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NECESSARY CONDITIONS OF OPTIMALITY FOR OPTIMAL PROBLEMS WITH DELAYS AND WITH A DISCONTINUOUS INITIAL CONDITION

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Let $J = [a, b]$ be a finite interval; $O \subset \mathbb{R}^n, G \subset \mathbb{R}^r$ be open sets and let the function $f : J \times O^s \times G^\nu \to \mathbb{R}^n$ satisfy the following conditions:

1) for a fixed $t \in J$ the function $f(t, x_1, \dots, x_s, u_1, \dots, u_\nu)$ is continuous with respect to $(x_1, \dots, x_s, u_1, \dots, u_\nu) \in O^s \times G^\nu$ and continuously differentiable with respect to $(x_1, \dots, x_s) \in O^s$;

2) for a fixed $(x_1, \dots, x_s, u_1, \dots, u_\nu) \in O^s \times G^\nu$ the functions $f,\ f_{x_i},\ i = 1, \dots, s$, are measurable with respect to t. For arbitrary compacts $K \subset O, V \subset G$ there exists a function $m_{K,V}(\cdot) \in L(J, \mathbb{R}_0^+)$, $\mathbb{R}_0^+ = [0, \infty)$ such that

$$|f(t, x_1, \dots, x_s, u_1, \dots, u_\nu)| + \sum_{i=1}^{s} |f_{x_i}(\cdot)| \le m_{K,V}(t),$$
$$\forall (t, x_1, \dots, x_s, u_1, \dots, u_\nu) \in J \times K^s \times V^\nu.$$

Let now $\tau_i(t),\ i = 1, \dots, s,\ t \in J$, be absolutely continuous functions, satisfying the conditions: $\tau_i(t) \le t,\ \ \dot{\tau}_i(t) > 0$; Δ be a space of piecewise continuous functions $\varphi : J_1 = [\tau, b] \to N$, $\tau = \min(\tau_1(a), \dots, \tau_s(a))$, with a finite number of discontinuity points of the first kind; The functions $\theta_i(t)$, $i = 1, \dots, \nu$, $t \in R$, satisfy commeasurability condition i.e. there exists absolutely continuos function $\theta(t) < t,\ \dot{\theta}(t) > 0$ such that $\theta_i(t) = \theta^{k_i}(t)$, where $k_\nu > \cdots k_1 \ge 0$ are natural numbers, $\theta^i(t) = \theta(\theta^{i-1}(t))$, $\theta^0(t) = t$; Ω is the set of measurable functions $u : J_2 = [\theta, b] \to U$, $\theta = \min\{\theta_1(a), \dots, \theta_\nu(a)\}$, satisfiying the conditions $cl\{u(t) : t \in J_2\}$ is compact lying in G, $U \subset G$ is an arbitrary set, $J_2 = [\theta, b], \theta = \theta_\nu(a)$; $q^i : J^2 \times O^2 \to \mathbb{R}$, $i = 0, \dots, l$, are continuously differentiable functions.

We consider the differential equation in $\mathbb{R}^n$

$$\dot{x}(t) = f(t, x(\tau_1(t), \dots, x(\tau_s(t)), u(\theta_1(t), \dots, u(\theta_\nu(t))),$$
$$t \in [t_0, t_1] \subset J, \tag{1}$$

with the discontinuity condition

$$x(t) = \varphi(t), \quad t \in [\tau, t_0), \quad x(t_0) = x_0. \tag{2}$$

Definition 1. The function $x(t) = x(t; \sigma) \in O$, $\sigma = (t_0, t_1, x_0, \varphi, u) \in A = J \times J \times O \times \Delta \times \Omega$, $t_0 < t_1$, defined on the interval $[\tau, t_1]$, is said to be a solution corresponding to the element $\sigma \in A$, if on the interval $[\tau, t_0]$ it satisfies the condition (2), while on the interval $[t_0, t_1]$ it is absolutely continuous and almost everywhere satisfies the equation (1).

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Definition 2. The element $\sigma \in A$ is said to be admissible if the corresponding solution $x(t) = x(t;\sigma)$ satisfies the conditions

$$q^i(t_0, t_1, x(t_0), x(t_1)) = 0, \quad i = 0, \dots, l.$$

The set of admissible elements will denoted by A_0.

