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ON EXISTENCE OF A MEASURE UNBOUNDED EXPONENTIAL SPECTRAL QUANTIZATION ON SYMPLECTIC MANIFOLDS

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Consider the linear system

$$\varepsilon\dot{x} = A(t)x, \quad \varepsilon \in (0,1], \quad x \in \mathbb{R}^2, \quad t \geq 1, \qquad (1_{A/\varepsilon})$$

with piecewise continuous bounded coefficients, which has the characteristic exponents $\lambda_1(A) \leq \lambda_2(A)$ and the Grobman coefficient of inequality [1] $\sigma_G(A)$ at $\varepsilon = 1$. Starting with fundamental works of A. N. Tikhonov, monographs and papers by A. B. Vasil'eva, V. F. Butuzov, E. F. Mishchenko, N. Kh. Rozov, M. V. Fiedoryuk, I. S. Lomov and many others were devoted to the investigation of more general singularly perturbed systems (for details, see [2–8]). In the paper [9], a partial case of the n-dimensional system $(1_{(A+Q)/\varepsilon})$ was considered under perturbations $Q(\cdot)$ of sufficiently small norm. Therein sufficient and necessary conditions were obtained for tending to zero as $\varepsilon \to +0$ of all solutions of such system on any finite segment of positive half-axis.

A set $S_\sigma(A/\varepsilon) \equiv \bigcup\limits_{\lambda[Q]\leq-\sigma} (\lambda_1(\frac{A+Q}{\varepsilon}), \lambda_2(\frac{A+Q}{\varepsilon}))$, $\sigma = \text{const} > 0$, where $\lambda[Q] \equiv \varlimsup\limits_{t\to\infty} t^{-1}\ln\|Q(\cdot)\|$, is called a spectral sigma-set of the system $(1_{A/\varepsilon})$ (the definition of the Grobman spectral set see in [10, 11]). It holds the following

Theorem 1. *For any real numbers $\lambda_1 < \lambda_2$ and $\sigma_0 > 2(\lambda_2 - \lambda_1)$, there exists a two-dimensional system (1_A) with infinitely differentiable bounded coefficients and their derivatives which has the characteristic exponents $\lambda_i(A) = \lambda_i$, $i = 1, 2$, and the Grobman coefficient of inequality $\sigma_G(A) = \sigma_0$ and is such that the spectral sigma-set $S_\sigma(A/\varepsilon)$ of the system $(1_{(A+Q)/\varepsilon})$ for all $\sigma > 0$ and $0 < \varepsilon < (\sigma_0 + 2(\lambda_1 - \lambda_2))\sigma^{-1}$ contains the set of points $(\mu_1, \mu_2) \in \mathbb{R}^2$ defined by the inequalities. $\lambda_2 - \sigma_0(\theta-1)^{-1} \leq \varepsilon\mu_1 < \lambda_2 < \varepsilon\mu_2 \leq (\lambda_2 - \varepsilon\mu_1)\theta^{-2} + \lambda_2 + (\sigma_0 + \lambda_1 - \lambda_2 - \varepsilon\sigma)\theta^{-1}$, where $\theta > 2\sigma_0(\lambda_2 - \lambda_1)^{-1} - 1$.*

