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# PSEUDO-DIFFERENTIAL CRACK THEORY

**Abstract.** Crack problems are regarded as elements in a pseudo-differential algebra, where the two sides $\mathrm{int}\, S_{\pm}$ of the crack $S$ are treated as interior boundaries and the boundary $Y$ of the crack as an edge singularity. We employ the pseudo-differential calculus of boundary value problems with the transmission property near $\mathrm{int}\, S_{\pm}$ and the edge pseudo-differential calculus (in a variant with Douglis-Nirenberg orders) to construct parametrices of elliptic crack problems (with extra trace and potential conditions along $Y$) and to characterise asymptotics of solutions near $Y$ (expressed in the framework of continuous asymptotics). Our operator algebra with boundary and edge symbols contains new weight and order conventions that are necessary also for the more general calculus on manifolds with boundary and edges.

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# Introduction

Crack theory is motivated by models in mechanics, where a medium has a crack, described by a surface $S$ with boundary $Y = \partial S$, embedded in a domain $G$, where $S$ is of codimension 1 in $G$. Given an elliptic operator $A$ in $G$ (in fact, a system) and elliptic boundary conditions $T_\pm$ on both sides $S_\pm$ of $S$, a problem is then to characterise regularity and asymptotics of solutions $u$ of

$$Au = f \ \text{ in } \ G \backslash S, \quad T_\pm u = g_\pm \ \text{ on } \ \operatorname{int} S_\pm \tag{1}$$

in a neighbourhood of $Y$. The specific difficulty (even in such a linear model) is that the crack boundary $Y$ is a geometric singularity of the configuration and that transparent answers require the machinery of singular boundary value problems, here, for the case of an edge singularity $Y$. Very much has been done in the literature for this kind of problems, cf. Duduchava and Wendland [3] and the references there, or Schulze [26]. (In the present paper we content ourselves with references to pseudo-differential techniques from the calculus on manifolds with a singular geometry.) Results mainly concern the case of a smooth crack boundary $Y$. It is also interesting to admit singularities of $Y$, e.g., conical ones. Such a situation requires a systematic strategy from the theory of corner boundary value problems, especially, the pseudo-differential calculus for edge singularities in a particularly adapted form.

The purpose of the present paper is to develop this type of analysis under the assumption that the edge $Y$ is smooth. (The case when $Y$ has conical singularities will be treated in a forthcoming paper.) Our paper employs and further develops the achievements of the analysis of boundary value problems on manifolds with edges, cf. the monograph of Schulze [27] (for the simpler case of manifolds with edges and without boundary). Compared with the orientation of [26] we establish a new pseudo-differential algebra that contains boundary operators $T_\pm$ that represent, for instance, Dirichlet or Neumann conditions, as well as arbitrary conditions on $\operatorname{int} S_\pm$ that are elliptic in the Shapiro-Lopatinskij sense with respect to $A$. Ellipticity also concerns additional trace and potential conditions along the crack boundary $Y$, depending on chosen weights, and we construct parametrices of elliptic operators within our calculus. To describe asymptotics of solutions near the crack boundary $Y$ we mainly employ the scenario of continuous asymptotics that has been introduced for conical singularities (with a closed base) by Schulze [29] and then applied to the problem of variable and branching discrete asymptotics for edge problems [30] and to boundary value problems without the transmission property [31], [32], cf. also Rempel and Schulze [18], and [25], Section 1.4. In the present paper we develop the approach of continuous asymptotics for the case $\{\text{slit plane}\} \times Y$ which is a wedge with boundary that locally describes the crack singularity near $Y$, cf. Section 1.1. We obtain elliptic regularity of solutions to elliptic crack problems in weighted Sobolev spaces with (and without) continuous asymptotics. Concerning the weighted Sobolev spaces themselves we apply a formalism, based on strongly continuous groups of isomorphisms in spaces along the model cone transversal to the edge, as it has been introduced in Schulze in [34], together with an adapted pseudo-differential calculus with "twisted" homogeneity, cf. also the monographs [33], [27]. Further technicalities on a specific subclass of Green and Mellin operator-valued symbols (with constant discrete asymptotics) for boundary value problems are developed in Schrohe and Schulze [22], [24], [23]. We employ here similar symbols, though with some essential modifications, in particular, with continuous asymptotics and of Douglis-Nirenberg orders that are indispensable for a consistent wedege pseudo-differential calculus with classical differential boundary conditions along the $\pm$-sides of the crack $S$.

## 1. Modelling of Singular Boundary Value Problems

**1.1. Boundary value problems in crack configurations.** As noted in the introduction we start from an elliptic differential operator $A$ with smooth coefficients in a domain $G$ in $\mathbb{R}^n$, i.e.,

$$A = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha \tag{2}$$

with $N \times N$ -matrix-valued coefficients $a_\alpha \in C^\infty(G) \otimes \mathbb{C}^N \otimes \mathbb{C}^N$. Moreover, we consider an oriented surface $S$ embedded in $G$ and of codimension 1, that is assumed to be a smooth, compact manifold with boundary $Y$, not intersecting the boundary of $G$. The orientation of $S$ allows us to talk about plus and minus sides $S_\pm$ of $S$. We want to study solutions $u$ of $Au = f$ in $G \backslash S$ under elliptic boundary conditions on $\operatorname{int} S_\pm = S_\pm \backslash Y$ of the form $T_\pm u = g$, where

$$T_\pm u = \mathrm{r}'_\pm \, B_\pm u \tag{3}$$

with $M \times N$- matrices of differential operators $B_\pm$ with smooth coefficients that are given in a neighbourhood of $S$, and $\mathrm{r}'_\pm$ being the operators of restriction to $\operatorname{int} S_\pm$. If we want to distinguish orders in the boundary conditions we write $M = \sum_{j=1}^{l_\pm} M_{\pm,j}$ and represent $B_\pm$ as a vector of $(M_{\pm,j} \times N)$-block matrix operators $B_{\pm,j}$ of orders $m_{\pm,j}$, $j = 1, \dots, l_\pm$. (For simplicity, we assume $m_{\pm,j} < m$ for all $j$, though this is not really necessary for our methods in general). An example for this situation is the following second order $3 \times 3$- Lamé system in $G \subset \mathbb{R}^3$

$$\mu \Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u = f, \qquad \mu > 0, \lambda + \mu > 0, \tag{4}$$

with Dirichlet or Neumann boundary conditions from the plus/minus sides of $S$. The main properties of our problem are not affected by the boundary of $G$. Moreover, for the "non-smoothing" part of the expected asymptotics of solutions it suffices to study the problem locally in a neighbourhood of any point of $Y$. Then $G\backslash S$ corresponds locally to a “wedge” $(\mathbb{R}^2\backslash\mathbb{R}_+)\times\Omega$. Here, $\Omega\subseteq\mathbb{R}^q$ is an open set, and $\mathbb{R}^2$ is a plane transversal to the boundary of the crack. The intersection of $S\backslash Y$ with that plane corresponds locally to $\mathbb{R}_+$; according to the nature of the problem we have, in fact, two copies of $\mathbb{R}_+$ in $\mathbb{R}^2$. Under that point of view the complement of the slit in $\mathbb{R}^2$ can be identified in polar coordinates with $(r,\phi)\in\mathbb{R}_+\times[0,2\pi]$, where $\mathbb{R}_+\times\{0\}$ and $\mathbb{R}_+\times\{2\pi\}$ are interpreted as the $+$ and $-$ sides, respectively. We thus arrive at a stretched wedge $\mathbb{R}_+\times[0,2\pi]\times\Omega$. This will be regarded as the local model of our crack configuration. Incidentally we set $\iota_+=0$, $\iota_-=2\pi$. Let $(r,\phi,y)$ denote the corresponding variables and $(\varrho,\vartheta,\eta)$ the associated covariables. To reformulate the operator (1) in these coordinates we write $x\in\mathbb{R}^n$, $\mathbb{R}^n=\mathbb{R}^2\times\mathbb{R}^q$, $q=n-2$, interprete $(r,\phi)$ as polar coordinates in $\mathbb{R}^2\backslash 0$ and let $y$ vary on the open set $\Omega\subseteq\mathbb{R}^q$. Then $A$ takes the form

$$A=r^{-m}\sum_{k+|\beta|\le m}a_{k\beta}(r,y)\left(-r\frac{\partial}{\partial r}\right)^k(rD_y)^\beta \tag{5}$$

with operator-valued coefficients $a_{k\beta}(r,y)\in C^\infty(\overline{\mathbb{R}}_+\times\Omega,\mathrm{Diff}^{m-(k+|\beta|)}_{N\times N}(I))$; here $I=[0,2\pi]$, and $\mathrm{Diff}^l_{N\times N}(I)$ is the space of all $N\times N$- systems of differential operators of order $l$ on $I$ with smooth coefficients (up to $\phi=0$ and $\phi=2\pi$). In a similar manner we can reformulate the boundary operators. We then have

$$T_{\pm,j}=\mathrm{r}'_\pm\, r^{-m_{\pm,j}}\sum_{k+|\beta|\le m_{\pm,j}}b_{\pm,j;k\beta}(r,y)\left(-r\frac{\partial}{\partial r}\right)^k(rD_y)^\beta \tag{6}$$

with coefficients $b_{\pm,j;k\beta}(r,y)\in C^\infty(\overline{\mathbb{R}}_+\times\Omega,\ \mathrm{Diff}^{m_{\pm,j}-(k+|\beta|)}_{M_{\pm,j}\times N}(I))$, $j=1,\dots,l_\pm$. (Clearly, it suffices to know $b_{\pm,j;k\beta}$ in a neighbourhood of $\mathbb{R}_+\times\{\iota_\pm\}\times\Omega$). We shall reformulate (5) and (6) as (pseudo-differential) operators with respect to the Mellin transform with operator-valued symbols. The Mellin transform will be used in its classical form

$$(Mu)(z)=\int_0^\infty r^{z-1}u(r)\,\mathrm{d}r, \tag{7}$$

first on (vector-valued) functions $u(r)$ with compact support on $\mathbb{R}_+$ and then extended to various weighted distribution spaces. Let $\mathcal{A}(\mathbb{C})$ denote the space of entire functions in $\mathbb{C}$, and set $\Gamma_\beta=\{z\in\mathbb{C}:\mathrm{Re}\,z=\beta\}$ for any $\beta\in\mathbb{R}$. Then $u\in C_0^\infty(\mathbb{R}_+)$ implies $(Mu)(z)\in\mathcal{A}(\mathbb{C})$, and we have $Mu|_{\Gamma_\beta}\in\mathcal{S}(\Gamma_\beta)$ for every $\beta\in\mathbb{R}$, uniformly in compact $\beta$-intervals. Recall that the inverse of (7) has the form

$$\left(M^{-1}g\right)(r)=(2\pi i)^{-1}\int_{\Gamma_\beta}r^{-z}g(z)\,\mathrm{d}z, \tag{8}$$

(by Cauchy's theorem the choice of $\beta$ is unessential for $u \in C_0^\infty(\mathbb{R}_+)$). As is well-known the Mellin transform extends to an isomorphism

$$M_\gamma : r^\gamma L^2(\mathbb{R}_+) \longrightarrow L^2(\Gamma_{\frac{1}{2}-\gamma})$$

(here, $L^2$ denotes the spaces of square integrable functions with the respective standard scalar products); then (8) is to be evaluated for $\beta = \frac{1}{2} - \gamma$. Given any $f(r, r', z) \in S^\mu(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma})$ (= Hörmander's standard symbol class on $\mathbb{R}_+ \times \mathbb{R}_+$ of order $\mu$, with $\operatorname{Im} z$ as the covariable, $z$ running over $\Gamma_{\frac{1}{2}-\gamma}$) we can form the Mellin pseudo-differential operator

$$\operatorname{op}_M^\gamma(f)u(r) = M_{\gamma,z\to r}^{-1}\{M_{\gamma,r'\to z} f(r, r', z)u(r')\} \tag{9}$$

with $M_\gamma$ being the weighted Mellin transform $(M_\gamma u)(z) = M(r^{-\gamma}u)(z+\gamma)$ (and $M := M_0$). Similar notation is employed in the vector-and operator-valued situation. Then, in particular, we have to specify the nature of symbol spaces. This will be done in a systematic form in Section 1.3 below. For the moment, such things are evident for Fuchs type differential operators. With (5) we associate a $(y, \eta)$- dependent family of $\operatorname{Diff}_{N\times N}^m(I)$-valued Mellin symbols

$$f(r, y, z, \eta) = r^{-m} \sum_{k+|\beta|\le m} a_{k\beta}(r, y) z^k (r\eta)^\beta. \tag{10}$$

Then $A$ can be written in the form

$$A = \operatorname{Op}_y(\operatorname{op}_M^\delta(f))$$

for any weight $\delta \in \mathbb{R}$ (to be chosen below); here $\operatorname{Op}_y(a)$ is the pseudo-differential action with respect to the Fourier transform in $y$- variables, i.e.,

$$\operatorname{Op}_y(a)v(y) = \iint e^{i(y-y')\eta} a(y, y', \eta) v(y') \, dy' \, đ\eta,$$

for any (operator-valued) amplitude function $a(y, y', \eta)$, $đ\eta = (2\pi)^{-q} \, d\eta$. Similarly, we proceed with the boundary operators $T_\pm$. Setting

$$b_{\pm,j}(r, y, z, \eta) = \mathrm{r}'_\pm \, r^{-m_{\pm,j}} \sum_{k+|\beta|\le m_{\pm,j}} b_{\pm,j;k\beta}(r, y) z^k (r\eta)^\beta \tag{11}$$

we get $T_{\pm,j} = \operatorname{Op}_y(\operatorname{op}_M^\delta(b_{\pm,j}))$, $j = 1, \ldots, l_\pm$. It will also be interesting to consider operator-valued symbols

$$a(y, \eta) = \begin{pmatrix} \operatorname{op}_M^\delta(f)(y, \eta) \\ \big(\operatorname{op}_M^\delta(b_{\pm,j})(y, \eta)\big)_{j=1,\ldots,l_\pm} \end{pmatrix}, \tag{12}$$

i.e., $(y, \eta)$-dependent amplitude functions that take values in boundary value problems on the infinite stretched cone $\mathbb{R}_+ \times I$; then $\operatorname{Op}_y(a)$ represents the boundary value problem

$$Au = f \ \text{ in } \ \mathbb{R}_+ \times I \times \Omega, \qquad T_\pm u = g_\pm \ \text{ on } \ \mathbb{R}_+ \times \{\iota_\pm\} \times \Omega \tag{13}$$

where $T_\pm = (T_{\pm,j})_{j=1,\dots,l_\pm}$.

**1.2. Weighted cone and wedge Sobolev spaces.** Let us now introduce natural scales of Sobolev spaces on $\mathbb{R}_+ \times I$ and $\mathbb{R}_+ \times I \times \Omega$ (as well as on $\mathbb{R}_+$ and $\mathbb{R}_+ \times \Omega$, respectively). We shall often employ cut-off functions that are in this paper arbitrary elements $\omega \in C^\infty(\mathbb{R}_+)$ with $\omega(r) = 1$ for $0 \le r \le \varepsilon_0$, $\omega(r) = 0$ for $\varepsilon_1 \le r$ for certain $0 < \varepsilon_0 < \varepsilon_1$. Let $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$, and set

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+) = \{u(r) \in r^\gamma L^2(\mathbb{R}_+) : \left(r\frac{\partial}{\partial r}\right)^j u \in r^\gamma L^2(\mathbb{R}_+) \text{ for all } 0 \le j \le s\}. \quad (14)$$

Clearly, we then have $\mathcal{H}^{s,\gamma}(\mathbb{R}_+) = r^\gamma \mathcal{H}^{s,0}(\mathbb{R}_+)$. By duality with respect to the scalar product of $L^2(\mathbb{R}_+) = \mathcal{H}^{0,0}(\mathbb{R}_+)$ as a reference space we can introduce $\mathcal{H}^{s,0}(\mathbb{R}_+)$ for all integers $s$ and then define $\mathcal{H}^{s,0}(\mathbb{R}_+)$ for all real $s$ by interpolation. We then define $\mathcal{H}^{s,\gamma}(\mathbb{R}_+) = r^\gamma \mathcal{H}^{s,0}(\mathbb{R}_+)$ for all $s, \gamma \in \mathbb{R}$. Furthermore, we set

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+),\ v \in H^s(\mathbb{R}_+)\},$$

where $\omega$ is any cut-off function. Similarly, we can introduce the spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times I) := r^\gamma \mathcal{H}^{s,0}(\mathbb{R}_+ \times I)$, with $\mathcal{H}^{0,0}(\mathbb{R}_+ \times I) := r^{-\frac{1}{2}} L^2(\mathbb{R}_+ \times I)$ (where $L^2$ refers to $\mathrm{d}r\,\mathrm{d}\phi$) and for $s \in \mathbb{N}$

$$\mathcal{H}^{s,0}(\mathbb{R}_+ \times I) = \{u(r,\phi) \in r^{-\frac{1}{2}} L^2(\mathbb{R}_+ \times I) : (r\partial r)^j \partial_\phi^k u \in r^{-\frac{1}{2}} L^2(\mathbb{R}_+ \times I) \text{ for all } 0 \le j + k \le s\}.$$

To extend this definition to arbitrary $s \in \mathbb{R}$ we first set

$$(S_\beta u)(t,\phi) = e^{-(\frac{1}{2}-\beta)t} u(e^{-t}, \phi), \qquad \beta \in \mathbb{R},$$

which defines an isomorphism

$$S_\beta : C_0^\infty(\mathbb{R}_+ \times \mathrm{int}\ I) \longrightarrow C_0^\infty(\mathbb{R} \times \mathrm{int}\ I). \quad (15)$$

In particular, let $\beta = \gamma - \frac{1}{2}$; then $S_{\gamma-\frac{1}{2}}$ extends to an isomorphism

$$S_{\gamma-\frac{1}{2}} : \mathcal{H}^{s,\gamma}(\mathbb{R} \times I) \longrightarrow H^s(\mathbb{R} \times I) \quad (16)$$

where $H^s(\mathbb{R} \times I)$ is the Sobolev space of smoothness $s \in \mathbb{N}$ in the cylinder $\mathbb{R} \times I$, i.e.,

$$H^s(\mathbb{R}\times I) = \{v(t,\phi) \in L^2(\mathbb{R}\times I) : D_t^j \partial_\phi^k v \in L^2(\mathbb{R}\times I) \text{ for all } 0 \le j+k \le s\}.$$

The space $H^s(\mathbb{R} \times I)$ is completely standard also for arbitrary $s \in \mathbb{R}$, and we also have $H_0^s(\mathbb{R}_+ \times I), s \in \mathbb{R}$, the completion of $C_0^\infty(\mathbb{R}_+ \times \mathrm{int}\ I)$ in the $H^s(\mathbb{R} \times I)$-norm. Applying (15) and (16) we get corresponding versions of spaces on $\mathbb{R}_+ \times I$, i.e., $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times I)$ and $\mathcal{H}_0^{s,\gamma}(\mathbb{R}_+ \times I)$, the latter one being the completion of $C_0^\infty(\mathbb{R}_+ \times \mathrm{int}\ I)$ in the $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times I)$-norm. We have a non-degenerate sesquilinear pairing

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times I) \times \mathcal{H}_0^{-s,-\gamma}(\mathbb{R}_+ \times I) \longrightarrow \mathbb{C}$$

via the $\mathcal{H}^{0,0}(\mathbb{R}_+ \times I)$-scalar product, for all $s, \gamma \in \mathbb{R}$.

Let us set $I^\wedge = \mathbb{R}_+ \times I$ and introduce

$$\mathcal{K}^{s,\gamma}(I^\wedge) = \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(I^\wedge),\ v \in H^s(\mathbb{R}^2 \backslash \overline{\mathbb{R}}_+)\}, \quad (17)$$

$$\mathcal{K}_0^{s,\gamma}(I^\wedge) = \{\omega u + (1-\omega)v : u \in \mathcal{H}_0^{s,\gamma}(I^\wedge),\ v \in H_0^s(\mathbb{R}^2 \backslash \overline{\mathbb{R}}_+)\}. \quad (18)$$

Here,

$$H^s(\mathbb{R}^2 \backslash \overline{\mathbb{R}}_+) = \{u \in H^s_{\mathrm{loc}}(\mathbb{R}^2 \backslash \overline{\mathbb{R}}_+) : (1-\chi)u \in H^s(\mathbb{R}^2),\ \chi u|_{\mathbb{R}^2_\pm} \in H^s(\mathbb{R}^2_\pm)\}$$

for any $\chi \in C^\infty(\mathbb{R}^2)$ with supp $\chi \subset \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > -\varepsilon\}$ for some $\varepsilon > 0$, $\chi = 1$ in a neighbourhood of $\overline{\mathbb{R}}_+$ and $\chi(\lambda x_1, x_2) = \chi(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2 : x_1 > \varepsilon,\ \lambda \geq 1$. Moreover, as usual, $H^s(\mathbb{R}^2_\pm) = H^s(\mathbb{R}^2)|_{\mathbb{R}^2_\pm}$; $\mathbb{R}^2_\pm = \{x = (x_1, x_2) : x_1 \in \mathbb{R},\ x_2 \gtrless 0\}$ in a standard quotient norm topology, while $H_0^s(\mathbb{R}^2 \backslash \overline{\mathbb{R}}_+)$ is defined to be the closure of $C_0^\infty(\mathbb{R}^2 \backslash \overline{\mathbb{R}}_+)$ in $H^s(\mathbb{R}^2 \backslash \overline{\mathbb{R}}_+)$. Clearly, the spaces $H^s(\mathbb{R}^2 \backslash \overline{\mathbb{R}}_+)$ and $H_0^s(\mathbb{R}^2 \backslash \overline{\mathbb{R}}_+)$ are independent of the choice of $\chi$; also $\mathcal{K}^{s,\gamma}(I^\wedge)$ and $\mathcal{K}_0^{s,\gamma}(I^\wedge)$ are independent of the specific cut-off function $\omega$. The spaces $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ and $\mathcal{K}^{s,\gamma}(I^\wedge)$ are Banach spaces and can easily be equipped with Hilbert space scalar products that generate the norm. The $\mathcal{K}^{0,0}(I^\wedge)$-scalar product from $r^{-\frac{1}{2}}L^2(\mathbb{R}_+ \times I) = \mathcal{H}^{0,0}(I^\wedge)$ gives rise to non-degenerate sesqulinear pairings

$$\mathcal{K}^{s,\gamma}(I^\wedge) \times \mathcal{K}_0^{-s,-\gamma}(I^\wedge) \longrightarrow \mathbb{C}$$

for all $s, \gamma \in \mathbb{R}$. We have canonical continuous embeddings $\mathcal{K}^{s',\gamma'}(I^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}(I^\wedge)$ for arbitrary $s \geq s', \gamma' \geq \gamma$ (analogous embeddings hold for the corresponding spaces on $\mathbb{R}_+$). Similar properties hold for the spaces $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$. In particular, the non-degenerate sesquilinear pairing $\mathcal{K}^{s,\gamma}(\mathbb{R}_+) \times \mathcal{K}^{-s,-\gamma}(\mathbb{R}_+) \to \mathbb{C}$ refers to the scalar product of $\mathcal{K}^{0,0}(\mathbb{R}_+) = L^2(\mathbb{R}_+)$.

