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ON A BOUNDARY VALUE PROBLEM FOR THE TWO-DIMENSIONAL SYSTEM OF EVOLUTION FUNCTIONAL DIFFERENTIAL EQUATIONS

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Suppose $C([a,b];\mathbb{R}^2)$ is the space of two-dimensional continuous vector functions $(x_1,x_2):[a,b]\to\mathbb{R}^2$ with the norm

$$\|(x_1,x_2)\|_C=\max\Big\{|x_1(t)|+|x_2(t)|:\ a\le t\le b\Big\};$$

$M([a,b];\mathbb{R}^2_+)=\Big\{(x_1,x_2)\in C([a,b];\mathbb{R}^2):\ x_1$ and x_2 are nonnegative nondecreasing functions $\Big\}$;

$L([a,b];\mathbb{R})$ is the space of summable functions $y:[a,b]\to\mathbb{R}$ with the norm

$$\|y\|_L=\int_a^b|y(t)|\,dt.$$

Consider the two-dimensional evolution differential system

$$\frac{du_i(t)}{dt}=f_i(u_1,u_2)(t)\ \ (i=1,2) \tag{1}$$

with the boundary conditions

$$u_1(a)=\varphi_1\big(u_2(a)\big),\quad \varphi_2\big(u_1(b),u_2(b)\big)=0, \tag{2}$$

where $f_i:M([a,b];\mathbb{R}^2_+)\to L([a,b];\mathbb{R})$ $(i=1,2)$ are continuous operators, while $\varphi_1:\mathbb{R}_+\to\mathbb{R}$ and $\varphi_2:\mathbb{R}^2_+\to\mathbb{R}$ are continuous functions. We are interested in the case where

$$\sup\bigg\{\Big|f_i(x_1,x_2)(\cdot)\Big|:\ \|(x_1,x_2)\|_C\le\rho\bigg\}\in L([a,b];\mathbb{R})\quad\text{for}\quad 0<\rho<+\infty,$$

and the function φ_2 satisfies one of the following three conditions:

$$\varphi_2(0,0)<0,\quad \varphi_2(x,y)>0\quad\text{for}\quad x\ge0,\ \ y\ge0,\ \ x+y>r; \tag{3}$$

$$\varphi_2(0,0)<0,\quad \varphi_2(x,y)>0\quad\text{for}\quad x\ge0,\ \ y>r; \tag{4}$$

$$\varphi_2(0,0)<0,\quad \varphi_2(x,y)>0\quad\text{for}\quad x>r,\ \ y\ge0, \tag{5}$$

where r is a positive number.

For the case $f_i(u_1,u_2)(t)\equiv f_{0i}(t,u_1(t),u_2(t))$ $(i=1,2)$, where $f_{0i}:[a,b]\times\mathbb{R}^2\to\mathbb{R}$ $(i=1,2)$ are functions satisfying the local Carathéodory conditions, boundary value problems of the type (1), (2) are investigated in full detail (see [1], [2], [4], [9]–[14], and the references therein). In the general case this problem have not been studied enough. The results below fill to some extent the existing gap.

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Let $\delta_i : [a,b] \to \mathbb{R}$ $(i=1,2)$ be continuous functions such that

$$0 \le \delta_i(t) \le t-a \quad \text{for} \quad a \le t \le b \ \ (i=1,2).$$

$f : M([a,b];\mathbb{R}^2_+) \to L([a,b];\mathbb{R})$ is called the (δ_1,δ_2)-*Volterra operator* if for any $t \in \,]a,b]$ and for any vector functions (x_1,x_2) and $(y_1,y_2) \in M([a,b];\mathbb{R}^2_+)$ satisfying the equalities

$$x_1(s) = y_1(s) \ \text{ for } \ 0 \le s \le t-\delta_1(t), \quad x_2(s) = y_2(s) \ \text{ for } \ 0 \le s \le t-\delta_2(t),$$

we have

$$f(x_1,x_2)(s) = f(y_1,y_2)(s) \ \text{ for almost all } \ s \in [0,t].$$

f is called the *Volterra operator* if it is the $(0,0)$-Volterra operator.