Scheme of the proof. Denote $\tau_{j,l} \equiv \theta^l - j\Delta$, $t_l \equiv \theta^l = \tau_{0,l}$, where $l \in \mathbb{Z}_+$, $j = \overline{0,3}$, $\Delta \in (0, (\theta-1)/3)$. Consider the set of functions $\varphi^i_{k,l}(t)$, $k = \overline{1,4}$, $l \in \mathbb{Z}_+$, $i = 1, 2$, from the class $C^\infty_{[1,+\infty)}$ of infinitely differentiable functions defined as follows: $\varphi^i_{2j-1,l}(t) \equiv [[(-1)^{|i-j|}t_{2l}^{-1}t + (\theta+1)|i-j|](\lambda_i + \delta_i) - \theta\delta_i - \lambda_i](\theta-1)^{-1}$, $\varphi^i_{2j,l}(t) \equiv [\lambda_i - (\lambda_i + \delta_i)|i - j|]\tau_{2,2l+j}t^{-1}$, $i, j = 1, 2$, where $\delta_1 \equiv \sigma_0 + \lambda_1 - 2\lambda_2$, $\delta_2 \equiv \sigma_0 - \lambda_2$. One can see that $\delta_1 + \lambda_2 = \lambda_2 + \delta_1 = \sigma_0 + \lambda_1 - \lambda_2 \equiv \sigma_1 > 0$. ∎

By means of this set of functions and the infinitely differentiable Gelbaum function $g(t, a, b)$ with bounded derivatives of any order (see [12]) which is equal to zero at $t \in [1, a]$, equal to $\exp[-(t-a)^{-2}\exp[-(t-b)^{-2}]]$ at $t \in \in (a, b)$ and equal to 1 at $t \in [b, +\infty)$, $1 < a < b$, we define the functions $f_i(t) = \varphi^i_{1,0}(t) + \sum\limits_{l=0}^{\infty}\sum\limits_{m,k=1}^{2}[\varphi^i_{2m-k+2,l}(t) - \varphi^i_{2m-k+1,l}(t)]g(t, \tau_{2k-1,2l+m}, \tau_{2k-2,2l+m})$, $i = 1, 2$, $t \geq 1$, $\varphi^i_{5,l}(t) \equiv \varphi^i_{1,l+1}(t)$, which

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belong to the class $C^{\infty}_{[1,+\infty)}$ as a sum and a product of functions from that class. Besides, on any segment $[t_{2l}, t_{2l+2}]$ we have $-\delta_i - 2\Delta(|\lambda_i| + |\delta_i|)t_{2l}^{-1} \le f_i(t) \le \lambda_i + 2\Delta(|\lambda_i| + |\delta_i|)t_{2l}^{-1}$, $t \in [t_{2l}, t_{2l+2}]$, $l \in \mathbb{Z}_+$, therefore the functions $f_i(t)$ are bounded on $[1, +\infty)$.

The system (1_A) will be constructed by its Cauchy matrix $X_A(t,\tau) = \operatorname{diag}[\exp[tf_1(t) - \tau f_1(\tau)], \exp[tf_2(t) - \tau f_2(\tau)]]$. The coefficients $a_i(t)$, $i = = 1, 2$, of the matrix $A(t) = \operatorname{diag}[a_1(t), a_2(t)]$ of this system have the form $a_i(t) = tf_i'(t) + f_i(t)$ and belong to the class $C^{\infty}_{[1,+\infty)}$ as a sum and a product of functions of this class. By direct calculations, it is possible to show that the coefficients and their derivatives of any order are bounded on $[1, +\infty)$.

Calculate now the characteristic exponents $\lambda_i(A)$ and $\delta_i(A)$, $i = 1, 2$, of the initial system and of the conjugate one, respectively. Taking into acoount the above mentioned estimatens for $f_i(t)$ on $[t_{2l}, t_{2l+2}]$, we obtain $\lambda_i(A) = \overline{\lim\limits_{t\to\infty}}\, f_i(t) \le \lambda_1$, $\delta_i(A) = \lambda_i(-A^T) = \overline{\lim\limits_{t\to\infty}}\,[-f_i(t)] \le \delta_1$. Besides, these limits will be realized by the sequences $\{t_{2l+i}\} \uparrow +\infty$ and $\{t_{2l+i-1}\} \uparrow +\infty$. Thus we get $\lambda_i(A) = \lambda_i$, $\delta_i(A) = \delta_i$, $i = 1, 2$.