*Remark* 1.1. Setting $(\kappa_\lambda^{(0)} u)(r) = \lambda^{\frac{1}{2}} u(\lambda r)$ for $u \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$, $(\kappa_\lambda^{(1)} v)(r, \phi) = \lambda v(\lambda r, \phi)$ for $v \in \mathcal{K}^{s,\gamma}(I^\wedge)$ or $v \in \mathcal{K}_0^{s,\gamma}(I^\wedge)$, $\lambda \in \mathbb{R}_+$, we get groups $\{\kappa_\lambda^{(0)}\}_{\lambda \in \mathbb{R}_+}$ and $\{\kappa_\lambda^{(1)}\}_{\lambda \in \mathbb{R}_+}$ of isomorphisms on the respective spaces, strongly continuous with respect to $\lambda \in \mathbb{R}_+$.

To analyse our crack problem we mainly have to consider a neighbourhood of $r = 0$, though, freezing of coefficients gives us operators on the infinite cones $I^\wedge = \mathbb{R}_+ \times I$ and $\mathbb{R}_+ \times \{\iota_\pm\}$. To unify descriptions we assume that the coefficients $a_{k\beta}(r, y)$ in (10) and $b_{\pm,j;k\beta}$ in (11) are independent of $r$ for $r > R$ for some $R > 0$. Then (10) gives rise to a $(y, \eta)$-dependent family of continuous operators

$$\mathrm{op}_M^{\gamma - \frac{1}{2}}(f)(y, \eta) : \mathcal{K}^{s,\gamma}(I^\wedge, \mathbb{C}^N) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(I^\wedge, \mathbb{C}^N) \quad (19)$$

for all $s \in \mathbb{R}$ (and all $\gamma \in \mathbb{R}$). The same is true of $\mathrm{op}_M^{\gamma-\frac{1}{2}}(f_0)(y,\eta)$ when we set

$$f_0(r,y,z,\eta) = r^{-m} \sum_{k+|\beta|\leq m} a_{k\beta}(0,y) z^k (r\eta)^\beta. \tag{20}$$

Then we have

$$\mathrm{op}_M^{\gamma-\frac{1}{2}}(f_0)(y,\lambda\eta) = \lambda^m \kappa_\lambda^{(1)} \, \mathrm{op}_M^{\gamma-\frac{1}{2}}(f_0)(y,\eta)(\kappa_\lambda^{(1)})^{-1} \tag{21}$$

for all $\lambda \in \mathbb{R}_+$ and all $(y,\eta) \in \Omega \times \mathbb{R}^q$. Similar relations hold for the boundary operators. In fact, we get families of continuous operators

$$\mathrm{op}_M^{\gamma-\frac{1}{2}}(b_{\pm,j})(y,\eta) : \mathcal{K}^{s,\gamma}(I^\wedge, \mathbb{C}^N) \longrightarrow \mathcal{K}^{s-m_{\pm,j}-\frac{1}{2},\gamma-m_{\pm,j}-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{M_{\pm,j}})_\pm \tag{22}$$

for $s > m_{\pm,j} + \frac{1}{2}$ for all $j$. Subscript $\pm$ denotes copies of the corresponding spaces defined on $\mathbb{R}_+ \times \Omega$ belonging to $\{\iota_\pm\}$. Setting

$$b_{0;\pm,j}(r,y,z,\eta) = \mathrm{r}'_\pm \, r^{-m_{\pm,j}} \sum_{k+|\beta|\leq m_{\pm,j}} b_{\pm,j;k\beta}(0,y) z^k (r\eta)^\beta, \tag{23}$$

also $\mathrm{op}_M^{\gamma-\frac{1}{2}}(b_{0;\pm,j})(y,\eta)$ is continuous in that sense and we have the homogeneity

$$\mathrm{op}_M^{\gamma-\frac{1}{2}}(b_{0;\pm,j})(y,\lambda\eta) = \lambda^{m_{\pm,j}+\frac{1}{2}} \kappa_\lambda^{(0)} \, \mathrm{op}_M^{\gamma-\frac{1}{2}}(b_{0;\pm,j})(y,\eta)(\kappa_\lambda^{(1)})^{-1} \tag{24}$$

for all $\lambda \in \mathbb{R}_+$ and all $(y,\eta) \in \Omega \times \mathbb{R}^q$. The operator families (19) and (22) are symbols in the following sense.

**Defintion 1.2.** Let $E$ be a Hilbert space equipped with a strongly continuous group of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, $\kappa_\lambda : E \to E$. Then if $\tilde{E}$ is another Hilbert space with such a $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, the symbol space $S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$ for $U \subseteq \mathbb{R}^p$ open, $\mu \in \mathbb{R}$, is defined to be the set of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q; E, \tilde{E})$ satisfying the symbol estimates

$$||\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D_y^\alpha D_\eta^\beta a(y,\eta)\}\kappa_{\langle\eta\rangle}||_{\mathcal{L}(E,\tilde{E})} \leq c\langle\eta\rangle^{\mu-|\beta|}$$

for all $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$ and all $y \in K$ for arbitrary $K \Subset U$, and all $\eta \in \mathbb{R}^q$, with constants $c = c(\alpha,\beta,K) > 0$. Moreover, let $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \tilde{E})$ denote the subspace of all classical symbols $a(y,\eta)$, i.e., there are elements $a_{(\mu-j)}(y,\eta) \in C^\infty(U \times (\mathbb{R}^q\backslash 0), \mathcal{L}(E,\tilde{E}))$, $j \in \mathbb{N}$, with $a_{(\mu-j)}(y,\lambda\eta) = \lambda^{\mu-j}\tilde{\kappa}_\lambda a_{(\mu-j)}(y,\eta)\kappa_\lambda^{-1}$ for all $\lambda \in \mathbb{R}_+$ and $(y,\eta) \in U \times (\mathbb{R}^q\backslash 0)$, such that $a - \chi\sum_{j=0}^N a_{(\mu-j)} \in S^{\mu-(N+1)}(U \times (\mathbb{R}^q\backslash 0); E, \tilde{E})$ for any excision function $\chi(\eta)$ in $\mathbb{R}^q$ and every $N \in \mathbb{N}$.

Usually, we have $U = \Omega$ or $U = \Omega \times \Omega$ for open $\Omega \subseteq \mathbb{R}^q$ (in the latter case we denote the variables by $(y,y')$). For $a(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}^q; E, \tilde{E})$ we set

$$\sigma_\wedge(a)(y,\eta) = a_{(\mu)}(y,y',\eta)|_{y'=y},$$

called the homogeneous principal symbol of $a$ of order $\mu$.

In the present case we have $E = \mathcal{K}^{s,\gamma}(I^\wedge, \mathbb{C}^N)$ with $\kappa_\lambda = \kappa_\lambda^{(1)}$, $\lambda \in \mathbb{R}_+$, while $\tilde{E}$ is either $\mathcal{K}^{s-\mu,\gamma-\mu}(I^\wedge, \mathbb{C}^N)$ with $\kappa_\lambda = \kappa_\lambda^{(1)}$, $\lambda \in \mathbb{R}_+$ or one of the spaces
$\mathcal{K}^{s-m_{\pm,j}-\frac{1}{2},\gamma-m_{\pm,j}-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{M_{\pm,j}})_\pm$ with $\tilde{\kappa}_\lambda = \kappa_\lambda^{(0)}, \lambda \in \mathbb{R}_+$.

**Proposition 1.3.** *If $a(y,\eta)$ is operator family of the form* (12) *associated with our boundary value problem in the described way, we have*

$$\mathrm{op}_M^{\gamma-\frac{1}{2}}(f)(y,\eta) \in S^m(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(I^\wedge, \mathbb{C}^N), \mathcal{K}^{s-m,\gamma-m}(I^\wedge, \mathbb{C}^N))$$

*for all $s \in \mathbb{R}$, and*

$$\begin{aligned}\mathrm{op}_M^{\gamma-\frac{1}{2}}(b_{\pm,j})(y,\eta) \in S^{m_{\pm,j}+\frac{1}{2}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(I^\wedge, \mathbb{C}^N),\\ \mathcal{K}^{s-m_{\pm,j}-\frac{1}{2},\gamma-m_{\pm,j}-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{M_{\pm,j}})_\pm)\end{aligned}$$

*for all $s > m_{\pm,j}+\frac{1}{2}$ and all $j$. Moreover, $\mathrm{op}_M^{\gamma-\frac{1}{2}}(f_0)(y,\eta)$ and $\mathrm{op}_M^{\gamma-\frac{1}{2}}(b_{0;\pm,j})(y,\eta)$ are classical symbols of the corresponding orders (with respect to the same spaces), and they equal their own homogeneous principal parts for $\eta \neq 0$, cf. relations* (21) *and* (24)*, respectively.*

*Proof.* Let us first consider $\mathrm{op}_M^{\gamma-\frac{1}{2}}(f)$. To prove the result it suffices to consider the summands of (10) separately. Thus without loss of generality we may set

$$f(r,y,z,\eta) = r^{-m} a(r,y) z^k (r\eta)^\beta$$

for an $a(r,y) \in C^\infty(\overline{\mathbb{R}} \times \Omega, \mathrm{Diff}_{N\times N}^{m-(k+|\beta|)}(I))$. For simplicity, consider the case $N = 1$; the general case is completely analogous. Since $a(r,y)$ is a finite sum of expressions of the form $\varphi(r,\phi,y) D_\phi^l$ for certain $\varphi(r,\phi,y) \in C^\infty(\overline{\mathbb{R}}_+ \times I \times \Omega)$ and $0 \leq l \leq m-(k+|\beta|)$, it is suffices to set $a(r,y) = \varphi(r,\phi,y) D_\phi^l$. By a well-known result on a projective tensor products, here for $C^\infty(\overline{\mathbb{R}}_+ \times I \times \Omega) = C^\infty(\overline{\mathbb{R}}_+) \hat{\otimes}_\pi C^\infty(I \times \Omega)$, we can write $\varphi$ as a $\varphi(r,\phi,y) = \sum_{j=0}^\infty \lambda_j \varphi_j(r) \alpha_j(y,\phi)$ for sequences $\lambda_j \in \mathbb{C}, \sum_{j=0}^\infty |\lambda_j| < \infty$, and $\varphi_j \in C^\infty(\overline{\mathbb{R}}_+), \alpha_j \in C^\infty(I \times \Omega)$ tending to zero for $j \to \infty$ in the respective spaces. This reduces the assertion to

$$f(r,y,z,\eta) = \sum_{j=0}^\infty \lambda_j M_{\varphi_j} f_j(r,y,z,\eta) \tag{25}$$

with $f_j(r,y,z,\eta) = r^{-m} \alpha_j(y,\phi) D_\phi^j z^k (r\eta)^\beta$ and $M_{\varphi_j}$ being the operator of multiplication by $\varphi_j$. Now we have

$$\mathrm{op}_M^{\gamma-\frac{1}{2}}(f_j)(y,\eta) \in S_{\mathrm{cl}}^m(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(I^\wedge), \mathcal{K}^{s-m,\gamma-m}(I^\wedge))$$

(because it is even $\kappa_\lambda^{(1)}$-homogeneous) and we have $\mathrm{op}_M^{\gamma-\frac{1}{2}}(f_j)(y,\eta) \to 0$ for $j \to \infty$ in that symbol space which is a consequence of $\alpha_j \to 0$ in

$C^\infty(I \times \Omega)$. We now employ the assumption that the original coefficients $a(r,y)$ are independent of $r$ for $r > R$ for some $R > 0$. Then our $f(r,y,z,\eta)$ is a sum of an expression with constant coefficients and someone with compactly supported coefficients with respect to $r$. The first summand is $\kappa_\lambda^{(1)}$-homogeneous and as such a classical operator-valued symbol. Therefore, we may assume compact support in the coefficients with respect to $r$. That means, the above tensor product argument may be formulated with respect to the space $C^\infty([0,R]_0) := \{\varphi \in C^\infty(\overline{\mathbb{R}}_+) : \varphi(r) = 0 \text{ for } r \geq R\}$ which is also a Fréchet space. This gives us a sequence $\varphi_j \in C^\infty([0,R]_0)$ tending to zero in that space. The operator $M_{\varphi_j}$ then is an element of $S^0(\mathbb{R}^q_\eta; \mathcal{K}^{s,\gamma}(I^\wedge), \mathcal{K}^{s,\gamma}(I^\wedge))$ for each $s, \gamma \in \mathbb{R}$, and we have $M_{\varphi_j} \to 0$ in that space for $j \to \infty$ (details on such relations may be found in [33]). It follows that

$$M_{\varphi_j} \operatorname{op}_M^{\gamma-\frac{1}{2}}(f_j)(y,\eta) \in S^m(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(I^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(I^\wedge))$$

and

$$\operatorname{op}_M^{\gamma-\frac{1}{2}}(f)(y,\eta) = \sum_{j=0}^{\infty} \lambda_j M_{\varphi_j} \operatorname{op}_M^{\gamma-\frac{1}{2}}(f_j)(y,\eta)$$

with convergence in this symbol space. To prove the assertion for

$$\operatorname{op}_M^{\gamma-\frac{1}{2}}(b_{\pm,j})(y,\eta)$$

we can employ analogous arguments, combined with the observation that the restriction operators

$$\mathrm{r}'_\pm : \mathcal{K}^{s,\gamma}(I^\wedge) \longrightarrow \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+)_\pm$$

are homogeneous of order $\frac{1}{2}$ in the sense

$$\mathrm{r}'_\pm = \lambda^{\frac{1}{2}} \kappa_\lambda^{(0)} \, \mathrm{r}'_\pm (\kappa_\lambda^{(1)})^{-1}, \quad \lambda \in \mathbb{R}_+$$

and as such belong to $S^{\frac{1}{2}}_{\mathrm{cl}}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(I^\wedge), \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+)_\pm)$ for all $s > \frac{1}{2}$. $\square$

The form of the operator valued symbols in Proposition 1.3 suggests a simpler generalisation of Definition 1.2 to Douglis-Nirenberg (DN- )orders. In this connection we start with direct sums of Hilbert spaces $\boldsymbol{E} = \bigoplus_{j=1}^k E_j$ and $\tilde{\boldsymbol{E}} = \bigoplus_{l=1}^m \tilde{E}_l$ with

$$\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+} := \operatorname{diag}(\{\kappa_{\lambda,j}\}_{\lambda \in \mathbb{R}_+}), \{\tilde{\kappa}_\lambda\}_{\lambda \in \mathbb{R}_+} := \operatorname{diag}(\{\tilde{\kappa}_{\lambda,l}\}_{\lambda \in \mathbb{R}_+}),$$

with strongly continuous group actions $\{\kappa_{\lambda,j}\}_{\lambda \in \mathbb{R}_+}$ and $\{\tilde{\kappa}_{\lambda,l}\}_{\lambda \in \mathbb{R}_+}$ on $E_j$ and $\tilde{E}_l$, respectively. We then have the symbol spaces $S^{\mu_{lj}}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E_j, \tilde{E}_l)$ in the sense of Definition 1.2.

The boundary value problems (13) will be considered in weighted Sobolev spaces that can be subsumed under the following general definition:

**Defintion 1.4.** Let $E$ be a Hilbert space equipped with a group $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : E \to E$, strongly continuous in $\lambda \in \mathbb{R}_+$. Then $\mathcal{W}^s(\mathbb{R}^q, E)$ for $s \in \mathbb{R}$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm

$$\left\{ \int \langle\eta\rangle^{2s} ||\kappa^{-1}_{\langle\eta\rangle} \hat{u}(\eta)||^2_E \, \mathrm{d}\eta \right\}^{\frac{1}{2}} .$$

Here, $\langle\eta\rangle = (1 + |\eta|^2)^{\frac{1}{2}}$, and $\hat{u}(\eta) = (F_{y\to\eta} u)(\eta)$ is the Fourier transform in $\mathbb{R}^q$.

This definition directly extends to the case of Fréchet spaces $E = \operatorname{proj}\lim\{E^j : j \in \mathbb{N}\}$, where $(E^j)_{j\in\mathbb{N}}$ is a sequence of Hilbert spaces with continuous embeddings $E^{j+1} \hookrightarrow E^j$ for all $j$, where we assume that $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, first given on $E^0$, restricts to strongly continuous groups of isomorphisms on $E^j$ for all $j$. (Whenever a Fréchet space $E$ can be written as such a projective limit with $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ being given in the described way we will say that $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is a group action on $E$.) Then we get continuous embeddings $\mathcal{W}^s(\mathbb{R}^q, E^{j+1}) \hookrightarrow \mathcal{W}^s(\mathbb{R}^q, E^j)$ for all $j$, and we set

$$\mathcal{W}^s(\mathbb{R}^q, E) = \operatorname{proj}\lim\{\mathcal{W}^s(\mathbb{R}^q, E^j) : j \in \mathbb{N}\}. \tag{26}$$

Moreover, if $\Omega \subseteq \mathbb{R}^q$ is an open set, we have natural analogues of classical "comp" and "loc" Sobolev spaces on $\Omega$, namely

$$\mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \text{ and } \mathcal{W}^s_{\mathrm{loc}}(\Omega, E) \tag{27}$$

respectively, cf. [33]. Clearly, the spaces $\mathcal{W}^s(\mathbb{R}^q, E)$ depend on the choice of $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ (cf. also the following remark), but it is fixed and known in every concrete case.

*Remark* 1.5. The spaces $\mathcal{W}^s(\mathbb{R}^q, E)$ for any given $E$ together with a fixed choice of $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ are also called abstract wedge Sobolev spaces. For $\kappa_\lambda = \mathrm{id}_E$, $\lambda \in \mathbb{R}_+$, we get the standard Sobolev spaces of $E$-valued distributions, denoted by $H^s(\mathbb{R}^q, E)$. There is then a natural isomorphism

$$T = F^{-1}\kappa^{-1}_{\langle\eta\rangle} F : \mathcal{W}^s(\mathbb{R}^q, E) \longrightarrow H^s(\mathbb{R}^q, E) \tag{28}$$

for every $s$. This will be used below for an efficient description of singular functions of the edge asymptotics.

Let us set

$$\mathcal{W}^{s,\gamma}(I^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(I^\wedge)), \tag{29}$$

$$\mathcal{W}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(\mathbb{R}_+)), \tag{30}$$

where we employ $\kappa^{(1)}_\lambda$ and $\kappa^{(0)}_\lambda$, respectively, cf. Remark 1.1. The corresponding "comp" and "loc" versions are denoted by $\mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(I^\wedge \times \Omega)$, $\mathcal{W}^{s,\gamma}_{\mathrm{loc}(y)}(I^\wedge \times \Omega)$, etc.

Parallel to the abstract wedge Sobolev spaces we have the above-mentioned operator-valued symbols $a(y,\eta)$ that refer to a pair of spaces with strongly continuous groups of isomorphisms $(E,\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+})$, $(\tilde{E},\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+})$. The associated pseudo- differential operators $\mathrm{Op}(a):=\mathrm{Op}_y(a)$ with respect to $y\in\Omega$ (based on the Fourier transform in $y$) are then continuous between the corresponding spaces.

Applying this to the components of (12) we can write (13) as such a pseudo-differential operator $\mathcal{A}=\mathrm{Op}(a)$, i.e., (13) represents a continuous operator

$$\mathcal{A}:\mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(I^\wedge\times\Omega,\mathbb{C}^N)\longrightarrow\begin{array}{c}\mathcal{W}^{s-m,\gamma-m}_{\mathrm{loc}(y)}(I^\wedge\times\Omega,\mathbb{C}^N)\\ \oplus\\ \oplus_{j=0}^{l_-}\mathcal{W}^{s-m_{-,j}-\frac{1}{2},\gamma-m_{-,j}-\frac{1}{2}}_{\mathrm{loc}(y)}(\mathbb{R}_+\times\Omega,\mathbb{C}^{M_{-,j}})_-\\ \oplus\\ \oplus_{j=0}^{l_+}\mathcal{W}^{s-m_{+,j}-\frac{1}{2},\gamma-m_{+,j}-\frac{1}{2}}_{\mathrm{loc}(y)}(\mathbb{R}_+\times\Omega,\mathbb{C}^{M_{+,j}})_+\end{array}\quad(31)$$

for all $s\in\mathbb{R}$ with $s>\max\{m_{\pm,j}\}+\frac{1}{2}$. The edge spaces on the $\pm$- boundary components $\mathbb{R}_+\times\Omega$ refer to $\mathcal{K}^{s-m_{\pm,j}-\frac{1}{2},\gamma-m_{\pm,j}-\frac{1}{2}}(\mathbb{R}_+)_\pm$ that are nothing else than the corresponding spaces on $\mathbb{R}_+$. Clearly, in (31) all operators are local, such that we may write "comp$(y)$" and "loc$(y)$" on both sides. Under corresponding assumptions on ellipticity the discussion of the solvability of (13) will include the following aspects:

(i) the parametrix construction for (31) within a "wedge pseudo-differential calculus",

(ii) the characterisation of asymptotics of solutions near the boundary of the crack in suitable subspaces with asymptotics,

(iii) the description of (if necessary) additional trace and potential conditions along the boundary of the crack, satisfying an analogue of the Shapiro-Lopatinskij condition (this concerns the case $q=\dim\Omega\geq 1$).

Passing from the local representation on $\mathbb{R}_+\times I\times\Omega$ to the original operators on $G\backslash S$ we then obtain the Fredholm property (under the assumption of compactness of $\overline{G}$ and suitable Shapiro-Lopatinskij elliptic boundary conditions also on $\partial G$). In addition we get the invertibility within our operator spaces (in the case of unique solvability).

**1.3. Calculus with operator-valued symbols.** Definition 1.2 gives rise to a general pseudo-differential calculus with operator-valued symbols, cf. [33], [27]. Here, we only give some basic notation and results for future references. Let $\Omega\subset\mathbb{R}^q$ be open, and set

$$L^\mu_{(\mathrm{cl})}(\Omega;E,\tilde{E})=$$
$$=\{\mathrm{Op}(a)+C:a(y,y',\eta)\in S^\mu_{(\mathrm{cl})}(\Omega\times\Omega\times\mathbb{R}^q;E,\tilde{E}),C\in L^{-\infty}(\Omega;E,\tilde{E})\},\quad(32)$$

(subscript (cl) is used for classical or non-classical elements), where $L^{-\infty}(\Omega;E,\tilde{E})$ is the space of all integral operators on $\Omega$ with kernels in $C^\infty(\Omega\times$

$\Omega, \mathcal{L}(E, \tilde{E}))$. Notation (32) concerns the case that $E$ and $\tilde{E}$ are Hilbert spaces with strongly continuous groups of isomorphisms as well as the case of Fréchet spaces. Every $A \in L^{\mu}(\Omega; E, \tilde{E})$ induces a continuous operator

$$A : C_0^{\infty}(\Omega, E) \longrightarrow C^{\infty}(\Omega, \tilde{E})$$

that extends to a continuous operator

$$A : \mathcal{W}^{s}_{\mathrm{comp}(y)}(\Omega, E) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}(y)}(\Omega, \tilde{E}) \tag{33}$$

for every $s \in \mathbb{R}$.

