Mem. Differential Equations Math. Phys. 20(2000), 141–144

Z. Sokhadze

ON UNIQUENESS OF SOLUTION OF THE WEIGHTED INITIAL VALUE PROBLEM FOR HIGHER ORDER EVOLUTION SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

(Reported on November 8, 1999)

In the present paper optimal, in a certain sense, sufficient conditions for uniqueness of solution of the weighted initial value problem

$$u^{(n)}(t) = f(u)(t), \tag{1}$$

$$\lim_{t\to a} \frac{u^{(k)}(t)}{h^{(k)}(t)} = 0 \quad (k = 0,\dots,n-1) \tag{2}$$

are given, where $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{\mathrm{loc}}(]a,b];\mathbb{R}^m)$ is a continuous Volterra operator and $h : [a,b] \to [0,+\infty[$ is an $(n-1)$-times continuously differentiable function such that

$$h^{(k)}(a) = 0 \ \ (k = 0,\dots,n-2), \ \ h^{(n-1)}(t) > 0 \ \text{ for } \ a < t \le b.$$

A particular case of the equation (1) is the vector differential equation with delay

$$u^{(n)}(t) = \\ = f_0(t, u(\tau_{10}(t)),\dots,u^{(n-1)}(\tau_{1n-1}(t)),\dots,u(\tau_{l0}(t)),\dots,u^{(n-1)}(\tau_{ln-1}(t))), \tag{3}$$

where $f_0 :]a,b] \times \mathbb{R}^{lmn} \to \mathbb{R}^m$ satisfies the local Carathéodory conditions, and $\tau_{ik} : [a,b] \to [a,b]$ $(i = 1,\dots,l;\ k = 0,\dots,n-1)$ are measurable functions such that

$$\tau_{ik}(t) \le t \quad \text{for} \quad a \le t \le b \ \ (i = 1,\dots,l;\ k = 0,\dots,n-1).$$

Throughout the paper the use will be made of the following notation.

$\mathbb{R}^m$ is the space of m-dimensional column vectors $x = (x_i)_{i=1}^m$ with real components and the norm

$$\|x\| = \sum_{i=1}^m |x_i|.$$

$x \cdot y$ is the scalar product of the vectors x and $y \in \mathbb{R}^m$.

If $x = (x_i)_{i=1}^m \in \mathbb{R}^m$, then $\operatorname{sgn}(x) = (\operatorname{sgn} x_i)_{i=1}^m$.

$C^{n-1}([a,b];\mathbb{R}^m)$ is the space of $(n-1)$-times continuously differentiable vector functions $x : [a,b] \to \mathbb{R}^m$ with the norm

$$\|x\|_{C^{n-1}} = \max\left\{ \sum_{k=1}^{n-1} \|x^{(k-1)}(t)\| : a \le t \le b \right\}.$$

$C_h^{n-1}([a,b];\mathbb{R}^m)$ is the set of $u \in C^{n-1}([a,b];\mathbb{R}^m)$ such that

$$\sup\left\{ \frac{\|u^{(k)}(t)\|}{h^{(k)}(t)} : a < t \le b \right\} < +\infty \ \ (k = 0,\dots,n-1).$$

2000 *Mathematics Subject Classification.* 34K10.

Key words and phrases. Weighted initial value problem, singular functional equations, uniqueness.

If $x :]a,b] \to \mathbb{R}^m$ is a bounded function and $a \le s < t \le b$, then

$$\nu(x)(s,t) = \sup\{\|x(\xi)\| : s < \xi < t\}.$$

$L_{\rm loc}(]a,b];\mathbb{R}^m)$ is the space of locally summable vector functions $x :]a,b] \to \mathbb{R}^m$ with the topology of convergence in the mean on each segment from $]a,b]$.