Unless the contrary is specified, throughout the paper we will assume that f_1 and f_2 are the Volterra operators.

Definition. A vector function (u_1,u_2) with the absolutely continuous components $u_i : [a,b] \to \mathbb{R}$ $(i=1,2)$ is said to be a nonnegative nondecreasing solution of the problem (1), (2) if:

(i) $(u_1,u_2) \in M([a,b];\mathbb{R}^2_+)$ and almost everywhere on $[a,b]$ the equalities (1) are fulfilled;

(ii) (u_1,u_2) satisfies the boundary conditions (2).

Theorem 1. *Let*

$$f_i(0,0)(t) = 0, \quad f_i(x_1,x_2)(t) \ge 0 \ \ (i=1,2) \tag{6}$$
$$\text{for } \ a \le t \le b, \ \ (x_1,x_2) \in M([a,b];\mathbb{R}^2_+),$$
$$\varphi_1(0) = 0, \quad \varphi_1(x) \ge 0 \ \text{ for } \ x \ge 0, \tag{7}$$

and the condition (3) *be fulfilled. Then the problem* (1), (2) *has at least one nonnegative nondecreasing solution.*

Theorem 2. *Let the conditions* (4), (6), *and* (7) *hold. Let, moreover, there exist a summable function* $h : [a,b] \to \mathbb{R}_+$ *and a positive constant* ℓ *such that*

$$f_1(x_1,x_2)(t) \le \big[h(t) + \ell f_2(x_1,x_2)(t)\big]\big(1 + x_1(t)\big) \tag{8}$$
$$\text{for } \ a \le t \le b, \quad (x_1,x_2) \in M([a,b];\mathbb{R}^2_+), \quad \|x_2\|_C \le r.$$

Then the problem (1), (2) *has at least one nonnegative nondecreasing solution.*

Remark 1. The condition (8) in Theorem 2 cannot be replaced by the condition

$$f_1(x_1,x_2)(t) \le \big[h(t) + \ell f_2(x_1,x_2)(t)\big]\big(1 + x_1(t)\big)^{1+\varepsilon}$$
$$\text{for } \ a \le t \le b, \quad (x_1,x_2) \in M([a,b];\mathbb{R}^2_+), \quad \|x_2\|_C \le r$$

no matter how small $\varepsilon > 0$ would be. However, the condition (8) can be replaced by somewhat different type of condition. More precisely, the following theorem is valid.

Theorem 3. *Let the conditions* (4), (6), *and* (7) *hold. Let, moreover, there exist a continuous function* $\delta : [a,b] \to \mathbb{R}$ *such that*

$$0 < \delta(t) \le t-a \quad \text{for} \quad a < t \le b, \tag{9}$$

and f_1 *be the* $(\delta,0)$-*Volterra operator. Then the problem* (1), (2) *has at least one nonnegative nondecreasing solution.*

Theorem 4. *Let the conditions* (5)–(7) *be fulfilled. Let, moreover, there exist a summable in the first argument function* $g : [a,b] \times \mathbb{R}_+^2 \to \mathbb{R}_+$*, a summable function* $h : [a,b] \to \mathbb{R}_+$ *and a positive constant* ℓ *such that*

$$\limsup_{\rho \to +\infty} \Big[\varphi_1(\rho) + \int_a^b g\big(t, \varphi_1(\rho), \rho\big)\, dt\Big] > r, \tag{10}$$

$$f_1(x_1, x_2)(t) \geq g\big(t, x_1(a), x_2(a)\big) \quad \text{for} \quad a \leq t \leq b, \tag{11}$$
$$(x_1, x_2) \in M([a,b]; \mathbb{R}_+^2), \quad \|x_1\|_C \leq r,$$

and

$$f_2(x_1, x_2)(t) \leq \big[h(t) + \ell f_2(x_1, x_2)(t)\big]\big(1 + x_2(t)\big) \quad \text{for} \quad a \leq t \leq b,$$
$$(x_1, x_2) \in M([a,b]; \mathbb{R}_+^2), \quad \|x_1\|_C \leq r.$$

Then the problem (1), (2) *has at least one nonnegative nondecreasing solution.*

Remark 2. The condition (10) in Theorem 4 cannot be replaced by the condition

$$\limsup_{\rho \to +\infty} \Big[\varphi_1(\rho) + \int_a^b g\big(t, \varphi_1(\rho), \rho\big)\, dt\Big] \geq r. \tag{10$'$}$$

Indeed, it is clear that the problem

$$u_1'(t) = 0, \quad u_2'(t) = 0;$$
$$u_1(a) = r\,\frac{u_2(a)}{1 + u_2(a)}\,, \quad u_1(b) = r$$

has no solution, although all the conditions of Theorem 4, except of (10), are fulfilled. Instead of (10) the condition (10$'$) holds.