Note that there is a useful variant of the calculus globally in $\mathbb{R}^q$ with weighted symbols and weighted Sobolev spaces at infinity.

**Defintion 1.6.** Let $\mu, \delta \in \mathbb{R}$ and define $S^{\mu;\delta}(\mathbb{R}^q \times \mathbb{R}^q; E, \tilde{E})$ to be the set of all $a(y, \eta) \in C^{\infty}(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}))$ such that

$$||\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^{\alpha}_{y} D^{\beta}_{\eta} a(y,\eta)\}\kappa_{\langle\eta\rangle}||_{\mathcal{L}(E,\tilde{E})} \leq c\langle\eta\rangle^{\mu-|\beta|}\langle y\rangle^{\delta-|\alpha|} \tag{34}$$

for all $\alpha, \beta \in \mathbb{N}^q$, $y, \eta \in \mathbb{R}^q$, with constants $c = c(\alpha, \beta) > 0$.

There is also a subspace $S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^q \times \mathbb{R}^q; E, \tilde{E})$ of symbols that are classical both in $\eta$ and $y$, cf. Kapanadze and Schulze [14], where we have a triple of homogeneous principal symbols

$$\sigma(a) = (\sigma_{\partial}(a), \sigma_{\mathrm{e}'}(a), \sigma_{\partial,\mathrm{e}'}(a)). \tag{35}$$

Here, $\sigma_{\partial}(a)(y, \eta)$ for $(y, \eta) \in \mathbb{R}^q \times (\mathbb{R}^q \setminus 0)$ is the homogeneous principal symbol of $a(y, \eta) \in S^{\mu;\delta}_{\mathrm{cl}_{\eta;y}}(\mathbb{R}^q \times \mathbb{R}^q; E, \tilde{E})$ of order $\mu$ in $\eta \neq 0$, $\sigma_{\mathrm{e}'}(a)(y, \eta)$ for $(y, \eta) \in (\mathbb{R}^q \setminus 0) \times \mathbb{R}^q$ the homogeneous principal symbol of order $\delta$ in $y \neq 0$, and $\sigma_{\partial,\mathrm{e}'}(a)(y, \eta)$ for $(y, \eta) \in (\mathbb{R}^q \setminus 0) \times (\mathbb{R}^q \setminus 0)$ the homogeneous principal part of $\sigma_{\partial}(a)(y, \eta)$ in $\eta$ of order $\mu$ (that equals the homogeneous principal part of $\sigma_{\mathrm{e}'}(a)(y, \eta)$ in $y$ of order $\delta$). Let us set

$$L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E}) = \{\mathrm{Op}(a) : a(y, y', \eta) \in S^{\mu;\delta}_{(\mathrm{cl}_{\eta;y})}(\mathbb{R}^q \times \mathbb{R}^q; E, \tilde{E})\}.$$

The weighted Sobolev spaces are defined as

$$\mathcal{W}^{s;\varrho}(\mathbb{R}^q, E) = \langle y\rangle^{-\varrho}\mathcal{W}^{s}(\mathbb{R}^q, E)$$

$s \in \mathbb{R}$, with $\varrho \in \mathbb{R}$ being a weight at infinity, and

$$||u||_{\mathcal{W}^{s;\varrho}(\mathbb{R}^q,E)} := ||\langle y\rangle^{\varrho} u||_{\mathcal{W}^{s}(\mathbb{R}^q,E)}.$$

We then have

$$\mathcal{S}(\mathbb{R}^q, E) = \operatorname{proj}\lim\{\mathcal{W}^{N;N}(\mathbb{R}^q, E) : N \in \mathbb{N}\}.$$

**Theorem 1.7.** *Every* $A \in L^{\mu;\delta}(\mathbb{R}^q; E, E)$ *induces a continuous operator*

$$A : \mathcal{S}(\mathbb{R}^q, E) \longrightarrow \mathcal{S}(\mathbb{R}^q, \tilde{E}),$$

*that extends to a continuous operator*

$$A : \mathcal{W}^{s;\varrho}(\mathbb{R}^q, E) \longrightarrow \mathcal{W}^{s-\mu;\varrho-\delta}(\mathbb{R}^q, \tilde{E})$$

*for every* $s, \varrho \in \mathbb{R}$.

*Remark* 1.8. The map $\mathrm{Op} : a(y, \eta) \to \mathrm{Op}(a)$ induces an isomorphism

$$\mathrm{Op} : S^{\mu;\delta}_{(\mathrm{cl}_{\eta;y})}(\mathbb{R}^q \times \mathbb{R}^q; E, \tilde{E}) \longrightarrow L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E})$$

for every $\mu, \delta \in \mathbb{R}$.

*Remark* 1.9. In the case $E = \mathbb{C}^N$ (or $\tilde{E} = \mathbb{C}^{\tilde{N}}$) the corresponding strongly continuous groups of isomorphisms are chosen to be $\kappa_\lambda = \mathrm{id}_{\mathbb{C}^N}$ (or $\tilde{\kappa}_\lambda = \mathrm{id}_{\mathbb{C}^{\tilde{N}}}$) for all $\lambda \in \mathbb{R}_+$. In particular, for $N = \tilde{N} = 1$ the pseudo-differential calculi (classical or non-classical ones) specialise to the corresponding scalar theories. In particular, we have the spaces $S^{\mu;\delta}_{(\mathrm{cl}_{\eta;y})}(\mathbb{R}^q \times \mathbb{R}^q)$ and $L^{\mu;\delta}_{(\mathrm{cl})}(\mathbb{R}^q)$ of scalar symbols and operators, respectively, with exit behaviour, of order $\mu$ and weight $\delta$.

**1.4. Local pseudo-differential boundary value problems.** In this section we prepare some necessary material on (classical) pseudo-differential boundary value problems with the transmission property.

Consider $\Omega \times \mathbb{R} \ni x = (y, t)$ for an open set $\Omega \subseteq \mathbb{R}^{n-1}$, and let $\xi = (\eta, \tau)$ be the covariables of $x$. Define $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_{\mathrm{tr}}$ for $\mu \in \mathbb{Z}$ to be the subspace of all $a(x, \xi) \in S^\mu_{\mathrm{cl}}(\Omega_y \times \mathbb{R}_t \times \mathbb{R}^n_\xi)$ such that

$$D^k_t D^\alpha_\eta \{a_{(\mu-j)}(y, t, \eta, \tau) - (-1)^{\mu-j} a_{(\mu-j)}(y, t, -\eta, -\tau)\} = 0 \qquad (36)$$

on the set $\{(x, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n : y \in \Omega, t = 0, \eta = 0, \tau \in \mathbb{R}\backslash 0\}$, for all $k \in \mathbb{N}, \alpha \in \mathbb{N}^{n-1}$ and all $j \in \mathbb{N}$. There are (by notation) the symbols with the transmission property with respect to $t = 0$. Moreover, set $S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)_{\mathrm{tr}} = \{a = \tilde{a} \mid_{\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n} : \tilde{a}(x, \xi) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_{\mathrm{tr}}\}$. With any symbol in the latter space we associate a pseudo-differential operator in the "half-space" $\Omega \times \overline{\mathbb{R}}_+$ by

$$\mathrm{Op}^+(a)u(x) = \mathrm{r}^+ \, \mathrm{Op}(\tilde{a}) \, \mathrm{e}^+ \, u(x),$$

where $\mathrm{e}^+$ is the operator of extension by zero from $\Omega \times \mathbb{R}_+$ to $\Omega \times \mathbb{R}$ and $\mathrm{r}^+$ the operator of restriction from $\Omega \times \mathbb{R}$ to $\Omega \times \mathbb{R}_+$, and $a = \tilde{a}|_{\Omega \times \mathbb{R}_+ \times \mathbb{R}^n}$. Similarly, with $a(y, t, \eta, \tau)$ we can associate the family of operators

$$\mathrm{op}^+(a)(y, \eta) = \mathrm{r}^+ \, \mathrm{op}(\tilde{a})(y, \eta) \, \mathrm{e}^+$$

on $\mathbb{R}_+$, where $\mathrm{op}(a)(y,\eta)u(x) = \iint e^{i(t-t')\tau} a(y,t,\tau,\eta)u(t')\,dt'\,đ\tau$, $(y,\eta) \in \Omega \times \mathbb{R}^{n-1}$. Recall that the transmission property entails the continuity of the mappings

$$\mathrm{Op}^+(a) : C_0^\infty(\Omega \times \overline{\mathbb{R}}_+) \longrightarrow C^\infty(\Omega \times \overline{\mathbb{R}}_+),$$
$$\mathrm{op}^+(a)(y,\eta) : C_0^\infty(\overline{\mathbb{R}}_+) \longrightarrow C^\infty(\overline{\mathbb{R}}_+),$$

as well as continuity between Sobolev spaces of smoothness $s > -\frac{1}{2}$. That means (say, for the case that $a(y,t,\eta,\tau)$ is independent of $t$ for $t >$const for some constant $> 0$)

$$\mathrm{op}^+(a)(y,\eta) : H^s(\mathbb{R}_+) \longrightarrow H^{s-\mu}(\mathbb{R}_+) \tag{37}$$

is continuous for all $s > -\frac{1}{2}$ (and similarly for $\mathrm{Op}^+(a)$ on $\Omega \times \mathbb{R}_+$).

Boundary value problems will be generated locally in the half-space as pseudo-differential operators with operator-valued symbols. The symbols have their values in a space of block matrix valued operators $D^{\mu,d}(\overline{\mathbb{R}}_+; N_-, N_+)$, $(\mu,d) \in \mathbb{Z} \times \mathbb{N}$, $N_-, N_+ \in \mathbb{N}$, that constitute the boundary symbol calculus, to be defined below. First, let $\Gamma^d(\overline{\mathbb{R}}_+; N_-, N_+)$ denote the space of all operators

$$g = g_0 + \sum_{j=1}^{d} g_j \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix} \tag{38}$$

for arbitrary $g_j \in \mathcal{L}(L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_-}, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_+})$ with $g_j^* \in \mathcal{L}(L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+}, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_-})$, $\mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$ (and $*$ indicating the adjoint with respect to the $L^2(\mathbb{R}_+) \oplus \mathbb{C}^{N_\pm}$-scalar products).

Let $S_{\mathrm{cl}}^\mu(\mathbb{R})_{\mathrm{tr}}$ be the specialisation (for $n = 1$) of the above-mentioned space of symbols $a(\tau)$ with the transmission property (i.e., with constant coefficients). Define $D^{\mu,d}(\overline{\mathbb{R}}_+; N_-, N_+)$ to be the space of all operators of the form

$$\boldsymbol{a} = \begin{pmatrix} \mathrm{op}^+(a) & 0 \\ 0 & 0 \end{pmatrix} + g : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

for arbitrary $a \in S_{\mathrm{cl}}^\mu(\mathbb{R})_{\mathrm{tr}}$, $g \in \Gamma^d(\overline{\mathbb{R}}_+; N_-, N_+)$.

We will be interested, in fact, in $m \times k$-systems of operators. Let $\mathcal{R}_G^{\mu,0}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+)$ denote the space of so- called Green symbols $g(y,\eta)$ of order $\mu$ and type zero, defined by the properties

$$g(y,\eta) \in S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{N_-}, \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^m) \oplus \mathbb{C}^{N_+}), \tag{39}$$
$$g^*(y,\eta) \in S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+, \mathbb{C}^m) \oplus \mathbb{C}^{N_+}, \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^k) \oplus \mathbb{C}^{N_-}). \tag{40}$$

(Operator-valued symbols in this connection refer to the group actions $\mathrm{diag}(\kappa_\lambda \otimes \mathrm{id}_{\mathbb{C}^k}, \mathrm{id}_{\mathbb{C}^{N_-}})$ and $\mathrm{diag}(\kappa_\lambda \otimes \mathrm{id}_{\mathbb{C}^m}, \mathrm{id}_{\mathbb{C}^{N_+}})$, respectively, $(\kappa_\lambda u)(t) =$

$\lambda^{\frac{1}{2}}u(\lambda t)$, $\lambda \in \mathbb{R}_+$). In other words, $g(y,\eta)$ belongs $(y,\eta)$-wise to the $m \times k$-matrix-valued analogue (with respect to upper left corners) of $\Gamma^0(\overline{\mathbb{R}}_+; N_-, N_+)$, defined before. Moreover, we introduce the space $\mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+)$ of Green symbols of order $\mu$ and type $d$ to be the set of all

$$g(y,\eta) = g_0(y,\eta) + \sum_{j=1}^{d} g_j(y,\eta) \begin{pmatrix} \partial_t^j & 0 \\ 0 & 0 \end{pmatrix} \tag{41}$$

for arbitrary $g_j(y,\eta) \in \mathcal{R}_G^{\mu-j,0}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+)$.

The space $\mathcal{R}^{\mu,d}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+)$ of boundary symbols of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}$ is defined to be the set of all operator families

$$\boldsymbol{a}(y,\eta) = \begin{pmatrix} \mathrm{op}^+(a)(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + g(y,\eta) \tag{42}$$

for arbitrary $a(y,t,\eta,\tau) \in S_{\mathrm{cl}}^{\mu}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)_{\mathrm{tr}} \otimes \mathbb{C}^m \otimes \mathbb{C}^k$, independent of $t$ for $t > c$ for some $c > 0$, and $g(y,\eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+)$.

We then have

$$\mathcal{R}^{\mu,d}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+) \subseteq S^{\mu}(\Omega \times \mathbb{R}^{n-1}; E, \tilde{E}) \tag{43}$$

for

$$E = H^s(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{N_-}, \quad \tilde{E} = H^{s-\mu}(\mathbb{R}_+, \mathbb{C}^m) \oplus \mathbb{C}^{N_+},$$

$s > d - \frac{1}{2}$, with $\kappa_\lambda = \mathrm{diag}(\kappa_\lambda^{(0)}, \mathrm{id}_{\mathbb{C}^{N_\pm}})$ and $\kappa_\lambda^{(0)}$ acting on the first components by the rule $(\kappa_\lambda^{(0)}u)(t) = \lambda^{\frac{1}{2}}u(\lambda t)$, $\lambda \in \mathbb{R}_+$, cf. Remark 1.1. Using properties of the type $\mathcal{W}^s(\mathbb{R}^{n-1}, H^s(\mathbb{R}_+)) = H^s(\mathbb{R}^n_+)$ $(= H^s(\mathbb{R}^n)|_{\mathbb{R}^n_+})$, relation (43) entails the continuity

$$\mathrm{Op}(a) : \begin{matrix} H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_+, \mathbb{C}^k) \\ \oplus \\ H^s_{\mathrm{comp}}(\Omega, \mathbb{C}^{N_-}) \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}_+, \mathbb{C}^m) \\ \oplus \\ H^s_{\mathrm{loc}}(\Omega, \mathbb{C}^{N_+}) \end{matrix} \tag{44}$$

for all $s > d - \frac{1}{2}$. Here, $H^s_{\mathrm{comp}\,/\,\mathrm{loc}}(\Omega, \mathbb{C}^N)$ is understood as usual. Moreover,

$$H^s_{\mathrm{comp}(y)/\,\mathrm{loc}(y)}(\Omega \times \mathbb{R}_+, \mathbb{C}^l) := H^s_{\mathrm{comp}(y)/\,\mathrm{loc}(y)}(\Omega \times \mathbb{R}, \mathbb{C}^l)|_{\Omega \times \mathbb{R}_+},$$

where $H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_+, \mathbb{C}^l)$ is the subspace of all $u \in H^s(\mathbb{R}^n, \mathbb{C}^l)$ such that there is a compact subset $K = K(u) \subset \Omega$ with $\mathrm{supp}\, u \subset K \times \mathbb{R}$, while $H^s_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}, \mathbb{C}^l)$ is the subspace of all $u \in \mathcal{D}'(\Omega \times \mathbb{R}, \mathbb{C}^l)$ with $\varphi u \in H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}, \mathbb{C}^l)$ for every $\varphi \in C_0^\infty(\Omega)$.

Next we describe the smoothing operators in the space of all pseudo-differential boundary value problems on $\Omega \times \mathbb{R}$, first of type 0. These constitute the space $\mathcal{B}^{-\infty,0}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w})$, $\boldsymbol{w} := (k, m; N_-, N_+)$, of all operators

$$\mathcal{C} : \begin{matrix} C_0^\infty(\Omega \times \overline{\mathbb{R}}_+, \mathbb{C}^k) \\ \oplus \\ C_0^\infty(\Omega, \mathbb{C}^{N_-}) \end{matrix} \longrightarrow \begin{matrix} C^\infty(\Omega \times \overline{\mathbb{R}}_+, \mathbb{C}^m) \\ \oplus \\ C^\infty(\Omega, \mathbb{C}^{N_+}) \end{matrix}$$

that have $C^\infty$–kernels. More precisely, we have (by definition) $\mathcal{C}=(C_{ij})_{i,j=1,2}$, where $(C_{11}u)(x)=\int_{\Omega\times\mathbb{R}_+} c_{11}(x,x')u(x')\,\mathrm{d}x'$ for a $c_{11}(x,x')\in C^\infty(\Omega\times\overline{\mathbb{R}}_+\times\Omega\times\overline{\mathbb{R}}_+)\otimes\mathbb{C}^m\otimes\mathbb{C}^k$, $(C_{12}v)(x)=\int_\Omega c_{12}(x,y)v(y)\,\mathrm{d}y$ for a $c_{12}(x,y)\in C^\infty(\Omega\times\overline{\mathbb{R}}_+\times\Omega)\otimes\mathbb{C}^{N_-}\otimes\mathbb{C}^m$, etc. Moreover, the space $\mathcal{B}^{-\infty,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})$ of smoothing operators of the type $d$ is defined to be the set of all operators of the form

$$\mathcal{C}=\mathcal{C}_0+\sum_{j=1}^{d}\mathcal{C}_j\begin{pmatrix}\partial_t^j & 0\\ 0 & 0\end{pmatrix}$$

for arbitrary $\mathcal{C}_j\in\mathcal{B}^{-\infty,0}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w}),\ j=0,\dots,d.$

**Defintion 1.10.** The space $\mathcal{B}^{\mu,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})$ for $(\mu,d)\in\mathbb{Z}\times\mathbb{N}$ and dimension data $\boldsymbol{w}=(k,m;N_-,N_+)$ is defined to be the set of all operators

$$\mathcal{A}=\mathrm{Op}(\boldsymbol{a})+\mathcal{P}+\mathcal{C}\tag{45}$$

for arbitrary $\boldsymbol{a}(y,\eta)\in\mathcal{R}^{\mu,d}(\Omega\times\mathbb{R}^{n-1};\boldsymbol{w})$, $\mathcal{P}=\begin{pmatrix}P & 0\\ 0 & 0\end{pmatrix}$ for any $\mathcal{P}\in L^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}_+)\otimes\mathbb{C}^m\otimes\mathbb{C}^k$ with $\varphi P\psi=0$ for all $\varphi,\psi\in C^\infty(\overline{\mathbb{R}}_+)$ with $\varphi=\psi=0$ for $t>\varepsilon$ for some $\varepsilon>0$, and $\mathcal{C}\in\mathcal{B}^{-\infty,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})$

The elements $\mathcal{A}$ of $\mathcal{B}^{\mu,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})$ are called pseudo-differential boundary value problems in the "half-space" $\Omega\times\overline{\mathbb{R}}_+$. The principal symbol structure of operators $\mathcal{A}=(A_{ij})_{i,j=1,2}$ consists of two components, namely $\sigma_\psi(\mathcal{A})(x,\xi):=\sigma_\psi(A_{11})(x,\xi)$, the homogeneous principal interior symbol of order $\mu$, $(x,\xi)\in T^*(\Omega\times\overline{\mathbb{R}}_+)\backslash 0$ and $\sigma_\partial(\mathcal{A})(y,\eta)$, the homogeneous principal boundary symbol of order $\mu$ that is given by

$$\sigma_\partial(\mathcal{A})(y,\eta)=\begin{pmatrix}\mathrm{op}^+(a|_{t=0})(y,\eta) & 0\\ 0 & 0\end{pmatrix}+\sigma_\partial(g)(y,\eta),$$

cf. (40), $(y,\eta)\in T^*\Omega\backslash 0$, where $\sigma_\partial(g)(y,\eta)$ is the homogeneous principal symbol of $g(y,\eta)$ as a classical operator- valued symbol. We then have

$$\sigma_\partial(\mathcal{A})(y,\lambda\eta)=\lambda^\mu\begin{pmatrix}\kappa_\lambda & 0\\ 0 & 1\end{pmatrix}\sigma_\partial(\mathcal{A})(y,\eta)\begin{pmatrix}\kappa_\lambda & 0\\ 0 & 1\end{pmatrix}^{-1}\tag{46}$$

for all $\lambda\in\mathbb{R}_+,\ (y,\eta)\in T^*\Omega\backslash 0$ (clearly, the identities in the block matrices in (46) refer to the involved dimensions $N_-,N_+$, etc.).

*Remark* 1.11. $\mathcal{A}\in\mathcal{B}^{\mu,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})$ and $\sigma_\psi(\mathcal{A})=0$, $\sigma_\partial(\mathcal{A})=0$ imply $\mathcal{A}\in\mathcal{B}^{\mu-1,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})$.

*Remark* 1.12. Given arbitrary $\varphi_1\in C^\infty(\Omega\times\overline{\mathbb{R}}_+),\varphi_2\in C^\infty(\Omega)$ we set $\varphi=(\varphi_1,\varphi_2)$ and define $M_\varphi$ to be the operator of multiplication by $\mathrm{diag}(\varphi_1\otimes\mathrm{id}_{\mathbb{C}^N},\varphi_2\otimes\mathrm{id}_{C^l})$ for suitable dimensions $N$ and $l$ (that will be clear by the context). Then $\mathcal{A}\in\mathcal{B}^{\mu,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})$ implies $M_\varphi\mathcal{A},\mathcal{A}M_\varphi\in\mathcal{B}^{\mu,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})$.