The Grobman coefficient of inequality for this system $\sigma_G(A) \equiv \max\limits_i\{\lambda_i(A) + \delta_i(A)\} = \lambda_2 + \delta_2 = \sigma_0$ is equal to the given value σ_0.

For the singular system $(1_{A/\varepsilon})$ with a small parameter $\varepsilon > 0$ by the derivative, the characteristic exponents and the Grobman coefficient of inequality are $\lambda_i(A/\varepsilon) = \lambda_i/\varepsilon$, $i = 1, 2$, $\sigma_G(A/\varepsilon) = \sigma_0/\varepsilon$, respectively.

Take a point (α_1, α_2) from the domain $D \subset \mathbb{R}^2$ defined by

$$\sigma\theta - \frac{\sigma_1}{\varepsilon}(\theta + 1) \le \alpha_1 < \alpha_2 < \alpha_2(\theta^2 + 1) - \alpha_1 + \frac{\sigma_1}{\varepsilon}(\theta^2 - 1) \le \frac{\sigma_0}{\varepsilon}(\theta + 1) + \alpha_2. \qquad (2)$$

Take also a parameter $r \in (1, 1 + e^{-4}]$, and take a rather small angle $\rho = \rho(r) \in (0, r - 1)$, satisfying $r^{-1}\rho \le \sin\,\rho < \rho < \operatorname{tg}\,\rho \le r\rho$. Take also $l_0 \in \mathbb{Z}_+$ large enough for

$$t_{2l_0} \ge \ln[C_\varepsilon r^7 \rho^{-1}(r-1)^{-1}] / \min\{\sigma, \alpha_2 - \alpha_1, \alpha_2\theta^2 - \alpha_1 + \sigma_1\varepsilon^{-1}(\theta^2 - 1)\}, \qquad (3)$$

where $C_\varepsilon \equiv \exp[2\Delta\sigma_1\varepsilon^{-1}]$. Note that the inequality (3) is true for any $l \ge l_0$.

Let us show that there exists a piecewise continuous perturbation $Q(\cdot)\lambda[Q] \le -\sigma < 0$ such that for the singularly perturbed system $(1_{(A+Q)/\varepsilon})$ there are two solutions $y_i(t)$, $i = 1, 2$, such that the angles $\beta_i(t_{2l})$, $i = 1, 2$, between the straight lines containing these solutions and the axis Ox_2 are $\beta_1(t_{2l}) = \exp[\alpha_1 t_{2l}]$, $\beta_2(t_{2l}) = d(2l)\exp[\alpha_2 t_{2l}]$ at the moments $t = t_{2l}$, $l \ge l_0$, $l \in \mathbb{Z}_+$, where $r^{-4} \le d(2l) \le r^2$.

Let $Q(\cdot) = 0$ on $[1, t_{2l_0}]$. At the moment $t = t_{2l_0}$, take the vectors $y_i(t_{2l_0}) = ((-1)^{l_0+1}\sin\exp[\alpha_i t_{2l_0}], (-1)^{l_0}\cos\exp[\alpha_i t_{2l_0}])$. It is clear that these solutions may be extended to the left by Cauchy matrix $X_{A/\varepsilon}(t, 1) = \operatorname{diag}[\exp[\varepsilon^{-1}tf_1(t) + \delta_1\varepsilon^{-1}], \exp[\varepsilon^{-1}t f_2(t) - \lambda_2\varepsilon^{-1}]]$ of the system $(1_{A/\varepsilon})$ on $[1, t_{2l_0}]$.