Definition 1. $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{\rm loc}(]a,b];\mathbb{R}^m)$ is called a Volterra operator if the equality $f(x)(t) = f(y)(t)$ holds almost everywhere on $]a,t_0[$ for any $t_0 \in]a,b]$ and any vector functions x and $y \in C^{n-1}([a,b];\mathbb{R}^m)$ satisfying the condition $x(t) = y(t)$ for $a \le t \le t_0$.

Definition 2. We will say that the operator $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{\rm loc}(]a,b];\mathbb{R}^m)$ satisfies the local Carathéodory conditions if it is continuous and there exists a nondecreasing with respect to the second argument function $\gamma :]a,b] \times [0;+\infty[\to [0;+\infty[$ such that $\gamma(\cdot,\rho) \in L_{\rm loc}(]a,b];\mathbb{R})$ for any $\rho \in]0;+\infty[$ and the inequality

$$\|f(x)(t)\| \le \gamma(t, \|x\|_{C^{n-1}})$$

is fulfilled for any $x \in C^{n-1}([a,b];\mathbb{R}^n)$ almost everywhere on $]a,b[$.

Definition 3. A function $u : [a,b] \to \mathbb{R}^m$ is called a solution of the problem (1), (2) if:

(i) $u \in C^{n-1}([a,b];\mathbb{R}^m)$ and $u^{(n-1)}$ is absolutely continuous on each segment contained in $]a,b]$;

(ii) u satisfies (1) almost everywhere on $]a,b[$;

(iii) u satisfies initial conditions (2).

The following theorem is valid.

Theorem 1. *Let there exist summable functions $p_k : [a,b] \to [0,+\infty[$ $(k=0,\dots,n-1)$ such that*

$$\limsup_{t\to a} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \int_a^t p_k(s)\,ds\right) < 1 \tag{4}$$

and for any $u_i \in C_h^{n-1}([a,b];\mathbb{R}^m)$ the inequality

$$\begin{aligned}\big(f(u_1)(t) - f(u_2)(t)\big)\,&\mathrm{sgn}(u_1(t) - u_2(t)) \le \\ &\le \sum_{k=0}^{n-1} p_k(t)\nu\left(\frac{u_1-u_2}{h^{(k)}}\right)(a,t)\end{aligned} \tag{5}$$

be fulfilled almost everywhere on $]a,b[$. Then the problem (1), (2) *has at least one solution.*

From the above theorem and Theorem 2 from [3] follows

Theorem 2. *Let the conditions of Theorem* 1 *hold and*

$$\lim_{t\to a} \left(\frac{1}{h^{(n-1)}(t)} \int_a^t \|f(0)(s)\| ds\right) = 0. \tag{6}$$

Then the problem (1), (2) *has one and only one solution.*

Theorems 1 and 2 for the problem (3), (2) take the following form.

Corollary 1. *Let $\tau_{1n-1}(t) \equiv t$ and let there exist summable functions $p_{ik} : [a,b] \to [0,+\infty[$ $(i = 1,\dots,\ell; k = 0,\dots,n-1)$ such that*

$$\limsup_{t\to a} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \sum_{i=1}^{\ell} \int_a^t p_{ik}(s)\,ds \right) < 1 \tag{7}$$

and the inequality

$$\begin{gathered}\Big(f_0(t, h(\tau_{10}(t))x_{10}, \dots, h^{(n-1)}(\tau_{\ell n-1}(t))x_{\ell n-1}) - \\ - f_0(t, h(\tau_{10}(t))y_{10}, \dots, h^{(n-1)}(\tau_{\ell n-1}(t))y_{\ell n-1})\Big) \operatorname{sgn}(x_{\ell n-1} - y_{\ell n-1}) \le \\ \le \sum_{k=0}^{n-1} \sum_{i=1}^{\ell} p_{ik}(t)\|x_{ik} - y_{ik}\|\end{gathered}$$

is fulfilled for any $t \in]a,b[$, $x_{ik} \in \mathbb{R}^m$ and $y_{ik} \in \mathbb{R}^m$ $(i = 1,\dots,\ell;\ k = 0,\dots,n-1)$. Then the problem (3), (2) *has at most one solution.*