**Theorem 1.13.** *$\mathcal{A} \in \mathcal{B}^{\mu,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{v})$, $\boldsymbol{v} = (l,m;\tilde{N},N_+)$, and $\mathcal{B} \in \mathcal{B}^{\nu,e}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})$, $\boldsymbol{w} = (k,l;N_-,\tilde{N})$, implies $\mathcal{A}M_\varphi\mathcal{B} \in \mathcal{B}^{\mu+\nu,h}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{v}\circ\boldsymbol{w})$ for every $\varphi = (\varphi_1,\varphi_2) \in C_0^\infty(\Omega\times\overline{\mathbb{R}}_+)\times C_0^\infty(\Omega)$, where $h = \max(\nu+d,e)$, $\boldsymbol{v}\circ\boldsymbol{w} = (k,m;N_-,N_+)$, and we have*

$$\sigma_\psi(\mathcal{A}M_\varphi\mathcal{B}) = \sigma_\psi(\mathcal{A}M_\varphi)\sigma_\psi(\mathcal{B}), \quad \sigma_\partial(\mathcal{A}M_\varphi\mathcal{B}) = \sigma_\partial(\mathcal{A}M_\varphi)\sigma_\partial(\mathcal{B}).$$

For reasons that become clear below in the edge pseudo-differential calculus we want to slightly modify the order conventions in our block-matrices $\mathcal{A} = (A_{ij})_{i,j=1,2}$. As we saw in the beginnig it is natural to accept a shift of orders by $\frac{1}{2}$ in the operators $A_{12}$ and $A_{21}$ and also different orders from the original boundary conditions). Theorem 1.13 allows us to reduce orders as follows. Instead of $\mathcal{B}^{\mu,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})$ for $\boldsymbol{w} = (k,m;N_-,N_+)$ we take the space

$$\boldsymbol{\mathcal{B}}^{\mu,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w}) = \\ = \{\mathcal{R}_1\mathcal{A}\mathcal{R}_2^{-1} + \mathcal{C} : \mathcal{A} \in \mathcal{B}^{\mu,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w}),\ \mathcal{C}\in\mathcal{B}^{-\infty,d}(\Omega\times\overline{\mathbb{R}}_+;\boldsymbol{w})\} \quad (47)$$

for $\mathcal{R}_1 = \mathrm{diag}(\mathrm{id}_{\mathbb{C}^m}, R\otimes\mathrm{id}_{\mathbb{C}^{N_+}})$, $\mathcal{R}_2 = \mathrm{diag}(\mathrm{id}_{\mathbb{C}^k}, R\otimes\mathrm{id}_{\mathbb{C}^{N_-}})$ with $R \in L_{\mathrm{cl}}^{\frac{1}{2}}(\Omega)$ being any properly supported elliptic pseudo-differential operator of order $\frac{1}{2}$ on $\Omega$. Clearly, (47) is independent of the specific choice of the order reducing operator $R$. The space (47) could also be defined directly by using a corresponding space of amplitude functions that is defined in terms of a corresponding "DN-analogue" $\boldsymbol{\mathcal{R}}_G^{\mu,d}(\Omega\times\mathbb{R}^{n-1};\boldsymbol{w})$ instead of $\mathcal{R}_G^{\mu,d}(\Omega\times\mathbb{R}^{n-1};\boldsymbol{w})$. Here, "DN" stands for Douglis-Nirenberg orders, where in this case for $g = (g_{ij})_{i,j=1,2} \in \boldsymbol{\mathcal{R}}_G^{\mu,d}(\Omega\times\mathbb{R}^{n-1};\boldsymbol{w})$ we have

$$\mathrm{ord}\ g_{11} = \mathrm{ord}\ g_{22} = \mu,\ \mathrm{ord}\ g_{12} = \mu - \frac{1}{2}, \mathrm{ord}\ g_{21} = \mu + \frac{1}{2}$$

with orders being interpreted in the operator-valued sense (39) (that to be applied for entries separately). It is easy to formulate DN-orders in general, not only for the boundary components of the block matrices but also for the systems of operators in the upper left corners. We mainly content ourselves with DN-orders for the boundary operators. The explicit orders in the operators (6) suggest to admit operators of the form

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix}$$

where, according to the former notation, $A := A_{11}$, $K := A_{12}$, $T := A_{21}$, $Q := A_{22}$, and

$$T = \begin{pmatrix} T_1 \\ \vdots \\ T_{N_+} \end{pmatrix}, \quad K = (K_1,\ldots,K_{N_-}), \quad Q = (Q_{ij})_{\substack{i=1,\ldots,N_- \\ j=1,\ldots,N_+}}$$

with tuples of reals $\beta = (\beta_1, \dots, \beta_{N_-}), \gamma = (\gamma_1, \dots, \gamma_{N_+})$ and

$$\text{ord } T_j = \mu + \gamma_j + \frac{1}{2}, \text{ ord } K_i = \mu - \beta_i - \frac{1}{2}, \text{ ord } Q_{ij} = \mu + \gamma_j - \beta_i$$

for $i = 1, \dots, N_-$, $j = 1, \dots, N_+$. Then, analogously to (47), we could introduce a space $\boldsymbol{\mathcal{B}}^{\mu,d}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w})_{(\gamma,\beta)}$, where (47) corresponds to $\beta = (0, \dots, 0)$ and $\gamma = (0, \dots, 0)$. The classes $\boldsymbol{\mathcal{B}}^{\mu,d}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w})_{(\gamma,\beta)}$ and $\boldsymbol{\mathcal{B}}^{\mu,d}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w})_{(\tilde{\gamma},\tilde{\beta})}$ for different choices of $(\gamma, \beta)$, $(\tilde{\gamma}, \tilde{\beta})$ can be transformed to each other by corresponding reductions of orders on the boundary $\Omega$. It may be advisable in concrete applications to formulate the results for the true orders from the problem rather than the reduced ones (cf. Section 3.2 below); nevertheless, to give a transparant description of the main ideas, in the general calculus we refer to the case (47), i.e., $\beta_i = \gamma_j = 0$ for all $i, j$.

The spaces $\boldsymbol{\mathcal{B}}^{\mu,d}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w})$ are also needed in a parameter-dependent version, where the amplitude functions as well as the smoothing operators contain parameters $\lambda \in \mathbb{R}^l$ that are treated as additional covariables. First we have the symbol class $S^{\mu}_{\text{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^l)_{\text{tr}}$ defined by relations analogous to (36), where $(\eta, \tau)$ is to be replaced by $(\eta, \tau, \lambda) \in \mathbb{R}^{n+l}$. This gives rise to $S^{\mu}_{\text{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^l)_{\text{tr}}$, and instead of $\text{op}^+(a)(y, \eta)$ we now have the operator families $\text{op}^+(a)(y, \eta, \lambda)$ for $a(y, t, \eta, \tau, \lambda) \in S^{\mu}_{\text{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n+l})_{\text{tr}}$. Moreover, there is a straightforward generalisation of the spaces of operator-valued symbols, cf. Definition 1.2, namely $S^{\mu}_{(\text{cl})}(\Omega \times \mathbb{R}^q; E, \tilde{E}; \mathbb{R}^l) := S^{\mu}_{(\text{cl})}(\Omega \times \mathbb{R}^{q+l}; E, \tilde{E})$. This gives us a corresponding parameter-dependent analogue $\boldsymbol{\mathcal{R}}^{\mu,d}_G(\Omega \times \mathbb{R}^{n-1}; \boldsymbol{w}; \mathbb{R}^l)$ of the Green symbols (in DN-orders) and, similarly to (42), the space of parameter-dependent amplitude functions $\boldsymbol{\mathcal{R}}^{\mu,d}(\Omega \times \mathbb{R}^{n-1}; \boldsymbol{w}; \mathbb{R}^l)$, $\boldsymbol{w} = (k, m; N_-, N_+)$. To define parameter-dependent boundary value problems we also need to introduce parameter-dependent smoothing operators. These are nothing else than

$$\mathcal{B}^{-\infty,d}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w}; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, \mathcal{B}^{-\infty,d}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w})).$$

Here, $\mathcal{B}^{-\infty,d}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w})$ is used in its canonical Fréchet topology. Finally, we have the spaces $L^{\mu}_{\text{cl}}(\Omega \times \mathbb{R}_+; \mathbb{R}^l)$, $L^{\mu}_{\text{cl}}(\Omega; \mathbb{R}^l)$ of parameter–dependent pseudo–differential operators on $\Omega \times \mathbb{R}_+$ and $\Omega$, respectively (cf., e.g. [25]). In particular, there is a standard notion of parameter-dependent ellipticity.

Summing up we have introduced all ingredients to define $\mathcal{B}^{\mu,d}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w}; \mathbb{R}^l)$ analogously to Definition 1.10. Then $\boldsymbol{\mathcal{B}}^{\mu,d}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w}; \mathbb{R}^l)$, is defined similarly to (47) where we employ a parameter-dependent elliptic reduction of orders $R(\lambda) \in L^{\frac{1}{2}}_{\text{cl}}(\Omega; \mathbb{R}^l)$.

**1.5. Global operators on manifolds with boundary.** Crack problems in our theory localise outside the crack boundary to standard boundary value problems on certain (non-compact) manifolds with boundary. In the present case $M_{\text{crack}} := (\overline{G} \setminus S) \cup \text{int } S_- \cup \text{int } S_+$ is such a manifold with smooth boundary components $\partial G$, int $S_-$ and int $S$, cf. the notation of the introduction. For references below we now give a brief definition of global pseudo-differential boundary value problems on a smooth (not necessarily compact) manifold $M$ with boundary $\partial M$. Note that the case $M = \Omega \times \overline{\mathbb{R}}_+$ for open $\Omega \subseteq \mathbb{R}^{n-1}$ is also included; thus the assertions that we formulate for $M$ in general are also valid for the local situation of the preceding section. Let Vect(.) denote the set of smooth complex vector bundles on the space in the brackets. Let $E, F \in \mathrm{Vect}(M), J^-, J^+ \in \mathrm{Vect}(\partial M)$, and set $\boldsymbol{v} = (E, F; J^-, J^+)$. We then have the space $\mathcal{B}^{-\infty,0}(M; \boldsymbol{v})$ of all smoothing operators of type 0

$$\begin{array}{ccc} C_0^\infty(M, E) & & C^\infty(M, F) \\ \oplus & \longrightarrow & \oplus \\ C_0^\infty(\partial M, J^-) & & C^\infty(\partial M, J^+) \end{array} \tag{48}$$

that are given by corresponding $C^\infty$ kernels, smooth up to boundary (in the corresponding variables on $M$). Integrations refer to Riemannian metrics on $M$ and $\partial M$ that we keep fixed in the sequel, further to Hermitian metrics in the occurring vector bundles. Assume that the Riemannian metric on $M$ induces the product metric of $(\partial M) \times [0, 1)$ in a collar neighbourhood of $\partial M$. Incidentally we employ $2M$, the double of $M$, obtained by gluing together two copies of $M$ along $\partial M$ by an identification diffeomorphism. On $M$ we have the space $\mathrm{Diff}^j(M; E, F)$ of all differential operators of order $j$ acting between sections in the bundles $E$ and $F$. Then $\mathcal{B}^{-\infty,d}(M; \boldsymbol{v})$, the space of all smoothing operators on $M$ of type $d \in \mathbb{N}$, is defined to be the set of all

$$\mathcal{G} = \mathcal{G}_0 + \sum_{j=1}^{d} \mathcal{G}_j \begin{pmatrix} D^j & 0 \\ 0 & 0 \end{pmatrix}$$

for arbitrary $\mathcal{G}_0, \ldots, \mathcal{G}_d \in \mathcal{B}^{-\infty,0}(M; \boldsymbol{v})$ and $D^j \in \mathrm{Diff}^j(M; E, F)$. To introduce the space of Green operators on $M$ we employ the space $\mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+)$ of Green symbols of order $\mu$ type $d$, cf. Section 1.4. Here, $k, m, N_-$ and $N_+$ are the fibre dimensions of $E, F, J^-$ and $J^+$, respectively. Now $\mathcal{B}_G^{\mu,d}(M; \boldsymbol{v})$ is defined to be the set of all operators of the form $\mathcal{G}_0 + \mathcal{C}$ for arbitrary $\mathcal{C} \in \mathcal{B}^{-\infty,d}(M; \boldsymbol{v})$ and operators $\mathcal{G}_0$ that are concentrated in a collar neighbourhood of $\partial M$ and are locally finite sums of operators of the form $\mathrm{Op}(g)$ for certain $g(y, \eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^{n-1}; k, m; N_-, N_+)$. The pull- backs refer to charts $U \to \Omega \times \overline{\mathbb{R}}_+$ for coordinate patches $U$ near $\partial M$ and trivialisations of the involved bundles; "$\mathcal{G}_0$ concentrated near $\partial M$" means that for certain functions $\varphi, \psi \in C^\infty(M)$ that equal 1 in a collar neighbourhood of $\partial M$ and 0 outside another collar neighbourhood of $\partial M$ we have $\mathcal{G}_0 = M_\varphi \mathcal{G}_0 M_\psi$.

Finally, let $L_{\mathrm{cl}}^\mu(2M; \tilde{E}, \tilde{F})_{\mathrm{tr}}$ for $\tilde{E}, \tilde{F} \in \mathrm{Vect}(2M)$ denote the subspace of all $\tilde{A} \in L_{\mathrm{cl}}^\mu(2M; \tilde{E}, \tilde{F})$ (classical "in $\xi$-variables") pseudo-differential operators on $2M$ of order $\mu$ that have the transmission property with respect to $\partial M$, acting between sections of the bundles $\tilde{E}, \tilde{F}$. We employ the standard Sobolev spaces $H_{\mathrm{comp}}^s(M, E), H_{\mathrm{loc}}^s(M, E)$ of smoothness $s \in \mathbb{R}$ for bundles $E \in \mathrm{Vect}(M)$. "comp" and "loc" are understood in the sense

$H^s_{\mathrm{comp}}(M,E) = H^s_{\mathrm{comp}}(2M,\tilde{E})|_M$, $H^s_{\mathrm{loc}}(M,E) = H^s_{\mathrm{loc}}(2M,\tilde{E})|_M$ for any $\tilde{E} \in \mathrm{Vect}(2M)$ with $E = \tilde{E}|_M$. For every $\tilde{A} \in L^\mu_{\mathrm{cl}}(2M;\tilde{E},\tilde{F})_{\mathrm{tr}}$ and $E = \tilde{E}|_M, F = \tilde{F}|_M$ we can form $\mathrm{r}^+ \tilde{A}\,\mathrm{e}^+$, where $\mathrm{e}^+$ is the extension by zero from $\mathrm{int}\, M$ to $2M$ and $\mathrm{r}^+$ the restriction from $2M$ to $\mathrm{int}\, M$; this gives us continuous operators

$$\mathrm{r}^+ \tilde{A}\,\mathrm{e}^+ : H^s_{\mathrm{comp}}(M,E) \longrightarrow H^{s-\mu}_{\mathrm{loc}}(M,F)$$

for all $s > -\frac{1}{2}$.

**Defintion 1.14.** The space $\mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ for $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, $\boldsymbol{v} = (E,F;J^-,J^+)$, is defined to be the set of all operators

$$\mathcal{A} = \begin{pmatrix} \mathrm{r}^+ \tilde{A}\,\mathrm{e}^+ & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G} \tag{49}$$

for arbitrary $\tilde{A} \in L^\mu_{\mathrm{cl}}(2M;\tilde{E},\tilde{F})_{\mathrm{tr}}$ and $\mathcal{G} \in \mathcal{B}^{\mu,d}_G(M;\boldsymbol{v})$.

Let us now give an account of important properties of the space $\mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ of pseudo-differential boundary value problems of order $\mu$ and type $d$, with the transmission property. Concerning more details, cf. Rempel and Schulze [17], or Schulze [27], Chapter 4.

*Remark* 1.15. The space $\mathcal{B}^{\mu,d}(M,\boldsymbol{v})$ is Frechet in a natural semi-norm system.

**Theorem 1.16.** *Every $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ induces continuous operators*

$$\mathcal{A} : \begin{array}{c} H^s_{\mathrm{comp}}(M,E) \\ \oplus \\ H^s_{\mathrm{comp}}(\partial M,J^-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}_{\mathrm{loc}}(M,F) \\ \oplus \\ H^{s-\mu}_{\mathrm{loc}}(\partial M,J^+) \end{array} \tag{50}$$

*for all $s > d - \frac{1}{2}$ (this, in particular, implies the continuity between $C^\infty$ sections, cf.* (48)*). If $M$ is compact, we get continuous operators*

$$\mathcal{A} : \begin{array}{c} H^s(M,E) \\ \oplus \\ H^s(\partial M,J^-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(M,F) \\ \oplus \\ H^{s-\mu}(\partial M,J^+). \end{array} \tag{51}$$

The principal symbol structure of $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ consists of a pair

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})),$$

where $\sigma_\psi(\mathcal{A})$, the homogeneous principal interior symbol of order $\mu$, is a bundle homomorphism

$$\sigma_\psi(\mathcal{A}) := \sigma_\psi(\tilde{A})|_{T^*M\setminus 0} : \pi^*_\psi E \longrightarrow \pi^*_\psi F, \tag{52}$$

for $\pi_\psi : T^*M\backslash 0 \longrightarrow M$, and $\sigma_\partial(\mathcal{A})$, the homogeneous principal boundary symbol of order $\mu$, a bundle homomorphism

$$\sigma_\partial(\mathcal{A}) : \pi_\partial^* \begin{pmatrix} E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi_\partial^* \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J^+ \end{pmatrix} \tag{53}$$

for $\pi_\partial : T^*(\partial M)\backslash 0 \to \partial M$, and $E' = E|_{\partial M}$, $F' = F|_{\partial M}$. Alternatively, $\sigma_\partial(\mathcal{A})$ may be regarded as a homomorphism

$$\sigma_\partial(\mathcal{A}) : \pi_\partial^* \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi_\partial^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J^+ \end{pmatrix} \tag{54}$$

for all $s > d - \frac{1}{2}$. Setting symb $\mathcal{B}^{\mu,d}(M;\boldsymbol{v}) = \{\sigma(\mathcal{A}) : \mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})\}$ there is a map

$$\text{op} : \text{symb}\ \mathcal{B}^{\mu,d}(M;\boldsymbol{v}) \longrightarrow \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$$

with $\sigma \circ \text{op} = \text{id}$ on the symbol space. We have $\sigma(\mathcal{A}) = 0 \Rightarrow \mathcal{A} \in \mathcal{B}^{\mu-1,d}(M;\boldsymbol{v})$; if $M$ is compact, the operator (51) is compact when its symbol vanishes.

**Theorem 1.17.** *Let $M$ be compact; then $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v}), \boldsymbol{v} = (E_0, F; J_0, J^+)$, and $\mathcal{B} \in \mathcal{B}^{\nu,e}(M;\boldsymbol{w}), \boldsymbol{w} = (E, E_0; J^-, J_0)$, implies $\mathcal{AB} \in \mathcal{B}^{\mu+\nu,h}(M; \boldsymbol{v} \circ \boldsymbol{w})$, for $h = \max(\nu + d, e)$, $\boldsymbol{v} \circ \boldsymbol{w} = (E, F; J^-, J^+)$, and we have $\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ (with componentwise multiplication). An analogous result holds for general $M$ when we replace the composition by $\mathcal{A}M_\varphi\mathcal{B}$ for a compactly supported $\varphi \in C^\infty(M)$ where $\sigma(\mathcal{A}M_\varphi\mathcal{B}) = \sigma(\mathcal{A}M_\varphi)\sigma(\mathcal{B})$.*

An operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ is called elliptic, if both (52), and (53) are isomorphisms (the second condition is equivalent to the bijectivity of (54) for all $s > \max(\mu, d) - \frac{1}{2}$). Set $\nu^+ = \max(\nu, 0)$ for any $\nu \in \mathbb{R}$.

**Theorem 1.18.** *Let $M$ be compact. Then the following conditions are equivalent:*
(i) *$\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ is elliptic,*
(ii) *the operator (51) is Fredholm for some $s = s_0 > \max(\mu, d) - \frac{1}{2}$.*
*If $\mathcal{A}$ is elliptic, then (51) is a Fredholm operator for all $s > \max(\mu, d) - \frac{1}{2}$, and there is a parametrix $\mathcal{P} \in \mathcal{B}^{-\mu,(d-\mu)^+}(M;\boldsymbol{v}^{-1})$ of $\mathcal{A}$ in the sense*

$$\mathcal{PA} - \mathcal{I} \in \mathcal{B}^{-\infty,d_l}(M;\boldsymbol{v}_l), \quad \mathcal{AP} - \mathcal{I} \in \mathcal{B}^{-\infty,d_r}(M;\boldsymbol{v}_r) \tag{55}$$

*for $d_l = \max(\mu, d)$, $\boldsymbol{v}_l = (E, E; J^-, J^-)$, $d_r = (d - \mu)^+$, $\boldsymbol{v}_r = (F, F; J^+, J^+)$.*

*Remark* 1.19. Ellipticity of $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ for non-compact $M$ entails the existence of a parametrix $\mathcal{P} \in \mathcal{B}^{-\mu,(d-\mu)^+}(M;\boldsymbol{v}^{-1})$, where (55) is to be replaced by

$$M_\psi\mathcal{P}M_\varphi\mathcal{A} - M_\varphi \in \mathcal{B}^{-\infty,d_l}(M;\boldsymbol{v}_l), \quad M_\varphi\mathcal{A}M_\psi\mathcal{P} - M_\varphi \in \mathcal{B}^{-\infty,d_r}(M;\boldsymbol{v}_r)$$

for arbitrary $\varphi, \psi \in C_0^\infty(M)$ with $\varphi\psi = \varphi$ (and $M_\varphi, M_\psi$ being the multiplication operators, containing evident tensor products with identity maps in the respective vector bundles).

Next let $M$ be a smooth manifold with smooth boundary, not necessarily compact. There is then a direct parameter-dependent analogue of the class of pseudo-differential boundary value problems $\mathcal{B}^{\mu,d}(M;\boldsymbol{v})$, cf. Definition 1.14, namely

$$\mathcal{B}^{\mu,d}(M;\boldsymbol{v};\mathbb{R}^l). \tag{56}$$

To define (56) we simply have to replace the ingredients of (49) by the corresponding parameter-dependent versions $\mathrm{r}^+ \tilde{A}(\lambda)\, \mathrm{e}^+$ and $\mathcal{G}(\lambda)$, respectively. Here, $\tilde{A}(\lambda) \in L^\mu_{\mathrm{cl}}(2M;\tilde{E},\tilde{F};\mathbb{R}^l)_{\mathrm{tr}}$ with obvious meaning of notation (recall that "cl″ here only means "classical" in the covariables, though $M$ may be non-compact) and $\mathcal{G}(\lambda)$ is an element of $\mathcal{B}^{\mu,d}_G(M;\boldsymbol{v};\mathbb{R}^l)$, also being defined along the lines of the class without parameters (all symbols simply contain $\lambda$ as an extra covariable, i.e., $(\xi,\lambda)$ instead of $\xi$ in the interior and $(\eta,\lambda)$ instead of $\eta$ near the boundary), and the parameter- dependent smoothing operators are given by

$$\mathcal{B}^{-\infty,d}(M;\boldsymbol{v};\mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, \mathcal{B}^{-\infty,d}(M;\boldsymbol{v})), \tag{57}$$

where $\mathcal{B}^{-\infty,d}(M;\boldsymbol{v})$ is equipped with its standard Fréchet topology, cf. Remark 1.15.

For $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v};\mathbb{R}^l)$ we have parameter-dependent homogeneous principal symbols, namely

$$\sigma_\psi(\mathcal{A}) : \pi_\psi^* E \longrightarrow \pi_\psi^* F, \quad \pi_\psi : (T^*M \times \mathbb{R}^l)\setminus 0 \longrightarrow M, \tag{58}$$

$$\sigma_\partial(\mathcal{A}) : \pi_\partial^* \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi_\partial^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J^+ \end{pmatrix}, \tag{59}$$

with $\pi_\partial : (T^*(\partial M) \times \mathbb{R}^l)\setminus 0 \to \partial M$, for $s$ sufficiently large as above. $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v};\mathbb{R}^l)$ implies $\mathcal{A}(\lambda_0) \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v})$ for every fixed $\lambda_0 \in \mathbb{R}^l$, and we call (58), (59) the parameter-dependent principal symbols of $\mathcal{A}(\lambda)$, if we want to distinguish them from the usual ones of $\mathcal{A}(\lambda_0)$ that are independent of $\lambda_0$.