Definition 3. The element $\tilde{\sigma} = (\tilde{t_0}, \tilde{t_1}, \tilde{x_0}, \tilde{\varphi}, \tilde{u}) \in A_0$ is said to be optimal if for an arbitrary element $\sigma \in A_0$ the inequality

$$q^0(\tilde{t_0}, \tilde{t_1}, \tilde{x(\tilde{t_0})}, \tilde{x(\tilde{t_1})}) \le q^0(t_0, t_1, x(t_0), x(t_1)); \quad \tilde{x(t)} = x(t;\tilde{\sigma})$$

holds.

The problem of optimal control consists in finding optimal element. In order to formulate the main results, we will need the following notations:

$$\dot{\gamma}_i^- = \dot{\gamma}_i(\tilde{t}_0-), \ i = 1, \dots, s, \gamma_i(t) \text{ is the function inverse to } \tau_i(t); \gamma_i = \gamma_i(\tilde{t}_0);$$

$$\omega_i^- = (\tilde{t}_0, \underbrace{\tilde{x}_0, \dots, \tilde{x}_0}_{\text{i-times}}, \underbrace{\tilde{\varphi}(\tilde{t}_0-)\dots, \tilde{\varphi}(\tilde{t}_0-)}_{\text{(p- i)-times}}, \tilde{\varphi}(\tau_{p+1}(\tilde{t}_0-)), \dots, \tilde{\varphi}(\tau_s(\tilde{t}_0-))),$$

$$i = 0, \dots, p;$$

$$\omega_i^- = (\gamma_i, \tilde{x}(\tau_1(\gamma_i)), \dots, \tilde{x}(\tau_{i-1}(\gamma_i)), \tilde{x}_0, \tilde{\varphi}(\tau_{i+1}(\gamma_i-)), \dots, \tilde{\varphi}(\tau_s(\gamma_i-))),$$

$$\overset{\circ}{\omega}{}_i^- = (\gamma_i, \tilde{x}(\tau_1(\gamma_i)), \dots, \tilde{x}(\tau_{i-1}(\gamma_i)), \tilde{\varphi}(\tilde{t}_0-), \tilde{\varphi}(\tau_{i+1}(\gamma_i-)), \dots, \tilde{\varphi}(\tau_s(\gamma_i-))),$$

$$i = p+1, \dots, s.$$

Analogously is defined $\dot{\gamma}_i^+$, ω_i^+, $\overset{\circ}{\omega}{}_i^+$.

Theorem 1. *Let $\tilde{\sigma} \in A_0$ be optimal element, $\tilde{t}_0 \in (a, b)$, $\tilde{t}_1 \in (a, b]$ and the following conditions are hold:*

1. $\tau_i(\tilde{t_0}) = \tilde{t_0}$, $i = 1, \dots, p$; $\tau_i(\tilde{t_0}) < \tilde{t_0}$, $\tau_i(\tilde{t_1}) > \tilde{t_0}$, $i = p+1, \dots, s$; *there exists the left semi-neighborhood $V_{\tilde{t_0}}^-$ of the point $\tilde{t}_0$ such that*

$$t < \gamma_1(t) < \dots < \gamma_p(t), \quad \forall t \in V_{\tilde{t_0}}^-; \tag{3}$$

next, $\gamma_{p+1} < \dots < \gamma_s$;

2. *There exist the finite limits:*

$$\dot{\gamma}_i^-, \quad i = 1, \dots, s;$$

$$\lim_{\omega \to \omega_i^-} \tilde{f}(\omega) = f_i^-, \ \omega = (t, x_1, \dots, x_s) \in R_{\tilde{t_0}}^- \times O^s, \ i = 0, \dots, p,$$

$$\textit{where } \tilde{f}(\omega) = f(\omega, \tilde{u}(\theta_1(t)), \dots, \tilde{u}(\theta_\nu(t))),$$

$$\lim_{(\omega_1, \omega_2) \to (\omega_i^-, \overset{\circ}{\omega}{}_i^-)} [\tilde{f}(\omega_1) - \tilde{f}(\omega_2)] = f_i^-, \ \omega_1, \omega_2 \in R_{\gamma_i}^- \times O^s, \ i = p+1, \dots, s.$$

$$\lim_{\omega \to \omega_{s+1}} \tilde{f}(\omega) = f_{s+1}^-, \omega \in R_{\tilde{t_1}}^- \times O^s, \omega_{s+1} = (\tilde{t_1}, \tilde{x}(\tau_1(\tilde{t_1})), \dots, \tilde{x}(\tau_s(\tilde{t_1}))),$$

