# ON STRONG STARLIKENESS CRITERIA OF $p$-VALENT FUNCTIONS

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*Abstract.* H. Silverman (1999) investigated the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. Many research workers have been working on analytic functions to be strongly starlike like Obradović and Owa (1989), Takahashi and Nunokawa (2003), Lin (1993) etc. In this paper we obtain a sufficient condition for $p$-valent functions to be strongly starlike of order $\alpha$.

*Keywords*: analytic function, strongly starlike function

*MSC 2000*: 30C45

## Introduction

Let $A$ denote the class of functions that are analytic in the open unit disc $U = \{z \in C\colon\ |z| < 1\}$ and let $A_{n,p}$ be the subclass of $A$ consisting of the functions $f_{n,p}$ of the form

$$f_{n,p}(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (n \in \mathbb{N}) \tag{0.1}$$

where $p$ is a positive integer and $f_{n,p}$ is analytic and $p$-valent in $U$. Then a function $f_{n,p} \in A_{n,p}$ is said to be in class $S_n(p,\delta)$ if and only if

$$\operatorname{Re}\Big(\frac{zf'_{n,p}(z)}{f_{n,p}(z)}\Big) > \delta \ \ (z \in U) \tag{0.2}$$

for some $\delta$ $(0 \leqslant \delta < p)$.

A function $f_{n,p} \in S_n(p,\delta)$ is called $p$-valent starlike of order $\delta$. On the other hand, a function $f_{n,p} \in A_{n,p}$ is said to be in the class $K_n(p,\delta)$ iff

$$\text{Re}\left(1+\frac{zf''_{n,p}(z)}{f'_{n,p}(z)}\right) > \delta \quad (z \in U) \tag{0.3}$$

for some $\delta$ $(0 \leqslant \delta < p)$. A function $f_{n,p} \in K_n(p,\delta)$ is called a $p$-valent convex function of order $\delta$. On the other hand, a function$f_{n,p}(z)$ in $A_{n,p}$ is said to be strongly starlike of order $\alpha$ if it satisfies

$$\left|\arg\left\{\frac{zf'_{n,p}(z)}{f_{n,p}(z)}\right\}\right| < \frac{\pi\alpha}{2} \tag{0.4}$$

for some $\alpha$ $(0 < \alpha \leqslant p)$ and for all $z \in U$. We write $f_{n,p}(z) \in A_{n,p}S^*(\alpha)$ if $f_{n,p}(z)$ is strongly starlike of order $\alpha$ in $U$. Silverman [1] investigated the properties of functions defined in terms of the quotient of the analytic representation of convex and starlike functions. Let $G_b$ be the subclass of $A_{n,p}$ consisting of functions $f_{n,p} \in A_{n,p}$ which satisfy

$$\left|\frac{1+\dfrac{zf''_{n,p}(z)}{f'_{n,p}(z)}}{\dfrac{zf'_{n,p}(z)}{pf_{n,p}(z)}} - p\right| < b \quad (z \in U) \tag{0.5}$$

for some real $b$. For this class $G_b$, Silverman obtained the following result.

**Theorem 0.1** [1]**.** *If* $0 < b \leqslant 1$, *then*

$$G_b \subset S^*\left\{\frac{2}{1+\sqrt{(1+8b)}}\right\}. \tag{0.6}$$

*The result is sharp for all* $b$.

In this paper we consider the strong starlikeness for $p$-valent functions $f_{n,p}(z)$ belonging to $G_b$.

## 1. Strong starlikeness

For discussing the strong starlikeness of a function $f_{n,p}(z)$ in $G_b$, we have to recall the following result by Nunokawa [3].

**Lemma 1.2.** *Let $p(z)$ be analytic in $U$ with $p(0)=1$ and let $p(z)\neq 0$ $(z\in U)$. Suppose that there exists a point $z_0\in U$ such that*

$$|\arg p(z)|<\frac{\pi\beta}{2}\quad (|z|<|z_0|) \tag{1.7}$$

*and*

$$|\arg p(z_0)|=\frac{\pi\beta}{2} \tag{1.8}$$

*where $\beta>0$. Then we have*

$$\Bigl(\frac{z_0p'(z_0)}{p(z_0)}\Bigr)=\iota K\beta \tag{1.9}$$

*where $K\geqslant \frac12\{a+1/a\}$ when $\arg\{p(z_0)\}=\frac12\pi\beta$, and $K\leqslant -\frac12\{a+1/a\}$ when $\arg\{p(z_0)\}=-\frac12\pi\beta$ where $p\{z_0\}^{1/\beta}=\pm\iota a$ and $a>0$.*