An element $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v};\mathbb{R}^l)$ is called parameter- dependent elliptic if (58), (59) are isomorphisms.

**Theorem 1.20.** *Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v};\mathbb{R}^l)$ be parameter-dependent elliptic. Then there is a parameter-dependent parametrix $\mathcal{P} \in \mathcal{B}^{-\infty,(d-\mu)^+}(M;\boldsymbol{v}^{-1};\mathbb{R}^l)$ in a similar sense as in Remark* 1.19*; here, the remainders are smoothing in the sense of* (57).

**Theorem 1.21.** *Let $M$ be a compact smooth manifold with boundary, and let $\mathcal{A} \in \mathcal{B}^{\mu,d}(M;\boldsymbol{v};\mathbb{R}^l)$ be parameter-dependent elliptic. Then there is a $C > 0$ such that*

$$\mathcal{A}(\lambda) : \begin{matrix} H^s(M,E) \\ \oplus \\ H^s(\partial M, J^-) \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}(M,F) \\ \oplus \\ H^{s-\mu}(\partial M, J^+) \end{matrix}$$

*are isomorphisms for all $|\lambda| \geq C$ and all $s > \max(\mu, d) - \frac{1}{2}$.*

Theorem 1.21 is a direct consequence of Theorem 1.20.

*Remark* 1.22. Under the conditions of Theorem 1.21 we can easily conclude that the inverse maps belong $\lambda$-wise to the corresponding algebras in the non- parameter-dependent sense (as such they are reductions of orders in the algebras). It suffices to observe that when $1 + \{\text{smoothing operator}\}$ is invertible, the inverse is of analogous structure and can be composed with the parametrix. This can even be done in the parameter-dependent framework for large $|\lambda|$, such that, in fact, the inverses for large $|\lambda|$ are also in the corresponding parameter-dependendent class.

All definitions and results of this section have a natural analogue for the case of DN-orders that we again indicate by bold face letters. In other words, we have the spaces

$$\boldsymbol{\mathcal{B}}^{\mu,d}(M;\boldsymbol{v};\mathbb{R}^l),$$

especially $\boldsymbol{\mathcal{B}}^{\mu,d}(M;\boldsymbol{v})$ for the case $l = 0$. Instead of (50) for $\mathcal{A}(\lambda) \in \boldsymbol{\mathcal{B}}^{\mu,d}(M;\boldsymbol{v};\mathbb{R}^l)$ we get continuous operators

$$\mathcal{A}(\lambda) : \begin{matrix} H^s_{\text{comp}}(M,E) \\ \oplus \\ H^{s-\frac{1}{2}}_{\text{comp}}(\partial M, J^-) \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}_{\text{loc}}(M,F) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}_{\text{loc}}(\partial M, J^+) \end{matrix} \tag{60}$$

for all $s > d - \frac{1}{2}$, $\lambda \in \mathbb{R}^l$ (if $M$ is compact, the subscripts "comp" and "loc" are superfluous). Concerning the homogeneity of boundary symbols we refer to DN-orders. In contrast to (46) we now have

$$\sigma_\partial(\mathcal{A})(y, \tau\eta, \tau\lambda) = \tau^\mu \begin{pmatrix} \kappa_\tau & 0 \\ 0 & \tau^{\frac{1}{2}} \end{pmatrix} \sigma_\partial(\mathcal{A})(y, \eta, \lambda) \begin{pmatrix} \kappa_\tau & 0 \\ 0 & \tau^{\frac{1}{2}} \end{pmatrix}^{-1} \tag{61}$$

for all $\tau \in \mathbb{R}_+, (\eta, \lambda) \neq 0$.

*Remark* 1.23. The DN-analogue of Remark 1.19 gives us elliptic regularity of solutions to elliptic boundary value problems in the following sense. Let $\mathcal{A} \in \boldsymbol{\mathcal{B}}^{\mu,d}(M;\boldsymbol{v})$ be elliptic. Then

$$\mathcal{A}u \in H^{s-\mu}_{\text{loc}}(M,F) \oplus H^{s-\mu-\frac{1}{2}}_{\text{loc}}(\partial M, J^+)$$

for $s > \max(\mu, d) - \frac{1}{2}$ and $u \in H^r_{\text{loc}}(M,E) \oplus H^{r-\frac{1}{2}}_{\text{loc}}(\partial M, J^+)$ for any $r > \max(\mu, d) - \frac{1}{2}$ implies $u \in H^s_{\text{loc}}(M,E) \oplus H^{s-\frac{1}{2}}_{\text{loc}}(\partial M, J^-)$.

The calculus for compact $M$ with boundary will be applied, in particular, for $M = I$. In this case all bundles are, of course, trivial, and we write in this case $\boldsymbol{w} = (k, m; N_-, N_+), N_- = (N_-(\iota_-), N_-(\iota_+)),\ N_+ = (N_+(\iota_-), N_+(\iota_+))$; then the space $\boldsymbol{\mathcal{B}}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^l)$ consists of families of continuous operators

$$\mathcal{A}(\lambda) : \begin{array}{c} H^s(I, \mathbb{C}^k) \\ \oplus \\ \mathbb{C}^{N_-(\iota_-)} \\ \oplus \\ \mathbb{C}^{N_-(\iota_+)} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(I, \mathbb{C}^m) \\ \oplus \\ \mathbb{C}^{N_+(\iota_-)} \\ \oplus \\ \mathbb{C}^{N_+(\iota_+)} \end{array},$$

$s > d - \frac{1}{2}$. (Notice that the pairs of dimensions associated with the different end points $\iota_\pm$ of $I$ are a consequence of the fact that elliptic interior symbols may induce different numbers of trace and potential conditions on both sides.)

Let us assume $l \geq 1$ which is the case in our applications. Every $\mathcal{A}(\lambda) \in \mathcal{B}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^l)$ has a parameter-dependent homogeneous principal symbol (in the covariables $(\vartheta, \lambda) \in \mathbb{R}^{1+l} \backslash 0$)

$$\sigma_{\psi,\mathrm{p}}(\mathcal{A})(\phi, \vartheta, \lambda) : \mathbb{C}^k \longrightarrow \mathbb{C}^m, \tag{62}$$

$$\sigma_{\psi,\mathrm{p}}(\mathcal{A})(\phi, \tau\vartheta, \tau\lambda) = \tau^\mu \sigma_{\psi,\mathrm{p}}(\mathcal{A})(\phi, \vartheta, \lambda) \tag{63}$$

for all $\tau \in \mathbb{R}_+, (\phi, \vartheta, \lambda) \in I \times (\mathbb{R}^{1+l} \backslash 0)$, and parameter-dependent DN-homogeneous boundary symbols

$$\sigma_{\partial,\mathrm{p}}(\mathcal{A})_\pm(\lambda) : \begin{array}{c} H^s(\mathbb{R}_+, \mathbb{C}^k) \\ \oplus \\ \mathbb{C}^{N_-(\iota_\pm)} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(\mathbb{R}_+, \mathbb{C}^m) \\ \oplus \\ \mathbb{C}^{N_+(\iota_\pm)} \end{array} \tag{64}$$

where

$$\sigma_{\partial,\mathrm{p}}(\mathcal{A})_\pm(\tau\lambda) = \tau^\mu \begin{pmatrix} \kappa_\tau & 0 \\ 0 & \tau^{\frac{1}{2}} \end{pmatrix} \sigma_{\partial,\mathrm{p}}(\mathcal{A})_\pm(\lambda) \begin{pmatrix} \kappa_\tau & 0 \\ 0 & \tau^{\frac{1}{2}} \end{pmatrix}^{-1} \tag{65}$$

for all $\tau \in \mathbb{R}_+, \lambda \in \mathbb{R}^l \backslash 0$ (the meaning of these relations is that they hold both with respect to $\iota_+$ and $\iota_-$).

*Remark* 1.24. Analogous relations will be used below for

$$\mathcal{A}(r, r', y; \lambda) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega, \boldsymbol{\mathcal{B}}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^l))$$

for an open set $\Omega \subseteq \mathbb{R}^q$; then we have a corresponding dependence of $\sigma_{\psi,\mathrm{p}}$ and $\sigma_{\partial,\mathrm{p}}$ on the additional variables, i.e.,

$$\sigma_{\psi,\mathrm{p}}(\mathcal{A})(r, r', \phi, y, \vartheta, \lambda), \qquad \sigma_{\partial,\mathrm{p}}(\mathcal{A})_\pm(r, r', y, \lambda). \tag{66}$$

**1.6. The Mellin operator convention in the slit plane.** The constructions in Section 1.1 have produced parameter-dependent boundary value problems on the interval $I$, starting from our original differential crack problems. In the pseudo-differential calculus, e.g., to express parametrices, we have to pass to the pseudo-differential case and to consider, for instance, families of the class $C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \boldsymbol{\mathcal{B}}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^l))$ for $\boldsymbol{w} = (k, m; N_-, N_+)$, $N_- = (N_-(\iota_-), N_-(\iota_+))$, $N_+ = (N_+(\iota_-), N_+(\iota_+))$, $l = 1 + q$, $\lambda = (\varrho, \eta)$. In order to get continuous operators like (19) and (22) we need a suitable Mellin convention.

**Defintion 1.25.** Let $\boldsymbol{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^q)$ for $(\mu, d) \in \mathbb{Z} \times \mathbb{N}$ and $\boldsymbol{w} = (k, m; N_-, N_+)$, $N_- = (N_-(\iota_-), N_-(\iota_+))$, $N_+ = (N_+(\iota_-), N_+(\iota_+))$, denote the set of all $h(z, \eta) \in \mathcal{A}(\mathbb{C}_z, \boldsymbol{\mathcal{B}}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^q))$ such that

$$h(z, \eta)|_{\Gamma_\beta \times \mathbb{R}^q} \in \boldsymbol{\mathcal{B}}^{\mu,d}(I; \boldsymbol{w}; \Gamma_\beta \times \mathbb{R}^q)$$

for every real $\beta$, uniformly in $c \leq \beta \leq c'$ for every $c \leq c'$.

Recall that $\Gamma_\beta = \{z \in \mathbb{C} : \mathrm{Re}\, z = \beta\}$. In the notation of Definition 1.25 we identify $\Gamma_\beta$ with a real line and so apply corresponding notation on parameter-dependent operator spaces. Moreover, in Definition 1.25 we employ the canonical Fréchet topology that is given in $\boldsymbol{\mathcal{B}}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^q)$. The requirements in Definition 1.25 induce a natural semi-norm system in the space $\boldsymbol{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^q)$ under which this is a (nuclear) Fréchet space. Thus, it makes sense to talk about

$$C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \boldsymbol{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^q)).$$

Given an element $f(r, r', y, \varrho, \eta) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \Omega, \boldsymbol{\mathcal{B}}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^{1+q}))$ we can form a family of operators on $I^\wedge = \mathbb{R}_+ \times I$ by applying the pseudo- differential convention $\mathrm{op}_r$ in $r$-direction (with respect to the Fourier transform on the real axis)

$$\mathrm{op}_r(f)(y, \eta) : \begin{matrix} C_0^\infty(I^\wedge, \mathbb{C}^k) \\ \oplus \\ C_0^\infty((\partial I)^\wedge, \mathbb{C}^{N_-}) \end{matrix} \longrightarrow \begin{matrix} C^\infty(I^\wedge, \mathbb{C}^m) \\ \oplus \\ C^\infty((\partial I)^\wedge, \mathbb{C}^{N_+}) \end{matrix}. \tag{67}$$

According to $(\partial I)^\wedge = \mathbb{R}_+ \times (\partial I)$ with $\partial I = \{\iota_-\} \cup \{\iota_+\}$ we employ the abbreviation $C_0^\infty((\partial I)^\wedge, \mathbb{C}^{N_-}) = C_0^\infty(\mathbb{R}_+, \mathbb{C}^{N_-(\iota_-)}) \oplus C_0^\infty(\mathbb{R}_+, \mathbb{C}^{N_-(\iota_+)})$ as well as $C^\infty((\partial I)^\wedge, \mathbb{C}^{N_-}) = C^\infty(\mathbb{R}_+, \mathbb{C}^{N_-(\iota_-)}) \oplus C^\infty(\mathbb{R}_+, \mathbb{C}^{N_-(\iota_+)})$ and similarly for $N_+$, with the convention that $C_0^\infty$ or $C^\infty$ on $\mathbb{R}_+$ with values in $\mathbb{C}^0$ simply equals $\{0\}$. Moreover, with $h(r, r', y, z, \eta) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \Omega, \boldsymbol{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(I; \boldsymbol{w}; \mathbb{R}^q))$ we can associate weighted Mellin pseudo-differential operators $\mathrm{op}_M^\delta(h)(y, \eta)$, $\delta \in \mathbb{R}_+$, (cf. the notation (9)) first acting in the same spaces as in (67) and then extended to weighted Sobolev spaces.

**Theorem 1.26.** *Let* $\tilde{f}(r,r',y,\varrho,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega, \boldsymbol{\mathcal{B}}^{\mu,d}(I;\boldsymbol{w};\mathbb{R}^{1+q}))$ *and let* $f(r,r',y,\varrho,\eta) = \tilde{f}(r,r',y,r\varrho,\eta)$*. Then there is an* $h(r,r',y,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega, \boldsymbol{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(I;\boldsymbol{w};\mathbb{R}^q))$ *such that*

$$\mathrm{op}_M^\delta(h)(y,\eta) = \mathrm{op}_r(f)(y,\eta) \mod C^\infty(\Omega, \mathcal{B}^{-\infty,d}(I^\wedge;\boldsymbol{w};\mathbb{R}^q)) \tag{68}$$

*for every* $\delta \in \mathbb{R}$*, and* $h(r,r',y,z,\eta)$ *is unique* mod $C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega, \boldsymbol{\mathcal{M}}_{\mathcal{O}}^{-\infty,d}(I;\boldsymbol{w};\mathbb{R}^q))$.

*Remark* 1.27. We shall employ below an obvious analogue of Theorem 1.26 for families of operators of the form $\tilde{f}(r,r',y,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega, \boldsymbol{\mathcal{B}}^{\mu,d}(I;\boldsymbol{w};\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$ and $f(r,r',y,\varrho,\eta) = \tilde{f}(r,r',y,r\varrho,r\eta)$. Then there exists an $\tilde{h}(r,r',y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega, \boldsymbol{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(I;\boldsymbol{w};\mathbb{R}^q_{\tilde{\eta}}))$ such that (68) holds for $h(r,r',y,z,\eta) = \tilde{h}(r,r',y,z,r\eta)$. An inspection of the proof shows that when $\tilde{f}(r,r',y,\tilde{\varrho},\tilde{\eta})$ vanishes for $r > R, r' > R'$ for certain $R, R' > 0$ also $\tilde{h}(r,r',y,z,\tilde{\eta})$ can be chosen to be vanishing for $r > R, r' > R'$.

The proof of Theorem 1.26 and Remark 1.27 is similar to a corresponding assertion for a closed compact base of the cone, cf. [27], Section 3.2.2. More details for the case of boundary value problems may be found in [23], Theorem 3.29, or in [12].

**1.7. Discrete and continuous asymptotics.** The crack operator algebra that we develop in Chapter 2 below will contain (locally near the crack boundary) pseudo-differential operators with operator- valued symbols that take values in the (classical) cone algebra on the (stretched) cone $I^\wedge$. Because of the relevance of asymptotics of solutions to elliptic crack problems we refer to a version of the cone algebra with continuous asymptotics (that contains operators with discrete asymptotics as a substructure). Here, this is necessary for boundary value problems. The cone algebra for smooth and closed compact base manifolds with continuous asymptotics has been introduced in Schulze [29]. The case of boundary value problems with the transmission property is to some extent analogous, but it was not yet published before. Therefore, in this section we give the details.

First we introduce subspaces of $\mathcal{K}^{s,\gamma}(I^\wedge)$ with continuous asymptotics. Let us consider a weight interval $\Theta = (\vartheta, 0]$, $-\infty < \vartheta < 0$, and set

$$\mathcal{K}^{s,\gamma}_\Theta(I^\wedge) = \bigcap_{\varepsilon > 0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(I^\wedge), \tag{69}$$

endowed with the Fréchet topology of the projective limit. The elements of (69) are interpreted as functions that are flat "of order $\Theta$" for $r \to 0$, relative to the reference weight $\gamma$. For $\Theta = (-\infty, 0]$ we simply set $\mathcal{K}^{s,\gamma}_\Theta(I^\wedge) = \mathcal{K}^{s,\infty}(I^\wedge)$. We now introduce the so-called singular functions in terms of $C^\infty(I)$-valued analytic functionals, carried by compact sets $K$ in the complex plane of the Mellin covariable. If $U \subseteq \mathbb{C}$ is any open set and $E$ a Fréchet space, $\mathcal{A}(U,E)$ denotes the space of all holomorphic $E$-valued functions in $U$;

we then have $\mathcal{A}(U,E) = \mathcal{A}(U)\otimes_\pi E$; ($\otimes_\pi$ always means the (completed) projective tensor product between the respective spaces). Moreover, let $\mathcal{A}'(K)$ be the space of all (scalar) analytic functionals, carried by $K$ (cf. [11], Section 9.1) endowed with a natural Fréchet topology (cf. 25, Section 1.4.1 or [10]). Recall that every $\zeta \in \mathcal{A}'(K)$ can be represented in the form

$$\langle \zeta, h\rangle = \frac{1}{2\pi i}\int_C f_\zeta(z)h(z)\,\mathrm{d}z, \tag{70}$$

$h \in \mathcal{A}(\mathbb{C})$, for a certain $f_\zeta(z) \in \mathcal{A}(\mathbb{C}\backslash K)$, where $C$ is any (say smooth) compact curve surrounding $K$ counter-clockwise. For a Fréchet space $E$ we then set $\mathcal{A}'(K,E) = \mathcal{A}'(K)\otimes_\pi E$; the elements $\zeta$ in the latter space have the form (70) for suitable $f_\zeta(z) \in \mathcal{A}(\mathbb{C}\backslash K, E)$. Singular functions of the continuous asymptotics are of the form

$$u(r,\phi) = \omega(r)\langle \zeta, r^{-z}\rangle$$

for a cut-off function $\omega(r)$ and some $\zeta \in \mathcal{A}'(K, C^\infty(I))$, applied with respect to the complex variable $z$. Notice that for $K \subset \{z : \mathrm{Re}\ z < 1-\gamma\}$ we have $\omega(r)\langle\zeta, r^{-z}\rangle \in \mathcal{K}^{\infty,\gamma}(I^\wedge)$. In fact, the map

$$\zeta \longrightarrow \omega(r)\langle \zeta, r^{-z}\rangle$$

induces a bijection between $\mathcal{A}'(K, C^\infty(I))$ and a corresponding subspace of $\mathcal{K}^{\infty,\gamma}(I^\wedge)$, namely

$$\mathcal{E}_K(I^\wedge) := \{\omega(r)\langle\zeta, r^{-z}\rangle : \zeta \in \mathcal{A}'(K, C^\infty(I))\}. \tag{71}$$

Here, and in the sequel we assume $K = K^I$, where for any closed subset $A \subseteq \mathbb{C}$ the hull $A^I$ is defined to be the smallest set containing $A$ together with all intervals $\{\beta + i(\tau\alpha_1 + (1-\tau)\alpha_0) : 0 \le \tau \le 1\}$ for arbitrary points $\beta + i\alpha_0, \beta + i\alpha_1 \in A$.

Given Fréchet spaces $E_i, i = 0,1$, embedded in a topological Hausdorff vector space $H$, we form the non-direct sum $E_0 + E_1 = \{e_0 + e_1 : e_0 \in E_0, e_1 \in E_1\}$ as a subspace of $H$, endowed with the Fréchet topology from the isomorphism $E_0 + E_1 \cong E_0 \oplus E_1/\Delta$ for $\Delta = \{e \oplus (-e) : e \in E_0 \cap E_1\}$.

For any finite $\Theta$ we set

$$\mathcal{K}^{s,\gamma}_P(I^\wedge) = \mathcal{K}^{s,\gamma}_\Theta(I^\wedge) + \mathcal{E}_K(I^\wedge)$$

in the Fréchet topology of the non-direct sum. Here $P$ stands for the asymptotic information that remains from $\mathcal{A}'(K, C^\infty(I))$ via (71) in carrier $P := K \cap \{z : \mathrm{Re}\ z > 1-\gamma+\vartheta\}$. We call this $P$ a continuous asymptotic type associated with the weight data $\boldsymbol{g} = (\gamma, \Theta)$ and the base $I$ of the (stretched) cone $I^\wedge$. Let $\mathrm{As}(I, \boldsymbol{g})$ denote the set of all such continuous asymptotic types $P$. Every such asymptotic type can be interpreted as the quotient space $\mathcal{E}_K(I^\wedge)/\sim$, with the equivalence relation

$$u_1 \sim u_2 \Longleftrightarrow u_1 - u_2 \in \mathcal{K}^{\infty,\gamma}_\Theta(I^\wedge).$$

Recall that when $p \in \mathbb{C}$, Re $p < 1-\gamma$, is any point and $f(z)$ a meromorphic function in $\mathbb{C}$ with $p$ as a pole of multiplicity $m+1$ and Laurent coefficients $c_k$ at $(z-p)^{-(k+1)}$ in the Laurent expansion, $0 \leq k \leq m$, formula (70) for $f = f_\zeta$ gives us

$$\langle \zeta, h \rangle = \sum_{k=0}^{m} \frac{c_k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}z^k} h(z)|_{z=p} \tag{72}$$

and, in particular,

$$\langle \zeta, r^{-z} \rangle = r^{-p} \sum_{k=0}^{m} \frac{(-1)^k}{k!} c_k \log^k r. \tag{73}$$

Similar relations are true for $C^\infty(I)$-valued meromorphic functions; they are the motivation for the discrete asymptotics that will be defined below. Let us define continuous asymptotics also for the infinite weight interval $\Theta = (-\infty, 0]$. In this case we start with any closed subset $V \subset \{z : \text{Re } z > 1-\gamma\}$ such that $V^I = V$ and $V \cup \{z : c \leq \text{Re } z \geq c'\}$ compact for every $c \leq c'$, and form the compact set $V_k = V \cap \{z : \text{Re } z > -\gamma - k\}$ for every $k \in \mathbb{N}$. According to the above construction we get an element $P_k \in \text{As}(I, \boldsymbol{g}_k)$ for $\boldsymbol{g}_k = (\gamma, (-(k+1), 0]))$, and it is easy to verify that there are continuous embeddings $\mathcal{K}^{s,\gamma}_{P_{k+1}}(I^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}_{P_k}(I^\wedge)$ for all $k$. We then define

$$\mathcal{K}^{s,\gamma}_P(I^\wedge) = \bigcap_{k \in \mathbb{N}} \mathcal{K}^{s,\gamma}_{P_k}(I^\wedge)$$

with the Fréchet topology of the projective limit. In this notation $P$ is completely determined by the set $V$ =: carrier $P$ and stands for a corresponding continuous asymptotic type, associated with the weight data $\boldsymbol{g} = (\gamma, (-\infty, 0])$. We denote by $\text{As}(I, \boldsymbol{g})$ the set of all such $P$.