then there exists non-zero vector $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \le 0$, and a solution $\psi(t)$, $t \in [\tilde{t_0}, \gamma]$, $\gamma = \max(\gamma_1(b), \dots, \gamma_s(b))$ of the equation

$$\dot{\psi}(t) = -\sum_{i=1}^{s} \psi(\gamma_i(t)) \tilde{f_{x_i}}[\gamma_i(t)] \dot{\gamma}_i(t), \quad t \in [\tilde{t_0}, \tilde{t_1}],$$
$$\psi(t) = 0, \quad t \in (\tilde{t_1}, \gamma], \tag{4}$$

such that the following conditions are fulfilled:

$$\sum_{i=p+1}^{s} \int\limits_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} \psi(\gamma_i(t))\tilde{f}_{x_i}[\gamma_i(t)]\dot{\gamma}_{i(t)}\tilde{\varphi}(t)dt \geq$$

$$\geq \sum_{i=p+1}^{s} \int\limits_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} \psi(\gamma_i(t))\tilde{f}_{x_i}[\gamma_i(t)]\dot{\gamma}_{i(t)}\varphi(t)dt,\ \forall\varphi(\cdot) \in \triangle, \tag{5}$$

$$\int\limits_{\tilde{t}_0}^{\tilde{t}_1}\psi(t)\tilde{f}[t]dt \geq \int\limits_{\tilde{t}_0}^{\tilde{t}_1}\psi(t)f(t,\tilde{x}(\tau_1(t)),\ldots,\tilde{x}(\tau_s(t)),u(\theta_1(t)),\ldots,u(\theta_\nu(t)))dt,\ \forall u(\cdot)\in\Omega, \tag{6}$$

$$\pi\tilde{Q}_{x_0} = -\psi(\tilde{t_0}), \pi\tilde{Q}_{x_1} = \psi(\tilde{t_1}), \tag{7}$$

$$\pi\tilde{Q}_{t_0} \geq -\psi(\tilde{t_0})\sum_{i=0}^{p}(\hat{\gamma}_{i+1}^{-} - \hat{\gamma}_i^{-})f_i^{-} + \sum_{i=p+1}^{s}\psi(\gamma_i)f_i^{-}\dot{\gamma}_i^{-}, \tag{8}$$

$$\pi\tilde{Q}_{t_1} \geq -\psi(\tilde{t_1})f_{s+1}^{-}. \tag{9}$$

Here $\tilde{f[t]} = \tilde{f}(t,\tilde{x}(\tau_1(t)),\ldots,\tilde{x}(\tau_s(t)),$ $\tilde{f_{x_i}}[t] = \tilde{f_{x_i}}(t,\tilde{x}(\tau_1(t)),\ldots,\tilde{x}(\tau_s(t));$ $\hat{\gamma}_0^{-} = 1,$ $\hat{\gamma}_i^{-} = \dot{\gamma}_i^{-},\ i = 1,\ldots,p,\ \hat{\gamma}_{p+1} = 0;$

The tilde over $Q = (q^0,\ldots,q^l)^T$ *means that the corresponding gradient is calculated at the point* $(\tilde{t_0},\tilde{t_1},\tilde{x(\tilde{t_0})},\tilde{x(\tilde{t_1})})$

Remark 1. If

$$\operatorname{rank}(\tilde{Q}_{x_0},\tilde{Q}_{x_1}) = 1 + l,$$

then in theorem 1 $\psi(t) \not\equiv 0$. If $\tilde{\varphi}(\tilde{t}_0-) = \tilde{x}_0$, then $f_0^{-} = \cdots = f_p^{-}$, $f_i^{-} = 0$, $i = p+1,\ldots,s$, the condition (8) has the form

$$\pi\tilde{Q}_{t_0} \geq \psi(\tilde{t_0})f_0^{-}.$$

If $\dot{\gamma}_p^{-} < \cdots < \dot{\gamma}_1^{-} < 1$, then the condition (3) is held.