The further proof will be carried out by induction. Assume that at a moment $t = t_{2l}$, $l \ge l_0$, $l \in \mathbb{Z}_+$, we have obtained the vectors

$$y_1(t_{2l}) = ((-1)^{l+1}\sin\exp[\alpha_1 t_{2l}], (-1)^l\cos\exp[\alpha_1 t_{2l}])\|y_1(t_{2l})\|,$$
$$y_2(t_{2l}) = ((-1)^{l+1} \times \sin[d(2l)\exp[\alpha_2 t_{2l}]], (-1)^l\cos[d(2l)\exp[\alpha_2 t_{2l}]])\|y_2(t_{2l})\|.$$

We will construct on the segment $[t_{2l}, t_{2l+2}]$ a perturbation $Q(\cdot)$, $\lambda[Q] \le -\sigma < 0$, such that at the moment $t = t_{2(l+1)}$, the solutions $y(t_{2(l+1)})$ will be represented in the same form with l substituted by $l + 1$.

Let $Q(\cdot) = 0$ on the segment $[t_{2l}, \tau_{2,2l+1}]$. The Cauchy matrix of the system $(1_{(A+Q)/\varepsilon})$ is $X_{A/\varepsilon}(\tau_{2,2l+1}, t_{2l}) = \operatorname{diag}[\exp[\lambda_1\varepsilon^{-1}\tau_{2,2l+1} + \delta_1\varepsilon^{-1}t_{2l}], \exp\,[-\delta_2\varepsilon^{-1}\tau_{2,2l+1} - \lambda_2\varepsilon^{-1}t_{2l}]]$. Therefore the angles $\beta_i(\tau_{2,2l+1})$, $i = 1, 2$, between the straight lines containing the solutions $y_i(\tau_{2,2l+1})$, and axes Ox_1 will be represented as follows:

$$\operatorname{tg}\beta_i(\tau_{2,2l+1}) = C_\varepsilon\exp[-\sigma_1\varepsilon^{-1}(\theta + 1)t_{2l}]\operatorname{ctg}\beta_i(t_{2l})$$

at the moment $t = \tau_{2,2l+1}$. Hence, using (2), we obtain

$$r^{-2}C_\varepsilon \le \beta_1(\tau_{2,2l+1})\exp[\sigma_1\varepsilon^{-1}(\theta+1)t_{2l} + \alpha_1 t_{2l}] \le C_\varepsilon, \tag{4}$$

$$\beta_2(\tau_{2,2l+1}) \le r^4 C_\varepsilon \exp[-\sigma_1\varepsilon^{-1}(\theta+1)t_{2l} - \alpha_2 t_{2l}]. \tag{5}$$

On the following segment $[\tau_{2,2l+1}, \tau_{1,2l+1}]$, we perform the rotation of the solutions $y_i(t)$, $i = 1,2$, from the axes Ox_1 to the axis Ox_2 by the angle $\omega_{1,l} = \beta_1(\tau_{2,2l+1}) + \operatorname{arctg}(C_\varepsilon \operatorname{tg}\exp[\gamma t_{2l+1}])$, where $\gamma \equiv -\alpha_2\theta - \sigma_1\varepsilon(\theta+1)$. It may be realized by the rotation matrix $Q(\cdot) = Q_{1,l}$ with the elements $q_{11} = q_{22} = 0$ and $q_{12} = -q_{21} = \varepsilon\omega_{1,l}\Delta^{-1}$. If we note that $\gamma < -\sigma$, then using (3), (4) we get $\omega_{1,l} \le (r+1)C_\varepsilon \exp[-\sigma t_{2l}]$. Therefore the exponent of the matrix $Q_{1,l}(\cdot)$ satisfies the required condition $\lambda[Q_{1,l}] \le -\sigma < 0$.

As a result of this rotation, we get the angles $\beta_i(\tau_{1,2l+1}) = \omega_{1,l} - \beta_i(\tau_{2,2l+1})$, $i = 1,2$, at the moment $t = \tau_{1,2l+1}$ between the axes Ox_1 and the staight lines containing the solutions $y_i(\tau_{1,2l+1})$.