Consider a weight interval $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$, and let $\text{As}(I, \boldsymbol{g}^\bullet)$ for $\boldsymbol{g} = (\gamma, \Theta), \gamma \in \mathbb{R}$, denote the set of all sequences

$$P = \{(p_j, m_j, L_j)\}_{j=0,\ldots,N}$$

for some $N = N(P) \leq \infty$ (where $N(P) < \infty$ for finite $\Theta$) with $\pi_{\mathbb{C}} P := \{p_j\}_{j=0,\ldots,N} \subset \{z : 1 + \vartheta - \gamma < \text{Re } z < 1 - \gamma\}, m_j \in \mathbb{N}$, and $L_j \subset C^\infty(I)$ being a finite-dimensional subspace for every $j$. To define $\mathcal{K}^{s,\gamma}_P(I^\wedge)$ we first assume that $\Theta$ is finite. Then

$$\mathcal{E}_P(I^\wedge) := Big\{\omega(r) \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r : \\ c_{jk} \in L_j, \;\; 0 \leq k \leq m_j, \;\; 0 \leq j \leq N \Big\}$$

is a finite-dimensional subspace of $\mathcal{K}^{s,\gamma}(I^\wedge)$, and we set

$$\mathcal{K}^{s,\gamma}_P(I^\wedge) = \mathcal{K}^{s,\gamma}_\Theta(I^\wedge) + \mathcal{E}_P(I^\wedge)$$

which is a direct sum, endowed with the corresponding Fréchet topology. For $\Theta = (-\infty, 0]$ and $P \in \mathrm{As}(I, \boldsymbol{g}^\bullet)$ we form $P_k = \{(p, m, L) \in P : \mathrm{Re}\, p > -\gamma - k\} \in \mathrm{As}(I, \boldsymbol{g}_k^\bullet)$ for $\boldsymbol{g}_k = (\gamma, (-(k+1), 0])$. Then there are continuous embeddings $\mathcal{K}^{s,\gamma}_{P_{k+1}}(I^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}_{P_k}(I^\wedge)$ for all $k$, and we define

$$\mathcal{K}^{s,\gamma}_P(I^\wedge) = \bigcap_{k \in \mathbb{N}} \mathcal{K}^{s,\gamma}_{P_k}(I^\wedge)$$

with the Fréchet topology of the projective limit. It is clear that every $P \in \mathrm{As}(I, \boldsymbol{g}^\bullet)$ gives rise to a continuous asymptotic type $Q$, represented by the set $\pi_{\mathbb{C}} P \cap \{z : 1 - \gamma + \vartheta < \mathrm{Re}\, z\}$. Then we have a continuous embedding

$$\mathcal{K}^{s,\gamma}_P(I^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}_Q(I^\wedge).$$

Analogous constructions make sense for the spaces on $(\partial I)^\wedge = \mathbb{R}_+ \cup_{\mathrm{d}} \mathbb{R}_+$ (disjoint union), the corresponding material can be found in [27], Section 2.3.3. In other words, we have corresponding sets of discrete and continuous asymptotic types $\mathrm{As}(\boldsymbol{g}^\bullet)$ and $\mathrm{As}(\boldsymbol{g})$, respectively, for functions in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$, associated with weight data $\boldsymbol{g} = (\gamma, \Theta)$. For $P \in \mathrm{As}(\boldsymbol{g}^\bullet) (\in \mathrm{As}(\boldsymbol{g}))$ we then have the subspace $\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+) \subset \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ with asymptotics of type $P$.

*Remark* 1.28. For every $P \in \mathrm{As}(I, \boldsymbol{g})$ $(\in \mathrm{As}(I, \boldsymbol{g}^\bullet))$ the group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ on the Hilbert space $\mathcal{K}^{s,\gamma}(I^\wedge)$ induces a group action on the Fréchet subspace $\mathcal{K}^{s,\gamma}_P(I^\wedge)$ cf. the terminology after Definition 1.4 above. A similar remark holds for the spaces $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ and $\mathcal{K}^{s,\gamma}_R(\mathbb{R}_+)$, respectively, $R \in \mathrm{As}(\boldsymbol{g})$ $(\in \mathrm{As}(\boldsymbol{g}^\bullet))$. Then, applying formula (26) we get corresponding wedge spaces with asymptotics, namely

$$\mathcal{W}^{s,\gamma}_P(I^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_P(I^\wedge))$$

and

$$\mathcal{W}^{s,\gamma}_R(\mathbb{R}_+ \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_R(\mathbb{R}_+)),$$

respectively.

We shall employ this scenario for vector-valued functions. Set $\boldsymbol{n} = (k, N)$ for $N = (N(\iota_-), N(\iota_+)) \in \mathbb{N} \times \mathbb{N}$ and let

$$\mathcal{K}^{s,\gamma}((I, \partial I)^\wedge; \boldsymbol{n}) := \\ = \mathcal{K}^{s,\gamma}(I^\wedge, \mathbb{C}^k) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{N(\iota_-)}) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{N(\iota_+)}), \quad (74)$$

$$\mathcal{K}^{s,\gamma}_0((I, \partial I)^\wedge; \boldsymbol{n}) := \\ = \mathcal{K}^{s,\gamma}_0(I^\wedge, \mathbb{C}^k) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{N(\iota_-)}) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{N(\iota_+)}) \quad (75)$$

where the spaces on $\mathbb{R}_+$ correspond to the two component of $(\partial I)^\wedge$. Clearly, the second and third components of spaces on the right hand sides of (74) and (75) belong to the $-$ and $+$ sides of the crack, according to the meaning of $\iota_-$ and $\iota_+$. In contrast to Section 1.2 for convenience from now on we omit extra subscripts "$-$" and "$+$" in the notation of corresponding spaces.

Note that the $\mathcal{K}^{0,0}(I^\wedge, \mathbb{C}^k) \oplus \mathcal{K}^{0,0}(\mathbb{R}_+, \mathbb{C}^{N(\iota_-)}) \oplus \mathcal{K}^{0,0}(\mathbb{R}_+, \mathbb{C}^{N(\iota_+)})$-scalar product induces non- degenerate sesquilinear pairings

$$\mathcal{K}^{s,\gamma}((I,\partial I)^\wedge; \boldsymbol{n}) \times \mathcal{K}_0^{(-s)^*,(-\gamma)^*}((I,\partial I)^\wedge; \boldsymbol{n}) \longrightarrow \mathbb{C} \tag{76}$$

for all $s, \gamma \in \mathbb{R}$ where

$$\mathcal{K}^{r^*,\delta^*}((I,\partial I)^\wedge; \boldsymbol{n}) :=$$
$$= \mathcal{K}^{r,\delta}(I^\wedge, \mathbb{C}^k) \oplus \mathcal{K}^{r+\frac{1}{2},\delta+\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{N(\iota_-)}) \oplus \mathcal{K}^{r+\frac{1}{2},\delta+\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{N(\iota_+)}); \tag{77}$$

$\mathcal{K}_0^{r^*,\delta^*}((I,\partial I)^\wedge; \boldsymbol{n})$ is defined in a similar manner. Moreover, let $\mathrm{As}((I,\partial I), \boldsymbol{g}; \boldsymbol{n})$ for $\boldsymbol{n} = (k, N)$ be the set of all sequences of continuous asymptotic types

$$P = (P_1, \dots, P_k; P_{1,-}, \dots, P_{N(\iota_-),-}; P_{1,+}, \dots, P_{N(\iota_+),+})$$

for $P_j \in \mathrm{As}(I, (\gamma, \Theta)), j = 1, \dots, k$, and $P_{l,\pm} \in \mathrm{As}((\gamma, \Theta)), l = 1, \dots, N(\iota_\pm)$. In a similar sense we employ $\mathrm{As}((I,\partial I), \boldsymbol{g}^\bullet; \boldsymbol{n})$ for the discrete case. Then, both for $P \in \mathrm{As}((I,\partial I), \boldsymbol{g}; \boldsymbol{n})$ and $P \in \mathrm{As}((I,\partial I), \boldsymbol{g}^\bullet; \boldsymbol{n})$ we set

$$\mathcal{K}_P^{s,\gamma}((I,\partial I)^\wedge; \boldsymbol{n}) =$$
$$= \bigoplus_{j=1}^{k} \mathcal{K}_{P_j}^{s,\gamma}(I^\wedge) \oplus \bigoplus_{l=1}^{N(\iota_-)} \mathcal{K}_{P_{l,-}}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+) \oplus \bigoplus_{m=1}^{N(\iota_+)} \mathcal{K}_{P_{m,+}}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+). \tag{78}$$

To simplify notation we do not control individual asymptotic types for the components (though we can easily do it through the calculus) but assume from now on $P_1 = P_j, j = 1, \dots, k$, and $P_\pm = P_{l,\pm}$ for all $l = 1, \dots, N(\iota_\pm)$ and simply write

$$\mathcal{K}_P^{s,\gamma}((I,\partial I)^\wedge; \boldsymbol{n}) =$$
$$= \mathcal{K}_{P_1}^{s,\gamma}(I^\wedge; \mathbb{C}^k) \oplus \mathcal{K}_{P_-}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+; \mathbb{C}^{N(\iota_-)}) \oplus \mathcal{K}_{P_+}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+; \mathbb{C}^{N(\iota_+)}). \tag{79}$$

Finally, it will be useful to deal with the spaces that have an additional weight $\varrho$ for $r \to \infty$, namely

$$\langle r \rangle^{-\varrho} \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge; \boldsymbol{n}) \qquad \text{and} \qquad \langle r \rangle^{-\varrho} \mathcal{K}_P^{s,\gamma}((I,\partial I)^\wedge; \boldsymbol{n}),$$

respectively. We then set

$$\mathcal{S}_P^\gamma((I,\partial I)^\wedge; \boldsymbol{n}) := \bigcap_{l \in \mathbb{N}} \langle r \rangle^{-l} \mathcal{K}_P^{l,\gamma}((I,\partial I)^\wedge; \boldsymbol{n}) \tag{80}$$

endowed with the Fréchet topology of the projective limit. Similarly to the notation in (77) we set

$$\mathcal{S}_{P^*}^{\gamma^*}((I,\partial I)^\wedge; \boldsymbol{n}) := \bigcap_{l \in \mathbb{N}} \langle r \rangle^{-l} \mathcal{K}_{P^*}^{l,\gamma^*}((I,\partial I)^\wedge; \boldsymbol{n})$$

for any sequence $P^*$ of continuous asymptotic types with respect to $I$ and $\partial I$, where the position of carriers corresponds to the weights of the spaces, similarly as that in (80) corresponds to $\gamma$. The set of all such $P^*$ will be denoted by $\mathrm{As}((I,\partial I),(\gamma^*,\Theta);\boldsymbol{n})$.

*Remark* 1.29. Setting

$$\kappa_\lambda^{(\boldsymbol{n})}u=\big(\lambda u_j(\lambda r,\phi)\big)_{j=1,\dots,k}\oplus\big(\lambda^{\frac{1}{2}}u_l^-(\lambda r)\big)_{l=1,\dots,N(\iota_-)}\oplus\big(\lambda^{\frac{1}{2}}u_l^+(\lambda r)\big)_{l=1,\dots,N(\iota_+)}$$

for $u=(u_1,\dots,u_k;u_1^-,\dots,u_{N(\iota_-)}^-;u_1^+,\dots,u_{N(\iota_+)}^+)\in\mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}),\lambda\in\mathbb{R}_+$, we get a strongly continuous group $\{\kappa_\lambda^{(\boldsymbol{n})}\}_{\lambda\in\mathbb{R}_+}$ of isomorphisms on the space $\mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n})$. Moreover, the spaces $\mathcal{K}_P^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n})$ and $\mathcal{S}_P^\gamma((I,\partial I)^\wedge;\boldsymbol{n})$ can be written as countable projective limits of Hilbert spaces (continuously embedded in $\mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n})$), where $\{\kappa_\lambda^{(\boldsymbol{n})}\}_{\lambda\in\mathbb{R}_+}$ acts as a strongly continuous groups of isomorphisms in every space of the corresponding scale.

Asymptotic types will also be needed in a version for Mellin symbols. Let us start with the discrete case. A $\chi\in C^\infty(\mathbb{C})$ is called an $A$-excision function for a closed subset $A\subset\mathbb{C}$ if $\chi(z)=1$ for $\mathrm{dist}(z,A)<\varepsilon_0$, $\chi(z)=0$ for $\mathrm{dist}(z,A)>\varepsilon_1$ for certain $0<\varepsilon_0<\varepsilon_1$. Let $d\in\mathbb{N}$ and $\boldsymbol{w}=(k,m;N_-,N_+)$ like in Definition 1.25, and let $\mathbf{As}^d((I,\partial I);\boldsymbol{w}^\bullet)$ denote the set of all (so-called) discrete asymptotic types for Mellin symbols

$$R=\{(r_j,n_j,M_j)\}_{j\in\mathbb{Z}},\tag{81}$$

defined by the properties $r_j\in\mathbb{C}$, $n_j\in\mathbb{N}$ such that $\pi_{\mathbb{C}}R\cap\{z:c\le\mathrm{Re}\,z\le c'\}$ is finite for every $c\le c'$, where $\pi_{\mathbb{C}}R:=\{r_j\}_{j\in\mathbb{Z}}$, and where $M_j\subset\mathcal{B}^{-\infty,d}(I;\boldsymbol{w})$ is a finite-dimensional subspace of operators of finite rank, $j\in\mathbb{Z}$.

Now $M_R^{-\infty,d}(I;\boldsymbol{w})$ for any $R\in\mathbf{As}^d((I,\partial I);\boldsymbol{w}^\bullet)$ is defined to be the subspace of all

$$f(z)\in\mathcal{A}(\mathbb{C}\backslash\pi_{\mathbb{C}}R,\mathcal{B}^{-\infty,d}(I;\boldsymbol{w}))\tag{82}$$

that are meromorphic with poles at $r_j$ of multiplicities $n_j+1$ and Laurent coefficients at $(z-r_j)^{-(k+1)}$ belonging to $M_j$ for all $0\le k\le n_j, j\in\mathbb{Z}$, and with the property that for every $\pi_{\mathbb{C}}R$-excision function $\chi(z)$ we have

$$\chi(z)f(z)|_{\Gamma_\beta}\in\mathcal{B}^{-\infty,d}(I;\boldsymbol{w};\Gamma_\beta)$$

for every real $\beta$, uniformly in $c\le\beta\le c'$ for arbitrary $c\le c'$. The spaces $M_R^{-\infty,d}(I;\boldsymbol{w})$ are (nuclear) Fréchet in a canonical way.

An analogous definition makes sense in the case of continuous asymptotics $R$. These are represented by closed sets $V\subset\mathbb{C}$ satisfying the condition $V^I=V$ and $V\cap\{z:c\le\mathrm{Re}\,z\le c'\}$ compact for every $c\le c'$, where $V$ is regarded as the carrier of $R$. To every $c<c'$ we define the space $\mathcal{M}_{\mathcal{O},(c,c')}^{-\infty,d}(I;\boldsymbol{w})$ of all $h\in\mathcal{A}(\{z:c<\mathrm{Re}\,z<c'\},\mathcal{B}^{-\infty,d}(I;\boldsymbol{w}))$ with $h|_{\Gamma_\beta}\in\mathcal{B}^{-\infty,d}(I;\boldsymbol{w};\Gamma_\beta)$ for every $c<\beta<c'$, uniformly with respect to $\beta$ in compact subintervals. Clearly, $M_{\mathcal{O},(c,c')}^{-\infty,d}(I;\boldsymbol{w})$ is a nuclear Fréchet space. Moreover, observe, that

when $K \subset \mathbb{C}$ is a compact set, $K^I = K$, and $\tilde{\gamma} \in \mathbb{R}$ any weight with $\sup\{\mathrm{Re}\, z : z \in K\} < \frac{1}{2} - \tilde{\gamma}$, for every $\zeta \in \mathcal{A}'(K, \mathcal{B}^{-\infty,d}(I; \boldsymbol{w}))$ the function

$$f_\zeta(z) = M_{\tilde{\gamma}; r \to z}\{\omega(r)\langle \zeta, r^{-z}\rangle\} \tag{83}$$

with $\omega$ being any fixed cut-off function, has the property $f_\zeta(z) \in \mathcal{A}(\mathbb{C}\backslash K, \mathcal{B}^{-\infty,d}(I; \boldsymbol{w}))$ and $\chi(z) f_\zeta(z)|_{\Gamma_\beta} \in \mathcal{B}^{-\infty,d}(I; \boldsymbol{w}; \Gamma_\beta)$ for every $K$-excision function $\chi$ and every $\beta \in \mathbb{R}$, uniformly (with respect to $\beta$) in compact subintervals. Having fixed a cut-off function $\omega(r)$, relation (83) gives rise to an isomorphism between the space $\mathcal{A}'(K, \mathcal{B}^{-\infty,d}(I; \boldsymbol{w}))$ and

$$\boldsymbol{\mathcal{E}}^d_K(I; \boldsymbol{w}) := \{f_\zeta : \zeta \in \mathcal{A}'(K, \mathcal{B}^{-\infty,d}(I; \boldsymbol{w}))\}$$

with the induced topology. Now if $V \subset \mathbb{C}$ is a closed set of the above kind, to every $k \in \mathbb{N}$ we form the intersection $V_k := V \cap \{z : -(k+1) < \mathrm{Re}\, z < k+1\}$ and call $V_k$ the carrier of a continuous asymptotic type $R_k$ for Mellin symbols in the strip $\{z : -(k+1) < \mathrm{Re}\, z < k+1\}$, where $R_k$ can be identified with the quotient space $\boldsymbol{\mathcal{E}}^d_{V_{k+l+1}}(I; \boldsymbol{w})/\sim$ for some $l \in \mathbb{N}$, where $f_1 \sim f_2$ means $(f_1 - f_2)|_{\{z : -(k+1) < \mathrm{Re}\, z < k+1\}} \in \mathcal{M}^{-\infty,d}_{\mathcal{O};(-(k+1),k+1)}(I; \boldsymbol{w})$. This is, in fact, independent of the choice of $l$. Setting $\mathcal{M}^{-\infty,d}_{R_k;(-(k+1),k+1)}(I; \boldsymbol{w}) := \mathcal{M}^{-\infty,d}_{\mathcal{O};(-(k+1),k+1)}(I; \boldsymbol{w}) + \boldsymbol{\mathcal{E}}^d_{V_{k+l+1}}(I; \boldsymbol{w})$ for some $l \in \mathbb{N}$ in the Fréchet topology of the non-direct sum (this space is also independent of $l$) we get a sequence of spaces with continuous embeddings

$$\mathcal{M}^{-\infty,d}_{R_{k+1};(-(k+2),k+2)}(I; \boldsymbol{w}) \hookrightarrow \mathcal{M}^{-\infty,d}_{R_k;(-(k+1),k+1)}(I; \boldsymbol{w})$$

for all $k$, and we then form

$$\mathcal{M}^{-\infty,d}_R(I; \boldsymbol{w}) = \bigcap_{k \in \mathbb{N}} \mathcal{M}^{-\infty,d}_{R_k;(-(k+1),k+1)}(I; \boldsymbol{w}), \tag{84}$$

endowed with the Fréchet topology of the projective limit. In this notation $R$ stands for the sequence $(R_k)_{k \in \mathbb{N}}$, though it can be easily be proved that the space (84) is independent of the choice of the sequence of strips with breadth tending to infinity. So we identify $R$ with a continuous asymptotic type for Mellin symbols, associated with the set $V = \mathrm{carrier}\, R$. Let $\mathbf{As}^d((I, \partial I); \boldsymbol{w})$ denote the set of all such continuous asymptotic types $R$.

## 2. The Crack Operator Algebra

**2.1. Crack symbols.** The pseudo-differential crack theory will be formulated locally in terms of operator-valued symbols along the crack boundary. These are parameter-dependent operators on a cone, namely the slit plane transversal to the crack boundary. In the present section we shall develop the corresponding calculus for continuous asymptotics; the discrete case simpler (in the case of constant discrete asymptotic types) and may be regarded as a substructure.

Let us introduce symbol spaces with Douglis-Nirenberg (DN-)orders as follows. Let us consider Hilbert spaces $E_1$, $E_2$, $E_3$ and $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$ with strongly continuous groups of ismorphisms

$$\kappa_{(j),\lambda} : E_j \longrightarrow E_j, \qquad \tilde{\kappa}_{(j),\lambda} : \tilde{E}_j \longrightarrow \tilde{E}_j, \quad \lambda \in \mathbb{R}_+,$$

for $j = 1, 2, 3$ and set

$$E = E_1 \oplus E_2 \oplus E_3 \ \text{ and } \ \tilde{E} = \tilde{E}_1 \oplus \tilde{E}_2 \oplus \tilde{E}_3.$$

We denote by $\boldsymbol{S}^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; E, \tilde{E})$ the space of all block-matrices $(\boldsymbol{a}_{ij}(y, \eta))_{i,j=1,2,3}$ such that $\boldsymbol{a}_{ij}(y, \eta) \in S^{\mu_{ij}}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E_j, \tilde{E}_i)$ and

$$(\mu_{ij})_{i,j=1,2,3} = \begin{pmatrix} \mu & \mu - \frac{1}{2} & \mu - 1 \\ \mu + \frac{1}{2} & \mu & \mu - \frac{1}{2} \\ \mu + 1 & \mu + \frac{1}{2} & \mu \end{pmatrix}.$$

Similarly, let $\boldsymbol{S}^{\mu^*}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; \tilde{E}, E)$ denote the space of all block-matrices $(\boldsymbol{b}_{ij}(y, \eta))_{i,j=1,2,3}$ such that $\boldsymbol{b}_{ij}(y, \eta) \in S^{\mu^*_{ij}}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; \tilde{E}_j, E_i)$ for $(\mu^*_{ij})_{i,j=1,2,3} = \begin{pmatrix} \mu & \mu + \frac{1}{2} & \mu + 1 \\ \mu - \frac{1}{2} & \mu & \mu + \frac{1}{2} \\ \mu - 1 & \mu - \frac{1}{2} & \mu \end{pmatrix}$. We employ analogous notation also for the case of Fréchet spaces that are written as countable projective limits of Hilbert spaces, similarly to a corresponding definition for orders in the standard meaning.