Theorem 2. *Let* $\tilde{\sigma} \in A_0$ *be optimal element,* $\tilde{t}_0 \in [a,b)$, $\tilde{t}_1 \in (a,b)$ *and the following conditions hold:*

1. $\tau_i(\tilde{t_0}) = \tilde{t_0},\ i = 1,\ldots,p;$ $\tau_i(\tilde{t_0}) < \tilde{t_0},\ \tau_i(\tilde{t_1}) > \tilde{t_0},\ i = p+1,\ldots,s$; *there exists the right semi-neighborhood* $V^{+}(\tilde{t_0})$ *of the point* $\tilde{t}_0$ *such that*

$$t < \gamma_1(t) < \cdots < \gamma_p(t), \quad \forall t \in V_{\tilde{t}_0}^{+}; \tag{10}$$

next, $\gamma_{p+1} < \cdots < \gamma_s$;

2. *There exist the finite limits:*

$$\dot{\gamma}_i^{+}, \quad i = 1,\ldots,s,$$

$$\lim_{\omega\to\omega_i^{+}}\tilde{f}(\omega) = f_i^{+},\ \omega = (t,x_1,\ldots,x_s) \in R_{\tilde{t}_0}^{+} \times O^s,\ i = 0,\ldots,p,$$

$$\lim_{(\omega_1,\omega_2)\to(\omega_i^{+},\overset{\circ}{\omega}{}_i^{+})}[\tilde{f}(\omega_1) - \tilde{f}(\omega_2)] = f_i^{+},\ \omega_1,\omega_2 \in R_{\gamma_i}^{+} \times O^s,\ i = p+1,\ldots,s,$$

$$\lim_{\omega\to\omega_{s+1}}\tilde{f}(\omega) = f_{s+1}^{+},\omega \in R_{\tilde{t}_1}^{+} \times O^s,$$

then there exists a non-zero vector $\pi = (\pi_0, \ldots, \pi_l)$*,* $\pi_0 \le 0$*, and a solution* $\psi(t)$ *of the equation* (4) *such that the conditions* (5)–(7) *are fulfilled. Moreover,*

$$\pi \tilde{Q}_{t_0} \le -\psi(\tilde{t_0}) \sum_{i=0}^{p} (\hat{\gamma}_{i+1}^{+} - \hat{\gamma}_i^{+}) f_i^{+} + \sum_{i=p+1}^{s} \psi(\gamma_i) f_i^{+} \dot{\gamma}_i^{+}, \tag{11}$$

$$\pi \tilde{Q}_{t_1} \le -\psi(\tilde{t_1}) f_{s+1}^{+}, \tag{12}$$

where $\hat{\gamma}_0^{+} = 1$*,* $\hat{\gamma}_i^{+} = \dot{\gamma}_i^{+}$*,* $i = 1, \ldots, p$*,* $\hat{\gamma}_{p+1} = 0$*.*

Remark 2. If $\tilde{\varphi}(\tilde{t}_0+) = \tilde{x}_0$, then $f_0^{+} = \cdots = f_p^{+}$, $f_i^{+} = 0$, $i = p+1, \ldots, s$, the condition (11) has the form

$$\pi \tilde{Q}_{t_0} \le \psi(\tilde{t_0}) f_0^{+}.$$

If $1 < \dot{\gamma}_1^{+} < \cdots < \dot{\gamma}_p^{+}$, then the condition (10) holds.

Theorem 3. *Let* $\tilde{\sigma} \in A_0$ *be optimal element,* $\tilde{t_0}$*,* $\tilde{t_1} \in (a, b)$ *and the assumptions of theorems* 1, 2 *are hold. Let, besides*

$$\sum_{i=0}^{p} (\hat{\gamma}_{i+1}^{-} - \hat{\gamma}_i^{-}) f_i^{-} = \sum_{i=0}^{p} (\hat{\gamma}_{i+1}^{+} - \hat{\gamma}_i^{+}) f_i^{+} = f_0,$$
$$f_i^{-} \dot{\gamma_i}^{-} = f_i^{+} \dot{\gamma_i}^{+} = f_i, \; i = p+1, \ldots, s, \; f_{s+1}^{-} = f_{s+1}^{+} = f_{s+1},$$

then there exists non-zero vector $\pi = (\pi_0, \ldots, \pi_l)$*,* $\pi_0 \le 0$ *and a solution* $\psi(t)$ *of the equation* (4) *such that the condition* (5)–(7) *are fulfilled. Moreover,*

$$\pi \tilde{Q}_{t_0} = \psi(\tilde{t_0}) f_0 + \sum_{i=p+1}^{s} \psi(\gamma_i) f_i, \quad \pi \tilde{Q}_{t_1} = -\psi(\tilde{t_1}) f_{s+1}. \tag{13}$$