Theorem 5. *Let the conditions* (5)–(7) *be fulfilled. Let, moreover, there exist a summable in the first argument function* $g : [a,b] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ *and a continuous function* $\delta : [a,b] \to \mathbb{R}_+$ *such that the conditions* (9)–(11) *are fulfilled, and* f_2 *be the* $(0, \delta)$*-Volterra operator. Then the problem* (1), (2) *has at least one nonnegative nondecreasing solution.*

As an example, consider the boundary value problem

$$\frac{du_i(t)}{dt} = f_{0i}\big(t, u_1(\tau_1(t)), u_2(\tau_2(t))\big) \quad (i = 1, 2); \tag{12}$$

$$u_1(a) = \alpha u_2(a), \quad \beta_1 u_1(b) + \beta_2 u_2(b) = \gamma, \tag{13}$$

where $f_{0i} : [a,b] \times \mathbb{R}_+^2 \to \mathbb{R}$ $(i = 1, 2)$ are functions satisfying the local Carathéodory conditions, $\tau_i : [a,b] \to [a,b]$ $(i = 1, 2)$ are measurable functions satisfying the inequalities

$$\tau_i(t) \leq t \quad \text{for} \quad a \leq t \leq b \ \ (i = 1, 2),$$

$\alpha \geq 0$, $\beta_1 \geq 0$, $\beta_2 \geq 0$, $\beta_1 + \beta_2 > 0$ and $\gamma > 0$.

From Theorem 1–5 we have

Corollary. *Let*

$$f_{0i}(t, 0, 0) = 0, \quad f_i(t, x, y) \geq 0 \ \ (i = 1, 2) \quad \text{for} \quad a \leq t \leq b, \ \ x \geq 0, \ \ y \geq 0.$$

Then for the existence of at least one nonnegative nondecreasing solution of the problem (12), (13) *it is sufficient one of the following five conditions to be fulfilled:*

(i) $\beta_1 > 0$, $\beta_2 > 0$;

(ii) $\beta_1 = 0$, $\beta_2 = 1$, *and there exist a summable function* $h : [a,b] \to \mathbb{R}_+$ *and a positive constant* ℓ *such that*

$$f_{01}(t,x,y) \le \Big[h(t) + \ell f_{02}(t,x,y)\Big](1+x) \quad \textit{for} \quad a \le t \le b, \;\; x \ge 0, \;\; 0 \le y \le \gamma;$$

(iii) $\beta_1 = 0$, $\beta_2 = 1$, *and*

$$\operatorname{ess\,inf}\Big\{s - \tau_1(s) : \;\; t \le s \le b\Big\} > 0 \quad \textit{for} \quad a < t \le b;$$

(iv) $\alpha > 0$, $\beta_1 = 1$, $\beta_2 = 0$, *and there exist a summable function* $h : [a,b] \to \mathbb{R}_+$ *and a positive constant* ℓ *such that*

$$f_{02}(t,x,y) \le \Big[h(t) + \ell f_{01}(t,x,y)\Big](1+y) \quad \textit{for} \quad a \le t \le b, \;\; 0 \le x \le \gamma, \;\; y \ge 0;$$

(v) $\alpha > 0$, $\beta_1 = 1$, $\beta_2 = 0$, *and*

$$\operatorname{ess\,inf}\Big\{s - \tau_2(s) : \;\; t \le s \le b\Big\} > 0 \quad \textit{for} \quad a < t \le b.$$

The above-formulated theorems and their corollaries generalize some previous results from [3] and make the results from [5]–[8] more complete.

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