In our case we set, for instance,

$$E_1 = \mathcal{K}^{s,\gamma}(I^\wedge, \mathbb{C}^k), \qquad E_3 = \mathbb{C}^{L_-},$$
$$E_2 = \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{N_-(\iota_-)}) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{N_+(\iota_-)}),$$

where

$$\{\kappa_{(1),\lambda}\}_{\lambda \in \mathbb{R}_+} := \{\kappa^{(1)}_{\lambda}\}_{\lambda \in \mathbb{R}_+}$$
$$\{\kappa_{(2),\lambda}\}_{\lambda \in \mathbb{R}_+} := \mathrm{diag}(\{\kappa^{(0)}_{\lambda}\}_{\lambda \in \mathbb{R}_+}, \{\kappa^{(0)}_{\lambda}\}_{\lambda \in \mathbb{R}_+})$$

in the notation of Remark 1.1, and $\{\kappa_{(3),\lambda}\}_{\lambda \in \mathbb{R}_+} = \mathrm{id}_{\mathbb{C}^{L_-}}$, while $\tilde{E}_1$ and $\tilde{E}_2$ are Fréchet spaces with asymptotics on $I^\wedge$ and $(\partial I)^\wedge = \mathbb{R}_+ \cup_{\mathrm{d}} \mathbb{R}_+$, respectively, and $\tilde{E}_3 = \mathbb{C}^{L_+}$.

For an element $\boldsymbol{a}(y, \eta) \in \boldsymbol{S}^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; E, \tilde{E})$ we have the DN-homogeneous principal symbol $\sigma_\wedge(\boldsymbol{a})(y, \eta)$, defined to be the block matrix

$$\sigma_\wedge(\boldsymbol{a})(y, \eta) = \big(\boldsymbol{a}_{ij,(\mu_{ij})}(y, \eta)\big)_{i,j=1,2,3}\,, \tag{85}$$

$(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus 0)$, where $\boldsymbol{a}_{ij,(\mu_{ij})}(y, \eta)$ denotes the homogeneous principal component of $\boldsymbol{a}_{ij}(y, \eta) \in S^{\mu_{ij}}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; E_j, \tilde{E}_i)$ of order $\mu_{ij}$ (in the standard sense of classical operator-valued symbols).

**Defintion 2.1.** An operator function

$$g(y,\eta): \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}) \oplus \mathbb{C}^{L_-} \longrightarrow \mathcal{K}^{\infty,\delta}((I,\partial I)^\wedge;\boldsymbol{m}) \oplus \mathbb{C}^{L_+} \quad (86)$$

for $s > -\frac{1}{2}$, $\boldsymbol{n} = (k, N_-)$, $N_- = (N_-(\iota_-),\ N_-(\iota_+))$ and $\boldsymbol{m} = (m, N_+)$, $N_+ = (N_+(\iota_-),\ N_+(\iota_+))$ and given weights $\gamma, \delta \in \mathbb{R}$ is called a Green symbol of order $\mu \in \mathbb{R}$ and type 0 (with continuous asymptotics) if there are elements

$$P \in \mathrm{As}((I,\partial I),(\delta,\Theta);\boldsymbol{m}), \qquad Q \in \mathrm{As}((I,\partial I),((-\gamma)^*,\Theta);\boldsymbol{n})$$

such that

$$g(y,\eta) \in \bigcap_{s>-\frac{1}{2}} \boldsymbol{S}^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q, \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n})\oplus\mathbb{C}^{L_-}, \mathcal{S}^\delta_P((I,\partial I)^\wedge;\boldsymbol{m})\oplus\mathbb{C}^{L_+}) \quad (87)$$

and the point-wise adjoint $g^*(y,\eta)$ (with respect to the pairing between our spaces in question) represents an element

$$g^*(y,\eta) \in \bigcap_{s>-\frac{1}{2}} \boldsymbol{S}^{\mu^*}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q, \mathcal{K}^{s^*,(-\delta)^*}((I,\partial I)^\wedge;\boldsymbol{m}) \oplus$$
$$\oplus\mathbb{C}^{L_+}, \mathcal{S}^{(-\gamma)^*}_Q((I,\partial I)^\wedge;\boldsymbol{n})\oplus\mathbb{C}^{L_-}) \quad (88)$$

(cf. the remarks after this definition). Moreover, an operator function (86) for $s > d - \frac{1}{2}$ is called a Green symbol (with continuous asymptotics) of order $\mu \in \mathbb{R}$ and type $d \in \mathbb{N}$, if it has the form

$$g(y,\eta) = g_0(y,\eta) + \sum_{j=1}^{d} g_j(y,\eta) \begin{pmatrix} \partial^j_\phi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (89)$$

for certain Green symbols $g_j(y,\eta)$ (with continuous asymptotics) of order $\mu - j$ and type 0. Here $\partial^j_\phi$ acts in the space $\mathcal{K}^{s,\gamma}(I^\wedge,\mathbb{C}^k)$.

Condition (88) is to be interpreted as follows (consider, for simplicity, upper left corners, i.e., the case $L_- = L_+ = 0$) : Relation (87) means, in particular, that

$$g(y,\eta): \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}) \longrightarrow \mathcal{S}^\delta_P((I,\partial I)^\wedge;\boldsymbol{m})$$

is continuous. This can be also be regarded as a continuous map

$$g(y,\eta): \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}) \longrightarrow \mathcal{K}^{r,\gamma}_0((I,\partial I)^\wedge;\boldsymbol{m})$$

for each $r < \frac{1}{2}$, since $\mathcal{S}^\delta_P((I,\partial I)^\wedge;\boldsymbol{m}) \hookrightarrow \mathcal{K}^{r,\delta}((I,\partial I)^\wedge;\boldsymbol{m})$ is continuous (by definition) and $\mathcal{K}^{r,\delta}((I,\partial I)^\wedge;\boldsymbol{m}) \hookrightarrow \mathcal{K}^{r,\delta}_0((I,\partial I)^\wedge;\boldsymbol{m})$ is continuous for $r < \frac{1}{2}$. Then, applying the pairing $\mathcal{K}^{(-r)^*,(-\delta)^*}_0((I,\partial I)^\wedge;\boldsymbol{m}) \times \mathcal{K}^{r,\delta}((I,\partial I)^\wedge;\boldsymbol{m}) \to \mathbb{C}$ via the above-mentioned scalar product to the case $s = -r, s > -\frac{1}{2}$, we get $g^*(y,\eta)$ as a continuous map

$$g^*(y,\eta): \mathcal{K}^{(-s)^*,(-\delta)^*}((I,\partial I)^\wedge;\boldsymbol{m}) \longrightarrow \mathcal{K}^{(-s)^*,(-\delta)^*}_0((I,\partial I)^\wedge;\boldsymbol{n})$$

for all $s > -\frac{1}{2}$. Now condition (88) requires, in particular, that $g^*(y,\eta)$ is even continuous in the sense

$$g^*(y,\eta) : \mathcal{K}^{s^*,(-\delta)^*}((I,\partial I)^\wedge;\boldsymbol{m}) \longrightarrow \mathcal{S}_Q^{(-\gamma)^*}((I,\partial I)^\wedge;\boldsymbol{n})$$

and in addition defines a corresponding symbol. Here, $\mu^*$ in the symbol class $\boldsymbol{S}_{\mathrm{cl}}^{\mu^*}$ indicates the scheme of DN-orders $\begin{pmatrix} \mu & \mu+\frac{1}{2} \\ \mu-\frac{1}{2} & \mu \end{pmatrix}$ dual to the DN-orders $\mu = \begin{pmatrix} \mu & \mu-\frac{1}{2} \\ \mu+\frac{1}{2} & \mu \end{pmatrix}$ in $\boldsymbol{S}_{\mathrm{cl}}^{\mu}$.

Let $\boldsymbol{\mathcal{R}}_G^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$ for $\boldsymbol{g}=(\gamma,\delta,\Theta), \boldsymbol{v}=(\boldsymbol{n},\boldsymbol{m};L_-,L_+)$ denote the space of all Green symbols of order $\mu$ and type $d$. Moreover, let $\boldsymbol{R}_G^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$ for $\boldsymbol{w}=(\boldsymbol{n},\boldsymbol{m})$ be the space of upper left corners, i.e., when $L_-=L_+=0$.

Now we pass to the algebra of parameter-dependent cone operators (with continuous asymptotics). In the following definition $\omega,\omega_0,\omega_1$ are cut-off functions with $\omega\omega_0=\omega, \omega\omega_1=\omega_1$ and $\eta\to[\eta]$ is any strictly positive $C^\infty$ function in $\mathbb{R}^q$ with $[\eta]=|\eta|$ for $|\eta|\geq c$ for $c>0$.

**Defintion 2.2.** $\boldsymbol{R}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$ for $\mu\in\mathbb{Z}, d\in\mathbb{N}, \boldsymbol{g}=(\gamma,\gamma-\mu,\Theta)$ for $\Theta=(-(k+1),0], \boldsymbol{w}=(\boldsymbol{n},\boldsymbol{m})$, is defined to be the set of all operator families

$$\begin{aligned} a(y,\eta) &= r^{-\mu}\omega(r[\eta])\operatorname{op}_M^{\gamma-\frac{1}{2}}(h)(y,\eta)\omega_0(r[\eta]) \\ &+ r^{-\mu}(1-\omega(r[\eta]))\operatorname{op}_r(f)(y,\eta)(1-\omega_1(r[\eta])) + m(y,\eta) + g(y,\eta) \end{aligned} \tag{90}$$

where

(i) $f(r,r',y,\varrho,\eta)=\tilde f(r,r',y,r\varrho,r\eta)$ with a given $\tilde f(r,r',y,\tilde\varrho,\tilde\eta)\in C^\infty(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times\Omega,\boldsymbol{\mathcal{B}}^{\mu,d}(I;\boldsymbol{w};\mathbb{R}^{1+q}_{\tilde\varrho,\tilde\eta}))$, vanishing for large $r$ and $r'$,

(ii) $h(r,r',y,z,\eta)=\tilde h(r,r',y,z,r\eta)$ for an

$$\tilde h(r,r',y,z,\tilde\eta)\in C^\infty(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times\Omega,\boldsymbol{\mathcal{M}}_{\mathcal{O}}^{\mu,d}(I;\boldsymbol{w};\mathbb{R}^q_{\tilde\eta})),$$

vanishing for large $r$ and $r'$, where

$$\operatorname{op}_M^\delta(h)(y,\eta)=\operatorname{op}_r(f)(y,\eta) \mod C^\infty(\Omega,\mathcal{B}^{-\infty,d}(I^\wedge;\boldsymbol{w};\mathbb{R}^q_\eta))$$

for all $\delta$, cf. Theorem 1.26 and Remark 1.27,

(iii) $m(y,\eta)$ is a finite linear combination of operator families of the form

$$r^{-\mu+j}\omega(r[\eta])\operatorname{op}_M^{\gamma_{j\alpha}-\frac{1}{2}}(f_{j\alpha})(y)\eta^\alpha\tilde\omega(r[\eta]) \tag{91}$$

for arbitrary cut-off functions $\omega,\tilde\omega$ and $j\in\mathbb{N},\alpha\in\mathbb{N}^q$ with $|\alpha|\leq j$, and arbitrary elements $f_{j\alpha}(y,z)\in C^\infty(\Omega,\mathcal{M}_{R_{j\alpha}}^{-\infty,d}(I;\boldsymbol{w}))$ with $\Gamma_{\frac{1}{2}-\gamma_{j\alpha}}\cap$ carrier $R_{j\alpha}=\emptyset$ and $\gamma_{j\alpha}$, such that $\gamma-j\leq\gamma_{j\alpha}\leq\gamma$ for all $j,\alpha$,

(iv) $g(y,\eta)\in R_G^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$.

Moreover, let $\boldsymbol{\mathcal{R}}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$ for $\boldsymbol{v}=(\boldsymbol{n},\boldsymbol{m};L_-,L_+)$ be the space of all block-matrix-valued functions of the form

$$\boldsymbol{a}(y,\eta)=\begin{pmatrix} a(y,\eta) & 0 \\ 0 & 0 \end{pmatrix}+g(y,\eta) \tag{92}$$

for arbitrary $a(y,\eta)\in \boldsymbol{R}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w}),\boldsymbol{w}=(\boldsymbol{n},\boldsymbol{m})$, and $g(y,\eta)\in \boldsymbol{\mathcal{R}}_G^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$. Let $\boldsymbol{R}_{M+G}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$ ($\boldsymbol{\mathcal{R}}_{M+G}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$) denote the subspace of $\boldsymbol{R}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$ ($\boldsymbol{\mathcal{R}}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$) for which $h$ and $f$ in formula (90) vanish. These elements will also be called smoothing Mellin+Green edge symbols with continuous asymptotics.

*Remark* 2.3. The elements of $\boldsymbol{R}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$ are parameter- dependent pseudo-differential boundary value problems on the cone $I^\wedge$ with boundary $(\partial I)^\wedge$, with $(y,\eta)\in\Omega\times\mathbb{R}$ as parameters, and we have

$$a(y,\eta)\in C^\infty(\Omega,\boldsymbol{\mathcal{B}}^{\mu,d}(I^\wedge;\boldsymbol{w};\mathbb{R}^q)).$$

They are treated as operator-valued symbols along the boundary of the crack and (as the following assertions show) behave like edge symbols in a correspondings edge pseudo-differential calculus with $\Omega$ as edge and $I^\wedge$ as model cone. The additional entries in $\boldsymbol{a}(y,\eta)\in\boldsymbol{\mathcal{R}}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$ with $a(y,\eta)$ as upper left corners describe (on a symbol level) contributions to Green's function to elliptic edge problems and trace and potential operators on the edge that take part in an analogue of the Shapiro-Lopatinskij condition along the boundary of the crack in elliptic crack problems.

**Theorem 2.4.** *For every* $\boldsymbol{a}(y,\eta)\in\boldsymbol{\mathcal{R}}_{M+G}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$ *we have* $\boldsymbol{a}(y,\eta)\in \boldsymbol{S}_{\mathrm{cl}}^{\mu}(\Omega\times\mathbb{R}^q,E\oplus\mathbb{C}^{L_-},F\oplus\mathbb{C}^{L_+})$ *for* $E=\mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}),F=\mathcal{K}^{\infty,\gamma-\mu}((I,\partial I)^\wedge;\boldsymbol{m})$ *and* $E=\mathcal{K}_P^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}),F=\mathcal{S}_Q^{\gamma-\mu}((I,\partial I)^\wedge;\boldsymbol{m})$ *for all* $s>d-\frac{1}{2}$ *and every* $P\in\mathrm{As}((I,\partial I),(\gamma,\Theta);\boldsymbol{n})$ *with some resulting* $Q\in\mathrm{As}((I,\partial I),(\gamma-\mu,\Theta);\boldsymbol{m})$.

**Theorem 2.5.** *Every* $a(y,\eta)\in\boldsymbol{R}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w}),\boldsymbol{w}=(\boldsymbol{n},\boldsymbol{m})$, *belongs to* $\boldsymbol{S}^\mu(\Omega\times\mathbb{R}^q,E,F)$ *for* $E=\mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}),F=\mathcal{K}^{s-\mu,\gamma-\mu}((I,\partial I)^\wedge;\boldsymbol{m})$ *as well as* $E=\mathcal{K}_P^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}),F=\mathcal{K}_Q^{s-\mu,\gamma-\mu}((I,\partial I)^\wedge;\boldsymbol{m})$ *for every* $s>d-\frac{1}{2}$ *and every* $P\in\mathrm{As}((I,\partial I),(\gamma,\Theta);\boldsymbol{n})$ *with some resulting* $Q\in\mathrm{As}((I,\partial I),(\gamma-\mu,\Theta);\boldsymbol{m})$.

The details of the proof of Theorems 2.4 and 2.5 are rather voluminous, though elementary, except for the contributions from the upper left corners. The corresponding technique is contained in [22], Section 3. The smoothing Mellin+Green symbols are classical in the operator-valued sense. As such the arguments follow in terms of continuities of operators in cone Sobolev spaces and subspaces with asymptotics, dependent on variables $y$ and covariables $\eta$ on the unit sphere, and using "$\kappa_\lambda$-homogeneities".

*Remark* 2.6. There is an obvious analogue

$$\boldsymbol{\mathcal{R}}^{\mu,d}(\Omega\times\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v}) \tag{93}$$

of the operator-valued symbol class $\boldsymbol{\mathcal{R}}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$, where $y\in\Omega$ is replaced by $(y,y')\in\Omega\times\Omega$ in all ingredients. If $\boldsymbol{a}(y,y',\eta)\in\boldsymbol{\mathcal{R}}^{\mu,d}(\Omega\times\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$ is given, we can pass to an asymptotic sum (a "left symbol")

$$\boldsymbol{a}_L(y,\eta)\sim\sum_\alpha\frac{1}{\alpha!}D_y^\alpha\partial_{y'}^\alpha\boldsymbol{a}(y,y',\eta)|_{y'=y}$$

belonging to $\boldsymbol{\mathcal{R}}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$. On the level of operators $\mathrm{Op}(\boldsymbol{a})$, cf. Section 2.3 below, the difference $\mathrm{Op}(\boldsymbol{a})-\mathrm{Op}(\boldsymbol{a}_L)$ will be smoothing. For that reason we mainly consider the $y$-dependent case. Nevertheless, sometimes it is useful to consider symbols in (93) that have proper support near the diagonal in $(y,y')\in\Omega\times\Omega$ like amplitude functions in standard properly supported pseudo-differential operators. If $\boldsymbol{a}(y,y',\eta)$ is of that type we call $\mathrm{Op}(\boldsymbol{a})$ properly supported in the $y$-variables.

Let us now introduce the principal symbol structure of $\boldsymbol{\mathcal{R}}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{v})$, $\boldsymbol{v}=(\boldsymbol{n},\boldsymbol{m};L_-,L_+)$. It consists of a three-component hierarchy

$$\sigma(\boldsymbol{a})=(\sigma_\psi(\boldsymbol{a}),\sigma_\partial(\boldsymbol{a}),\sigma_\wedge(\boldsymbol{a})), \tag{94}$$

together with a subordinate principal conormal symbol $\sigma_M(\boldsymbol{a})$ that is determined by (the upper left corner of) $\sigma_\wedge(\boldsymbol{a})$ and responsible for the asymptotics of solutions in the case of ellipticity. The principal interior symbol $\sigma_\psi(\boldsymbol{a})$ and principal boundary symbol $\sigma_\partial(\boldsymbol{a})$ are completely determined by $f(r,r',y,\varrho,\eta)$ in Definition 2.2, (i). First, the element $\tilde f(r,r',y,\tilde\varrho,\tilde\eta)\in C^\infty(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times\Omega,\boldsymbol{\mathcal{B}}^{\mu,d}(I;\boldsymbol{w};\mathbb{R}^{1+q}))$ has the corresponding parameter-dependent principal interior and boundary symbols, cf. Remark 1.24,

$$\sigma_{\psi,\mathrm{p}}(\tilde f)(r,r',\phi,y,\tilde\varrho,\vartheta,\tilde\eta),\quad (r,r',\phi,y,\tilde\varrho,\vartheta,\tilde\eta)\in\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times I\times\Omega\times(\mathbb{R}^{2+q}\backslash 0),$$

and

$$\sigma_{\partial,\mathrm{p}}(\tilde f)(r,r',y,\tilde\varrho,\tilde\eta):=(\sigma_{\partial,\mathrm{p}}(\tilde f)_-(r,r',y,\tilde\varrho,\tilde\eta),\sigma_{\partial,\mathrm{p}}(\tilde f)_+(r,r',y,\tilde\varrho,\tilde\eta)),$$

$(r,r',y,\tilde\varrho,\tilde\eta)\in\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times\Omega\times(\mathbb{R}^{1+q}\backslash 0)$; the homogeneity of orders refers to the DN-convention, cf. formula (65). Now we may set

$$\sigma_\psi(\boldsymbol{a})(r,\phi,y,\varrho,\vartheta,\eta)=r^{-\mu}\sigma_{\psi,\mathrm{p}}(\tilde f)(r,r',\phi,y,r\varrho,\vartheta,r\eta)|_{r'=r}$$

and

$$\sigma_\partial(\boldsymbol{a})(r,y,\varrho,\eta)=r^{-\mu}\sigma_{\partial,\mathrm{p}}(\tilde f)(r,r',y,r\varrho,r\eta)|_{r'=r}.$$

Next we define the principal edge symbol $\sigma_\wedge(\boldsymbol{a})$ of $\boldsymbol{a}(y,\eta)$ in the sense of DN-homogeneities. First, for the Green summand $g(y,\eta)$ we have $\sigma_\wedge(g)(y,\eta)$ by Definition 2.1 and formula (85). Moreover, Theorem 2.4 tells us more generally that the space consists of classical symbols; thus to every $\boldsymbol{a}(y,\eta)$ in this space we get a corresponding DN-homogeneous principal edge symbol $\sigma_\wedge(\boldsymbol{a})(y,\eta)$, again by formula (85). In particular, $\sigma_\wedge(m)(y,\eta)$ for an operator familiy $m(y,\eta)$ like in (90) which is a finite sum of expressions (91), $\sigma_\wedge(m)(y,\eta)$ equals the sum of all operator families

$$r^{-\mu+j}\omega(r|\eta|)\,\mathrm{op}_M^{\gamma_{j\alpha}-\frac{1}{2}}(f_{j\alpha})(y)\eta^\alpha\tilde{\omega}(r|\eta|) \quad \text{for all} \quad j,\alpha$$

such that $j=|\alpha|$. It remains to define the edge symbol for the two summands in (90), i.e., for

$$\begin{aligned} c(y,\eta) &= r^{-\mu}\omega(r[\eta])\,\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)(y,\eta)\omega_0(r[\eta]) + \\ &+ r^{-\mu}(1-\omega(r[\eta]))\,\mathrm{op}_r(f)(y,\eta)(1-\omega_1(r[\eta])), \end{aligned}$$

$(y,\eta)\in\Omega\times(\mathbb{R}^q\backslash 0)$ (in this notation (90) equals $a(y,\eta)=c(y,\eta)+m(y,\eta)+g(y,\eta)$). Setting

$$h_0(r,y,z,\eta) := \tilde{h}(0,0,y,z,r\eta)$$

and

$$f_0(r,y,\varrho,\eta) := \tilde{f}(0,0,y,r\varrho,r\eta)$$

we simply form

$$\begin{aligned} \sigma_\wedge(c)(y,\eta) &= r^{-\mu}\omega(r|\eta|)\,\mathrm{op}_M^{\gamma-\frac{1}{2}}(h_0)(y,\eta)\omega_0(r|\eta|) \\ &+ r^{-\mu}(1-\omega(r|\eta|))\,\mathrm{op}_r(f_0)(y,\eta)(1-\omega_1(r|\eta|)). \end{aligned}$$

In other words, for (90) we set $\sigma_\wedge(a)(y,\eta) = \sigma_\wedge(c)(y,\eta) + \sigma_\wedge(m)(y,\eta) + \sigma_\wedge(g)(y,\eta)$. For $\boldsymbol{a}(y,\eta)\in\boldsymbol{\mathcal{R}}^{\mu,d}(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$, given in the form (92), we then define altogether

$$\sigma_\wedge(\boldsymbol{a})(y,\eta) = \begin{pmatrix} \sigma_\wedge(a)(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\wedge(g)(y,\eta), \tag{95}$$

$(y,\eta)\in\Omega\times(\mathbb{R}^q\backslash 0)$.