**Theorem 1.3.** *If $f_{n,p}(z)$ belong to the class $G_{b(\alpha)}$ with*

$$b(\alpha)=\left(\frac{\alpha}{p^2\sqrt{(p-\alpha)^{\frac{p-\alpha}{p}}(p+\alpha)^{\frac{p+\alpha}{p}}}}\right)\quad (0<\alpha\leqslant p) \tag{1.10}$$

*then $f_{n,p}(z)\in A_{n,p}S^*(\alpha)$.*

P r o o f. Let us define a function $\varphi_{n,p}(z)$ by

$$\varphi_{n,p}(z)=\Bigl(\frac{zf'_{n,p}(z)}{pf_{n,p}(z)}\Bigr). \tag{1.11}$$

Then it follows that

$$\left\{\frac{1+\dfrac{zf''_{n,p}(z)}{f'_{n,p}(z)}}{\dfrac{zf'_{n,p}(z)}{pf_{n,p}(z)}}-p\right\}=\Bigl\{\frac{z\varphi'_{n,p}(z)}{p^2[\varphi_{n,p}(z)]^2}\Bigr\}. \tag{1.12}$$

If there exists a point $z_0\in U$ such that

$$|\arg\{\varphi_{n,p}(z)\}|<\frac{\pi\alpha}{2p}\ \text{ for }\ |z|<|z_0|\,, \tag{1.13}$$

$$|\arg\{\varphi_{n,p}(z_0)\}|=\frac{\pi\alpha}{2p} \tag{1.14}$$

then, applying Lemma 1.2 we obtain that

$$\left|\frac{z\varphi'_{n,p}(z_0)}{p^2[\varphi_{n,p}(z_0)]^2}\right| = \left|\frac{\frac{1}{p}\iota K\alpha}{p^2(\pm\iota a)^{\alpha/p}}\right| = \frac{\alpha|K|a^{-\alpha/p}}{p^3} \geqslant \frac{\alpha[a^{(p-\alpha)/p} + a^{-(p+\alpha)/p}]}{2p^3}. \tag{1.15}$$

Define a function $F_p(a)$ by

$$F_p(a) = [a^{(p-\alpha)/p} + a^{-(p+\alpha)/p}] \quad (a > 0; 0 < \alpha \leqslant p), \tag{1.16}$$

$$F'_p(a) = \frac{1}{a^{(2p+\alpha)/p}}\Big[\frac{(p-\alpha)a^2 - (p+\alpha)}{p}\Big]. \tag{1.17}$$

$F_p(a)$ assumes its minimum value at $a = \sqrt{(p+\alpha)/(p-\alpha)}$. This implies that

$$\begin{aligned} \left|\frac{z\varphi'_{n,p}(z_0)}{p^2[\varphi_{n,p}(z_0)]^2}\right| &\geqslant \frac{\alpha}{2p^3}F_p\Big(\sqrt{\frac{p+\alpha}{p-\alpha}}\Big) \\ &= \frac{\alpha}{2p^3}\Big[\Big(\frac{p+\alpha}{p-\alpha}\Big)^{(p-\alpha)/2p} + \Big(\frac{p-\alpha}{p+\alpha}\Big)^{(p+\alpha)/2p}\Big] \\ &= \frac{\alpha}{p^2\sqrt{(p-\alpha)^{(p-\alpha)/p}(p+\alpha)^{(p+\alpha)/p}}}, \end{aligned} \tag{1.18}$$

which contradicts our condition $f_{n,p}(z) \in G_{b(\alpha)}$ of the theorem. Thus we complete the proof of the theorem. □

**Corollary 1.4.** *If we take $p = 1$ then the function $f_{n,p}(z)$ reduces to a univalent function $f_{n,1}(z) = f_n(z)$ and we get a sufficient condition of strong starlikeness of univalent functions.*

Considering the case of $\alpha = 1$ in Corollary 1.4, we have

**Corollary 1.5.** *If $f_n(z) \in G_b$ with $b = \frac{1}{2}$, then $f_n(z) \in AS^*(1)$ or $f_n(z)$ is strongly starlike in $U$*

Taking $\alpha = \frac{1}{2}$ in Corollary 1.4 ,we have

**Corollary 1.6.** *If $f_n(z) \in G_b$ with $b = 1/\sqrt{3\sqrt{3}} = 0.438691\ldots$, then $f_n(z) \in AS^*(\frac{1}{2})$.*

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