Corollary 2. *Let the conditions of Corollary* 1 *hold and*

$$\lim_{t\to a} \left(\frac{1}{h^{(n-1)}(t)} \int_a^t \|f_0(s,0,\dots,0)\|\,ds \right) = 0.$$

Then the problem (3), (2) *has one and only one solution.*

The above-formulated Theorems 1 and 2 and their corollaries generalize the results of [1] and make the results of §5 from monograph [2] more complete.

As an example, in the interval $]0,1/2]$ we consider the boundary value problem

$$u^{(n)}(t) = \sum_{k=0}^{n-1} g_k(t)|u^{(k)}(t)| + g(t), \tag{8}$$

$$\lim_{t\to 0} \frac{u^{(k)}(t)}{t^{n-k}} = 0 \quad (k = 0,\dots,n-1), \tag{9}$$

where

$$\begin{gathered} g(t) = w^{(n)}(t), \quad w(t) = t^n \big/ |\ln t|, \\ g_k(t) = \ell_k w^{(n)}(t) \big/ w^{(k)}(t), \quad \ell_k \ge 0 \ \ (k = 0,\dots,n-1). \end{gathered} \tag{10}$$

The problem (8), (9) is a particular case of the problem (1), (2), where $a = 0$, $b = 1/2$,

$$f(u)(t) = \sum_{k=0}^{n-1} g_k(t)|u^{(k)}(t)| + g(t)$$

and $h(t) = t^n$. Obviously, the operator f satisfies conditions (5) and (6), where

$$p_k(t) = \frac{n!}{(n-k)!} t^{n-k} g_k(t).$$

On the other hand,

$$\lim_{t\to 0} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \int_0^t p_k(s)\,ds \right) = \sum_{k=0}^{n-1} \ell_k.$$

According to Corollary 1, if

$$\sum_{k=0}^{n-1} \ell_k < 1,$$

then the problem (8), (9) has a unique solution.

Let us show that if

$$\sum_{k=0}^{n-1} \ell_k \geq 1, \tag{11}$$

then the problem (8), (9) has no solution. Assume the contrary that this problem has a solution u. Then with regard for (10) from (8) we obtain

$$u^{(k)}(t) \geq w^{(k)}(t) \quad (k = 0, \dots, n) \quad \text{for} \quad 0 < t \leq 1/2.$$

Put

$$\rho_0 = \inf\left\{ \frac{u^{(n-1)}(t)}{w^{(n-1)}(t)} : 0 < t \leq 1/2 \right\}. \tag{12}$$

Then we have

$$u^{(k)}(t) \geq \rho_0 w^{(k)}(t) \quad \text{for} \quad 0 < t \leq 1/2 \quad (k = 0, \dots, n-1).$$

If along with this we take into account (10) and (11), then from (8) we get

$$u^{(n)}(t) \geq (\rho_0 + 1) w^{(n)}(t) \quad \text{for} \quad 0 < t \leq 1/2.$$

Thus

$$u^{(n-1)}(t) \geq (\rho_0 + 1) w^{(n-1)}(t) \quad \text{for} \quad 0 < t \leq 1/2$$

which contradicts (12).

The above-constructed example shows that the condition (4) (the condition 7) in Theorems 1 and 2 (in Corollaries 1 and 2) cannot be replaced by the condition

$$\limsup_{t \to a} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \int_a^t p_k(s)\, ds \right) \leq 1$$

$$\left(\limsup_{t \to a} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \sum_{i=1}^{\ell} \int_a^t p_{ik}(s)\, ds \right) \leq 1 \right).$$

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Author's address:
A. Tsereteli Kutaisi State University
55, Queen Tamar St., Kutaisi 384000
Georgia