Let again $Q(\cdot) = 0$ on $[\tau_{1,2l+1}, t_{2l+1}]$. Therefore we obtain the Cauchy matrix $X_{A/\varepsilon}(t_{2l+1}, \tau_{1,2l+1}) = \operatorname{diag}[\exp[2\Delta\lambda_1\varepsilon^{-1}], \exp[-2\Delta\delta_2\varepsilon^{-1}]]$. By contracting to the axes Ox_1, the solutions $y_i(t)$ will be represented in the form

$$y_1(t_{2l+1}) = ((-1)^{l+1}\cos\exp[\gamma t_{2l+1}], (-1)^{l+1}\sin\exp[\gamma t_{2l+1}])||y_1(t_{2l+1})||$$

and

$$y_2(t_{2l+1}) = ((-1)^{l+1}\cos\operatorname{arctg}[C_\varepsilon^{-1}\operatorname{tg}\beta_2(\tau_{1,2l+1})], (-1)^{l+1}\sin\operatorname{arctg}[C_\varepsilon^{-1}\times\operatorname{tg}\beta_2(\tau_{1,2l+1})].$$

On the following "long" segment $[t_{2l+1}, \tau_{2,2l+2}]$, we take $Q(\cdot) = 0$. The Cauchy matrix of the system $(1_{(A+Q)/\varepsilon})$ has the form $X_{A/\varepsilon}(\tau_{2,2l+2}, t_{2l+1}) = \operatorname{diag}[\exp[-\delta_1\varepsilon^{-1}\tau_{2,2l+2} - \lambda_1\varepsilon^{-1}t_{2l+1}], \exp[\lambda_2\varepsilon^{-1}\tau_{2,2l+2} + \delta_2\varepsilon^{-1}t_{2l+1}]]$. The straight lines containing the solutions $y_i(\tau_{2,2l+2})$ $i = 1,2$, form with the axis Ox_2 the angles

$$\beta_1(\tau_{2,2l+2}) = \operatorname{arctg}[C_\varepsilon \exp[-\sigma_1\varepsilon^{-1}(\theta+1)t_{2l+1}]\operatorname{ctg}\exp[-\gamma t_{2l+1}]]$$

and

$$\beta_2(\tau_{2,2l+2}) = \operatorname{arctg} C_\varepsilon^2 \exp[-\sigma_1\varepsilon^{-1}(\theta+1)t_{2l+1}]$$

$\operatorname{ctg}\beta_2(\tau_{1,2l+1})]$, respectively. By virtue of the definition of γ and the inequalities (4), the first angle admits the estimates

$$r^{-2}C_\varepsilon \exp[\alpha_2 t_{2l+2}] \le \beta_1(\tau_{2,2l+2}) \le C_\varepsilon \exp[\alpha_2 t_{2l+2}] \le C_\varepsilon \exp[-\sigma t_{2l+2}]. \tag{6}$$

Using the inequalities (3), (4), (5), it is possible to obtain the estimate

$$\beta_2(\tau_{2,2l+2}) \le r^3 C_\varepsilon \exp[-\sigma_1\varepsilon^{-1}(\theta^2-1)t_{2l} + \alpha_1 t_{2l}] \tag{7}$$

for the angle $\beta_2(\tau_{2,2l+2})$.

Further, using the rotation matrices $Q(\cdot) = Q_{2,l}$ with the elements $q_{11} = q_{22} = 0$ and $q_{12} = -q_{21} = \varepsilon\omega_{2,l}\Delta^{-1}$, we turn the system $(1_{(A+Q)/\varepsilon})$ from the axis Ox_1 to thr axis Ox_2 by the angle $\omega_{2,l} = \beta_1(\tau_{2,2l+2}) + \operatorname{arctg}(C_\varepsilon \operatorname{tg}\exp[\alpha_1 t_{2l+2}])$ on $[\tau_{2,2l+1}, \tau_{1,2l+1}]$. Taking into account (4) and (7), we see that the value of $\omega_{2,l}$ admits the estimates

$$\omega_{2,l} \le C_\varepsilon \exp[\alpha_2 t_{2l+2}] + rC_\varepsilon \exp[\alpha_1 t_{2l+2}] \le rC_\varepsilon \exp[-\sigma t_{2l+2}], \tag{8}$$

and the exponent of this matrix $\lambda[Q_{2,l}] \le -\sigma < 0$.