If

$$\mathrm{rank}(\tilde{Q}_{t_0}, \tilde{Q}_{t_1}, \tilde{Q}_{x_0}, \tilde{Q}_{x_1}) = 1 + l,$$

then in theorem 3 $\psi(t) \not\equiv 0$. If $\tilde{\varphi}(\tilde{t}_0-) = \tilde{\varphi}(\tilde{t}_0+) = \tilde{x_0}$,, then $f_i = 0$, $i = p+1, \ldots, s$. For the case $s = \nu = 2$, $\tau_1(t) = \theta_1(t) = t$ the analogous theorems are given in [1].

Now we consider the case, when the functions $\theta_i(t)$, $i = 1, \ldots, \nu$, are absolutely continuous and $\theta_i(t) \le t$, $\dot{\theta}_i(t) > 0$. Next, $U \subset G$ is a convex set and the function $f(t, x_1, \ldots, x_s, u_1, \ldots, u_\nu)$ satisfies the following conditions: for a fixed $t \in J$ it is continuously differentiable with respect to $(x_1, \ldots, x_s, u_1, \ldots, u_\nu) \in O^s \times G^\nu$; for a fixed $(x_1, \ldots, x_s, u_1, \ldots, u_\nu) \in O^s \times G^\nu$ the functions f, f_{x_i}, $i = 1, \ldots, n$, f_{u_j}, $j = 1, \ldots, \nu$ are measuarable with respect to t; for arbitrary compacts $K \subset O$, $V \subset G$ there exists a function $m_{K,V}(\cdot) \in L(J, \mathbb{R}_0^{+})$ such that

$$|f(t, x_1, \ldots, x_s, u_1, \ldots, u_\nu)| + \sum_{i=1}^{s} |f_{x_i}(\cdot)| + \sum_{i=1}^{\nu} |f_{u_i}(\cdot)| \le m_{K,V}(t),$$
$$\forall (t, x_1, \ldots, x_s, u_1, \ldots, u_\nu) \in J \times K^s \times V^\nu.$$

Theorem 4. *Let* $\tilde{\sigma} \in A_0$ *be an optimal element,* $\tilde{t_0} \in (a, b)$*,* $\tilde{t_1} \in (a, b]$ *and the assumptions of Theorem* 1 *be fulfilled. Then there exist a non-zero vector* $\pi = (\pi_0, \ldots, \pi_l)$*,*

$\pi_0 \leq 0$ *and a solution* $\psi(t)$ *of the equation* (4) *such that the conditions* (5), (7)–(9) *are fulfilled. Moreover,*

$$\sum_{j=1}^{\nu} \int_{\tilde{t_0}}^{\tilde{t_1}} \psi(t) f_{u_j}[t] u(\tilde{\theta_j}(t))dt \geq \sum_{j=1}^{\nu} \int_{\tilde{t_0}}^{\tilde{t_1}} \psi(t) f_{u_j}[t] u(\theta_j(t))dt, \quad \forall u(\cdot) \in \Omega, \tag{14}$$

where

$$\tilde{f_{u_j}}[t] = f_{u_j}(t, \tilde{x}(\tau_1(t)), \dots, \tilde{x}(\tau_s(t)), \tilde{u}(\theta_1(t)), \dots \tilde{u}(\theta_\nu(t))).$$

Theorem 5. *Let* $\tilde{\sigma} \in A_0$ *be an optimal element,* $\tilde{t_0} \in [a, b)$, $\tilde{t_1} \in (a, b)$ *and the assumptions of Theorem* 2 *be fulfilled. Then there exist a non-zero vector* $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \leq 0$ *and a solution* $\psi(t)$ *of the equation* (4) *such that the conditions* (5), (7), (11), (12) *are fulfilled.*

Theorem 6. *Let* $\tilde{\sigma} \in A_0$ *be an optimal element,* $\tilde{t_0}$, $\tilde{t_1} \in (a, b)$ *and the assumptions of Theorem* 3 *be fulfilled. Then there exist a non-zero vector* $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \leq 0$, *and a solution* $\psi(t)$ *of the equation* (4) *such that the conditions* (5), (7), (13), (14) *hold.*

The case, when t_0 is fixed is considered in [2].

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