{g_m\} \subset C(\widetilde{X})$ and $g_m \downarrow \chi_Z$. This means $(g_m)_{|X} \downarrow 0$. By ([??]), $\tilde{\mu}$ can be considered a Baire measure on $\widetilde{X}$ and so $\tilde{\mu}(Z) = \lim \tilde{\mu}(g_m) = \mu((g_m)_{|X}) = 0$. Since E is weakly σ-distributive, $\tilde{\mu}$ is a regular Baire measure and so for any Baire set $B \subset \widetilde{X} \setminus X$, $\tilde{\mu}(B) = 0$. It is a simple verification that the class of Baire subsets of X is equal to the class of Baire subsets of $\widetilde{X}$ intersected with X. Now for any Baire subset B_0 of X, take a Baire subset B of $\widetilde{X}$ such that $B_0 = B \cap X$; define $\mu(B_0) = \tilde{\mu}(B)$. It is a simple verification the μ is well-defined and $\mu \in M^+_{(o,\sigma)}(X, E)$ ([??]). This proves the theorem. □

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Author's address: *Surjit Singh Khurana*, Department of Mathematics, University of Iowa, Iowa City, Iowa 52242, U.S.A., e-mail: `khurana@math.uiowa.edu`.