After the rotation, we obtain the angles $\beta_1(\tau_{1,2l+2}) = \operatorname{arctg} C_\varepsilon \operatorname{tg}\exp[\alpha_1 \times t_{2l+2}]$ and $\beta_2(\tau_{1,2l+2}) = \omega_{2,l} - \beta_2(\tau_{2,2l+2}$ between the axis Ox_2 and the straight lines containing the solutions $y_i(\tau_{1,2l+2})$, $i = 1,2$. By (3), (7) and (8), the angle $\beta_2(\tau_{1,2l+2})$ admits the estimates $r^{-3}C_\varepsilon \exp[\alpha_2 t_{2l+2}] \le \beta_2(\tau_{1,2l+2}) \le rC_\varepsilon \exp[\alpha_2 t_{2l+2}]$. Requiring that the obtained angle $\beta_2(\tau_{1,2l+2}) = p(l+1)C_\varepsilon \exp[\alpha_2 t_{2l+2}]$, where $r^{-3} \le p(l+1) \le r$, would be equal to $\operatorname{arctg}(C_\varepsilon \operatorname{tg}(d(2l+2)\exp[\alpha_2 t_{2l+2}]))$, we can define the constant $d(2l+2)$ as the least positive solution of the equation $(\operatorname{arctg}(C_\varepsilon \operatorname{tg} d(2l+2)\exp[\alpha_2 t_{2l+2}])) = p(l+1)C_\varepsilon \exp[\alpha_2 t_{2l+2}]$. Hence we obtain $r^{-4} \le d(2l+2) \le r^2$.

Note that by virtue of the arbitrary choice of r (it may be taken sufficiently close to 1), $d(2l) \to 1$ as $l \to +\infty$.

Putting $Q(\cdot) = 0$ on the last segment $[\tau_{1,2l+2}, t_{2l+2}]$, we obtain the Cauchy matrix $X_{A/\varepsilon}(t_{2l+2}, \tau_{1,2l+2}) = \operatorname{diag}[\exp[-2\Delta\delta_1\varepsilon^{-1}], \exp[2\Delta\lambda_2\varepsilon^{-1}]]$. The solutions $y_i(t)$, $i = 1, 2$, after being contracted to the axes Ox_2, will be represented in the required form:

$$y_1(t_{2l+2}) = ((-1)^{l+2}\sin\exp[\alpha_1 t_{2l+2}], (-1)^{l+1}\cos\times\exp[\alpha_1 t_{2l+2}])\|y_1(t_{2l+2})\|,$$

$$y_2(t_{2l+2}) = ((-1)^{l+2}\sin[d(2l+2)\times\exp[\alpha_2 t_{2l+2}]], (-1)^{l+1}\cos[d(2l+2)\exp[\alpha_2 t_{2l+2}]]).$$

Thus we obtain the angles $\beta_1(t_{2l}) = \exp[\alpha_1 t_{2l}]$, $\beta_2(t_{2l}) = d(2l)\exp[\alpha_2 t_{2l}]$, with the axis Ox_2 at the moment $t = t_{2l}$, $l \geq l_0$, and the angles $\beta_1(t_{2l+1}) = \exp[-\sigma_1\varepsilon^{-1}\times(\theta+1)t_{2l+1} - \alpha_2 t_{2l+2}]$, $\beta_2(t_{2l+1}) = d(2l+1)\exp[-\sigma_1\varepsilon^{-1}(\theta+1)t_{2l} - \alpha_1 t_{2l}]$, with the axis Ox_1 at the moment $t = t_{2l+1}$, where $d(k) \in [r^{-4}, r^2]$, $k \geq 2l_0$. These values allow us to calculate by induction the growth of the norms of the solutions $y_i(t)$, $i = 1, 2$, on the segments $[t_{2l_0}, t_{2l}]$ and $[t_{2l_0+1}, t_{2l+1}]$, $l \geq l_0$, and passing to limits as $l \to \infty$, we obtain the exponential growth of the norms of these solutions in the form of following partial exponents:

$$\lambda_e[y_1] \equiv \overline{\lim_{l\to\infty}}\, t_{2l}^{-1}\ln\|y_1(t_{2l})\| = \lambda_2\varepsilon^{-1} - \sigma_1\varepsilon^{-1} - (\alpha_2\theta^2 - \alpha_1)(\theta^2-1)^{-1},$$

$$\lambda_o[y_2] \equiv \overline{\lim_{l\to\infty}}\, t_{2l+1}^{-1}\ln\|y_2(t_{2l+1})\| = \lambda_1\varepsilon^{-1} + \sigma_1\varepsilon^{-1}\theta^{-1} + (\alpha_2\theta^2 - \alpha_1)/\theta(\theta^2-1),$$

$$\lambda_e[y_2] = \lambda_2\varepsilon^{-1} + (\alpha_2-\alpha_1)/(\theta^2-1), \quad \lambda_o[y_1] = \lambda_1\varepsilon^{-1} - (\alpha_2-\alpha_1)\theta/(\theta^2-1).$$

Note that $\lambda_e[y_i] \geq \lambda_o[y_i]$ for any point $(\alpha_1, \alpha_2) \in D$.

Bounding from above the functions

$$\psi^i(t) = \psi^i_{2l+j-1}(\tau) \equiv (t_{2l+j-1}\tau)^{-1}\times\ln\|y_i(t_{2l+j-1}\tau)\|,$$

$i, j = 1, 2$, $t = t_{2l+j-1}\tau$, $\tau \in [1, \theta]$, on segments $[t_{2l+j-1}, t_{2l+j}]$, $l \geq l_0$, and using the estimates for the norm growth of the solutions as well as the representation of the solutions $y_i(t) = X_{A/\varepsilon}(t, t_{2l+j-1})y_i(t_{2l+j-1})$ on these segments, we find that the characteristic exponents of the perturbed system $(1_{(A+Q)/\varepsilon})$ $\lambda_i[\frac{A+Q}{\varepsilon}] = \overline{\lim_{t\to\infty}}\,\psi^i(t) \leq \lambda_e[y_i]$. Besides, as is already shown, there is a sequence $t_{2l} \uparrow \infty$ such that these limits are realized.

So, the lowest and highest characteristic exponents of the system $(1_{(A+Q)/\varepsilon})$ are $\lambda_e[y_i]$, $i = 1, 2$, respectively.

The transformation

$$\mu_1 = \lambda_2\varepsilon^{-1} - \sigma_1\varepsilon^{-1} - (\alpha_2\theta^2 - \alpha_1)(\theta^2-1)^{-1},$$
$$\mu_2 = \lambda_2\varepsilon^{-1} + (\alpha_2-\alpha_1)(\theta^2-1)^{-1}$$

maps the domain D determined by (2) to the domain

$$S = \{(\mu_1, \mu_2) \in \mathbb{R}^2 : \lambda_2 - \sigma_0(\theta-1)^{-1} \leq \varepsilon\mu_1 < \lambda_2 < \varepsilon\mu_2 \leq$$
$$\leq (\lambda_2 - \varepsilon\mu_1)\theta^{-2} + \lambda_2 + (\sigma_0 + \lambda_1 - \lambda_2 - \varepsilon\sigma)\theta^{-1}\}.$$

This completes the proof of the theorem.

Corollary 1. $\operatorname{mes} S_\sigma(A/\varepsilon) \to +\infty$ *as* $\varepsilon \to +0$.

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