For the control of asymptotics of solutions also the principal conormal symbol is of importance, though this is a subordinate symbol level. It has the following form

$$\sigma_M(\boldsymbol{a})(y,z) = h(0,0,y,z,0) + \sigma_M(m)(y,z), \tag{96}$$

where $\sigma_M(m)(y,z)$ is defined to be the sum over all smoothing Mellin symbols $f_{00}(y,z)$, occurring in $m(y,\eta)$, cf. (91). As is known from the cone pseudo-differential calculus the principal conormal symbol is uniquely determined by $\sigma_\wedge(\boldsymbol{a})$, and it is independent of $\eta$. Under this point of view the principal conormal symbol of the wedge calculus is regarded as a subordinate symbol.

*Remark* 2.7. The structure of the upper left corner $\sigma_\wedge(a)(y,\eta)$ in (95) is very interesting. It is a family of continuous operators

$$\sigma_\wedge(a)(y,\eta) : \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}((I,\partial I)^\wedge;\boldsymbol{m})$$

for $s > d-\frac{1}{2}$, $(y,\eta)\in\Omega\times(\mathbb{R}^q\backslash 0)$, and DN-homogeneous in a typical way, cf. the $(2\times 2)$-upper left corner of formula (110) below. For fixed

$(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus 0)$ it belongs to a classical algebra of cone boundary value problems on the infinite (stretched) cone $I^\wedge$ with DN-order convention. More details on such cone algebras may be found in Kapanadze and Schulze [12]; a non-classical variant of a cone algebra of boundary value problems (near the tip of the cone and for "usual" orders) is elaborated in Schrohe and Schulze [19], [20]. A typical point in our theory is that $r \to \infty$ on $I^\wedge$ is to be interpreted as a conical exit to infinity, cf. Kapanadze and Schulze [14], and that $\sigma_\wedge(a)$ for fixed $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus 0)$ near the exits belongs to (a DN-version of) the operator space of order $\mu$, type $d$ and weight zero at infinity constructed in [14].

**Theorem 2.8.** *$\boldsymbol{a}(y,\eta) \in \boldsymbol{\mathcal{R}}^{\mu,d}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} = (\gamma-\nu, \gamma-\nu-\mu, \Theta)$, $\Theta = (-(k+1), 0]$, $\boldsymbol{v} = (\boldsymbol{n}_0, \boldsymbol{m}; L_0, L_+)$ and $\boldsymbol{b}(y,\eta) \in \boldsymbol{\mathcal{R}}^{\nu,e}(\Omega \times \mathbb{R}^q, \boldsymbol{f}; \boldsymbol{u})$ for $\boldsymbol{f} = (\gamma, \gamma-\nu, \Theta)$, $\boldsymbol{u} = (\boldsymbol{n}, \boldsymbol{n}_0; L_-, L_0)$ implies $\boldsymbol{a}(y,\eta)\boldsymbol{b}(y,\eta) \in \boldsymbol{\mathcal{R}}^{\mu+\nu,h}(\Omega \times \mathbb{R}^q, \boldsymbol{g} \circ \boldsymbol{f}; \boldsymbol{v} \circ \boldsymbol{u})$ for $h = \max(\nu+d, e)$, $\boldsymbol{g} \circ \boldsymbol{f} = (\gamma, \gamma-\nu-\mu, \Theta)$, $\boldsymbol{v} \circ \boldsymbol{u} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+)$, and we have $\sigma(\boldsymbol{ab}) = \sigma(\boldsymbol{a})\sigma(\boldsymbol{b})$ with componentwise multiplication (cf. formula (94)).*

*Remark* 2.9. If $\boldsymbol{a}$ and $\boldsymbol{b}$ in Theorem 2.8 belong to the corresponding subspace with subscript $M+G$ $(G)$, then the same is true of the composition.

The proof of the composition results, as far as it concerns the non-smoothing contribution in upper left corners, is essentially the content of [22], Section 3.9 (up to DN-orders that we are using here). Compositions, where one factor is smoothing Mellin+Green, can be treated in a similar manner as in the boundaryless case, cf. [27], Section 3.3.3.

**2.2. Wedge spaces and subspaces with asymptotics.** We now return to the spaces of Definition 1.4 and to the weighted wedge spaces (29), (30) and their "comp($y$)" and "loc($y$)"- variants. In the case of $m \times k$-systems of operators the spaces on $I^\wedge$ are to be replaced by $\mathbb{C}^k$-and $\mathbb{C}^m$-valued ones, for instance, $\mathcal{W}^{s,\gamma}_{\mathrm{loc}(y)}(I^\wedge \times \mathbb{R}^q, \mathbb{C}^k) := \mathcal{W}^{s,\gamma}_{\mathrm{loc}(y)}(I^\wedge \times \mathbb{R}^q) \otimes \mathbb{C}^k$, etc.

As noted after Definition 1.4 all constructions make sense for Fréchet spaces $E$ under the described natural assumptions. We will apply this, in particular, to the cone Sobolev spaces with asymptotics, cf. Remark 1.29.

Our theory is formulated for DN-orders.

Let $\boldsymbol{n} = (k, N)$, $N = (N(\iota_-), N(\iota_+))$, and set

$$\mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}((I,\partial I)^\wedge \times \Omega; \boldsymbol{n}) := \mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(I^\wedge \times \Omega, \mathbb{C}^k) \oplus$$
$$\oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{\mathrm{comp}(y)}(\mathbb{R}_+ \times \Omega, \mathbb{C}^{N(\iota_-)}) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{\mathrm{comp}(y)}(\mathbb{R}_+ \times \Omega, \mathbb{C}^{N(\iota_+)})$$

and

$$\mathcal{W}^{s,\gamma}_{P,\mathrm{comp}(y)}((I,\partial I)^\wedge \times \Omega; \boldsymbol{n}) := \mathcal{W}^{s,\gamma}_{P_1,\mathrm{comp}(y)}(I^\wedge \times \Omega, \mathbb{C}^k) \oplus$$
$$\oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{P_-,\mathrm{comp}(y)}(\mathbb{R}_+ \times \Omega, \mathbb{C}^{N(\iota_-)}) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{P_+,\mathrm{comp}(y)}(\mathbb{R}_+ \times \Omega, \mathbb{C}^{N(\iota_+)})$$

for $P=(P_1,P_+,P_-)\in \mathrm{As}((I,\partial I),\boldsymbol{g};\boldsymbol{n})$ (or $\in \mathrm{As}((I,\partial I),\boldsymbol{g}^\bullet;\boldsymbol{n})$), where the subscripts $P_1,P_\pm$ are interpreted analogously to those in formula (79), cf. also Remark 1.28. In a similar way we have the spaces

$$\mathcal{W}^{s,\gamma}_{\mathrm{loc}(y)}((I,\partial I)^\wedge\times\Omega;\boldsymbol{n}) \ \text{ and } \ \mathcal{W}^{s,\gamma}_{P,\mathrm{loc}(y)}((I,\partial I)^\wedge\times\Omega;\boldsymbol{n}),$$

respectively. In formula (77) we have defined spaces with smoothness and weight indices $(s^*,\gamma^*)$ instead of $(s,\gamma)$. This will be adopted here for our wedge spaces in analogous form, i.e., there are the spaces $\mathcal{W}^{s^*,\gamma^*}_{\mathrm{comp}(y)}((I,\partial I)^\wedge\times\Omega;\boldsymbol{n})$, $\mathcal{W}^{s^*,\gamma^*}_{P,\mathrm{comp}(y)}((I,\partial I)^\wedge\times\Omega;\boldsymbol{n})$, etc.

Let $\varphi_1\in C^\infty(\overline{\mathbb{R}}_+\times I\times\Omega),\varphi_\pm\in C^\infty(\overline{\mathbb{R}}_+\times\Omega)$, and let $M_\varphi$ for $\varphi=(\varphi_1,\varphi_-,\varphi_+)$ denote the operator of multiplication by $\mathrm{diag}(\varphi_1\otimes\mathrm{id}_{\mathbb{C}^k},\varphi_-\otimes\mathrm{id}_{\mathbb{C}^{N(\iota_-)}},\varphi_+\otimes\mathrm{id}_{\mathbb{C}^{N(\iota_+)}})$ (the involved dimensions $(k,N(\iota_-),N(\iota_+))$) will be clear in concrete formulas, so they are not indicated explicitly).

**Proposition 2.10.** *For arbitrary $\varphi_1\in C_0^\infty(\overline{\mathbb{R}}_+\times I\times\Omega),\varphi_\pm\in C_0^\infty(\overline{\mathbb{R}}_+\times\Omega)$ the multiplication by $M_\varphi$ induces continuous operators*

$$M_\varphi:\mathcal{W}^{s,\gamma}_{\mathrm{loc}(y)}((I,\partial I)^\wedge\times\Omega;\boldsymbol{n})\longrightarrow\mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}((I,\partial I)^\wedge\times\Omega;\boldsymbol{n})$$

*for all $s,\gamma\in\mathbb{R}$, and*

$$M_\varphi:\mathcal{W}^{s,\gamma}_{P,\mathrm{loc}(y)}((I,\partial I)^\wedge\times\Omega;\boldsymbol{n})\longrightarrow\mathcal{W}^{s,\gamma}_{\tilde{P},\mathrm{comp}(y)}((I,\partial I)^\wedge\times\Omega;\boldsymbol{n})$$

*for all $s,\gamma\in\mathbb{R}$ and every $P\in\mathrm{As}((I,\partial I),(\gamma,\Theta);\boldsymbol{n})(\in\mathrm{As}((I,\partial I),(\gamma,\Theta)^\bullet;\boldsymbol{n}))$ with some resulting $\tilde{P}\in\mathrm{As}((I,\partial I),(\gamma,\Theta);\boldsymbol{n})(\in\mathrm{As}((I,\partial I),(\gamma,\Theta)^\bullet;\boldsymbol{n}))$. If the components of $P$ satisfy a shadow condition (cf. [33], Section 2.1.1) in latter relation we may set $P=\tilde{P}$. Moreover, for $\omega_1,\omega_\pm\in C_0^\infty(\overline{\mathbb{R}}_+)$ we get continuous operators $M_\omega$ between corresponding spaces with "comp$(y)$" or "loc$(y)$" on both sides (in the latter case we also write $\omega$ in place of $M_\omega$).*

Proposition 2.10 is a consequence of corresponding results on wedge Sobolev spaces on closed base manifolds of the model cone, cf. [27], Theorem 3.1.24, and of the fact that wedge Sobolev spaces for base manifolds with boundary are restrictions of corresponding spaces over the double wedges (with corresponding doubles of the bases).

Let for a moment $\Omega=\mathbb{R}^q$. Then we have the above spaces without "comp$(y)$" or "loc$(y)$", i.e.,

$$\mathcal{W}^{s,\gamma}((I,\partial I)^\wedge\times\mathbb{R}^q;\boldsymbol{n})=\mathcal{W}^{s,\gamma}(I^\wedge\times\mathbb{R}^q,\mathbb{C}^k)\oplus\mathcal{W}^{s-\frac12,\gamma-\frac12}(\mathbb{R}_+\times\mathbb{R}^q,\mathbb{C}^{N(\iota_-)})$$
$$\oplus\mathcal{W}^{s-\frac12,\gamma-\frac12}(\mathbb{R}_+\times\mathbb{R}^q,\mathbb{C}^{N(\iota_+)})$$

as well as the corresponding subspaces with asymptotics, cf. Definition 1.4. Notice that

$$\mathcal{W}^{0,0}(I^\wedge\times\mathbb{R}^q,\mathbb{C}^k)=r^{-\frac12}L^2(\mathbb{R}_+\times I\times\mathbb{R}^q,\mathbb{C}^k),$$
$$\mathcal{W}^{0,0}(\mathbb{R}_+\times\mathbb{R}^q,\mathbb{C}^N)=L^2(\mathbb{R}_+\times\mathbb{R}^q,\mathbb{C}^N),$$

where the $L^2$-spaces refer to the measures $\mathrm{d}t\,\mathrm{d}\phi\,\mathrm{d}y$ and $\mathrm{d}t\,\mathrm{d}y$, respectively. Thus we can also form the spaces

$$\mathcal{W}^{0,0}(I^\wedge \times \Omega, \mathbb{C}^k) \text{ and } \mathcal{W}^{0,0}(\mathbb{R}_+ \times \Omega, \mathbb{C}^N)$$

for any open $\Omega \subseteq \mathbb{R}^q$ by restrictions, with the induced scalar products. Let $\boldsymbol{n} = (k, N(\iota_-), N(\iota_+)) \in \mathbb{N}^3$, $L \in \mathbb{N}$, set

$$C^\infty((\mathrm{int}\, I, \partial I)^\wedge \times \Omega; (\boldsymbol{n}; L)) = C^\infty(\mathbb{R}_+ \times (\mathrm{int}\, I) \times \Omega, \mathbb{C}^k)$$
$$\oplus C^\infty(\mathbb{R}_+ \times \Omega, \mathbb{C}^{N(\iota_-)}) \oplus C^\infty(\mathbb{R}_+ \times \Omega, \mathbb{C}^{N(\iota_+)}) \oplus C^\infty(\Omega, \mathbb{C}^L),$$

and define $C_0^\infty((\mathrm{int}\, I, \partial I)^\wedge \times \Omega; (\boldsymbol{n}; L))$ in an analogous manner. Similar notation will be used with $I$ instead of int $I$; in this case we mean smoothness on $I^\wedge$ up to the boundary. For $L = 0$ we simply write $\boldsymbol{n}$ instead of $(\boldsymbol{n}; 0)$. Given $P \in \mathrm{As}((I, \partial I), (\gamma, \Theta); \boldsymbol{n})$ we set

$$C_P^{\infty,\gamma}((I, \partial I)^\wedge \times \Omega; \boldsymbol{n}) = \{\omega u + (1-\omega)v : u \in \mathcal{W}_{P,\mathrm{loc}(y)}^{\infty,\gamma}((I, \partial I)^\wedge \times \Omega; \boldsymbol{n}),$$
$$v \in C^\infty((I, \partial I)^\wedge \times \Omega; \boldsymbol{n})\} \tag{97}$$

in the Fréchet topology of the non-direct sum. Given a continuous operator

$$\mathcal{C} : C_0^\infty((\mathrm{int}\, I, \partial I)^\wedge \times \Omega; (\boldsymbol{n}; L_-)) \longrightarrow C^\infty((\mathrm{int}\, I, \partial I)^\wedge \times \Omega; (\boldsymbol{m}; L_+)) \tag{98}$$

for $\boldsymbol{n} = (k, N_-(\iota_-), N_-(\iota_+)) \in \mathbb{N}^3$, $\boldsymbol{m} = (m, N_+(\iota_-), N_+(\iota_+)) \in \mathbb{N}^3$, $L_-, L_+ \in \mathbb{N}$, we can pass to the formal adjoint $\mathcal{C}^*$ by the relation

$$(\mathcal{C}u, v)_{(\boldsymbol{m}; L_+)} = (u, \mathcal{C}^* v)_{(\boldsymbol{n}; L_-)}$$

for all $u \in C_0^\infty((\mathrm{int}\, I, \partial I)^\wedge \times \Omega; (\boldsymbol{n}; L_-))$, $v \in C_0^\infty((\mathrm{int}\, I, \partial I)^\wedge \times \Omega; (\boldsymbol{m}; L_+))$, where $(.,.)_{(\boldsymbol{m}; L_+)}$ is the scalar product of

$$r^{-\frac{1}{2}} L^2(\mathbb{R}_+ \times I \times \Omega, \mathbb{C}^m) \oplus L^2(\mathbb{R}_+ \times \Omega, \mathbb{C}^{N_+(\iota_-)}) \oplus L^2(\mathbb{R}_+ \times \Omega, \mathbb{C}^{N_+(\iota_+)}) \oplus L^2(\Omega, \mathbb{C}^{L_+}),$$

and similarly $(.,.)_{(\boldsymbol{n}; L_-)}$. An operator (98) is said to belong to $\mathcal{V}^{-\infty,0}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} = (\gamma, \delta, \Theta)$ and $\boldsymbol{v} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+)$, if there are elements $P \in \mathrm{As}((I, \partial I), (\delta, \Theta); \boldsymbol{m})$ and $Q \in \mathrm{As}((I, \partial I), ((-\gamma)^*, \Theta); \boldsymbol{n})$ (cf. the notation in Section 1.7) such that $\mathcal{C}$ and $\mathcal{C}^*$ induce continuous operators

$$\mathcal{C}M_\varphi : \begin{array}{c} \mathcal{W}_{\mathrm{loc}(y)}^{s,\gamma}((I, \partial I)^\wedge \times \Omega; \boldsymbol{n}) \\ \oplus \\ H_{\mathrm{loc}}^{s-1}(\Omega, \mathbb{C}^{L_-}) \end{array} \longrightarrow \begin{array}{c} C_P^{\infty,\delta}((I, \partial I)^\wedge \times \Omega; \boldsymbol{m}) \\ \oplus \\ C^\infty(\Omega, \mathbb{C}^{L_+}) \end{array}$$

and

$$\mathcal{C}^* M_\varphi : \begin{array}{c} \mathcal{W}_{\mathrm{loc}(y)}^{s^*,(-\delta)^*}((I, \partial I)^\wedge \times \Omega; \boldsymbol{m}) \\ \oplus \\ H_{\mathrm{loc}}^{s+1}(\Omega, \mathbb{C}^{L_+}) \end{array} \longrightarrow \begin{array}{c} C_Q^{\infty,(-\gamma)^*}((I, \partial I)^\wedge \times \Omega; \boldsymbol{n}) \\ \oplus \\ C^\infty(\Omega, \mathbb{C}^{L_-}) \end{array}$$

for all $s > -\frac{1}{2}$ and arbitrary $\varphi = (\varphi_1, \varphi_-, \varphi_+, \varphi') \in C_0^\infty(\overline{\mathbb{R}}_+ \times I \times \Omega) \times C_0^\infty(\overline{\mathbb{R}}_+ \times \Omega) \times C_0^\infty(\overline{\mathbb{R}}_+ \times \Omega) \times C_0^\infty(\Omega)$, where $M_\varphi$ denotes the operator of multiplication by $\mathrm{diag}(\varphi_1, \varphi_-, \varphi_+, \varphi')$ (tensorised by identity operators as before).

An operator $\mathcal{C}$ is said to belong to $\boldsymbol{\mathcal{V}}^{-\infty,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$, the space of smoothing (local) crack operators (with continuous asymptotics) of type $d \in \mathbb{N}$, if it has the form

$$\mathcal{C} = \mathcal{C}_0 + \sum_{j=1}^{d} \mathcal{C}_j \begin{pmatrix} \partial_\phi^j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{99}$$

for certain $\mathcal{C}_j \in \mathcal{V}^{-\infty,0}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$, $j = 0, \ldots, d$; the meaning of $\partial_\phi^j$ on the right hand side is that it acts in the space $\mathcal{W}^{s,\gamma}_{\mathrm{loc}(y)}(I^\wedge \times \Omega, \mathbb{C}^k)$.

**2.3. Local crack operators.** As explained in the beginning our operator calculus is motivated by the structure of parametrices of elliptic differential crack problems. The structures that we have studied so far formulate various aspects of the solvability. In fact the weighted Sobolev spaces reflect the nature of elliptic regularity, as we shall see in the following section, while the operators themselves encode the nature of parametrices, including the specific behaviour of singular Green functions as well as of additional trace and potential conditions along the crack boundary. The operator class is as follows:

**Defintion 2.11.** Fix $(\mu, d) \in \mathbb{Z} \times \mathbb{N}$, weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, $\gamma \in \mathbb{R}$, $\Theta = (-(k+1), 0]$, and set $\boldsymbol{v} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+)$ for $\boldsymbol{n} = (k, N_-(\iota_-), N_-(\iota_+))$, $\boldsymbol{m} = (m, N_+(\iota_-), N_+(\iota_+))$. Then $\boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$, the space of all (local) crack operators of order $\mu$ and type $d$ is defined to be the set of all

$$\mathcal{A} = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \mathrm{Op}(\boldsymbol{a}) \begin{pmatrix} \tilde{\omega} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \chi \mathcal{B} \tilde{\chi} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{C} \tag{100}$$

for arbitrary $\boldsymbol{a}(y, \eta) \in \boldsymbol{\mathcal{R}}^{\mu,d}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$, $\mathcal{B} \in \boldsymbol{\mathcal{B}}^{\mu,d}(I^\wedge \times \Omega; \boldsymbol{w})$ for $\boldsymbol{w} = (\boldsymbol{n}, \boldsymbol{m})$ and $\mathcal{C} \in \boldsymbol{\mathcal{V}}^{-\infty,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$. Here, $\omega(r)$ and $\tilde{\omega}(r)$ are arbitrary cut-off functions with $\tilde{\omega} \equiv 1$ on $\mathrm{supp}\,\omega$ and $\chi(r) = 1 - \omega(r)$, $\tilde{\chi}(r) = 1 - \tilde{\tilde{\omega}}(r)$ for a cut-off function $\tilde{\tilde{\omega}}$ such that $\tilde{\chi} \equiv 1$ on $\mathrm{supp}\,\chi$.

Let $\boldsymbol{V}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{w})$, $\boldsymbol{w} = (\boldsymbol{n}, \boldsymbol{m})$ denote the space of upper left corners in the sense $\boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; (\boldsymbol{n}; 0), (\boldsymbol{m}; 0))$. We then have

$$\boldsymbol{V}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{w}) \subset \boldsymbol{\mathcal{B}}^{\mu,d}(I^\wedge \times \Omega; \boldsymbol{w}). \tag{101}$$

These operators still consists of block matrices $(\mathcal{A}_{ij})_{i,j=1,2}$ then

$$\mathcal{A}_{11} \in L^\mu_{\mathrm{cl}}((\mathrm{int}\, I) \times \Omega) \otimes \mathbb{C}^m \otimes \mathbb{C}^k. \tag{102}$$

Let $\boldsymbol{\mathcal{V}}^{\mu,d}_{M+G}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ ($\boldsymbol{\mathcal{V}}^{\mu,d}_{G}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$) denote the subspace of all $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ such that $\boldsymbol{a}(y, \eta)$ in representation (100) belongs to $\boldsymbol{\mathcal{R}}^{\mu,d}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$ ($\boldsymbol{\mathcal{R}}^{\mu,d}_{G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{v})$) and for which the operator

$\mathcal{B}$ vanishes. For the corresponding spaces of upper left corners we write analogously

$$\boldsymbol{V}^{\mu,d}_{M+G}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{w}) \text{ and } \boldsymbol{V}^{\mu,d}_{G}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{w}),$$

respectively. Note that

$$\boldsymbol{V}^{\mu,d}_{M+G}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{w}) \subset \mathcal{B}^{-\infty,d}(I^\wedge \times \Omega; \boldsymbol{w})$$

and, of course, $\mathcal{A}_{11} \in L^{-\infty}((\mathrm{int}\, I) \times \Omega) \otimes \mathbb{C}^m \otimes \mathbb{C}^k$ for every $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}_{M+G}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$.

Our crack operator classes also make sense for the infinite weight strip $\Theta = (-\infty, 0]$. If we set for a moment $\boldsymbol{g}_k = (\gamma, \gamma - \mu, (-(k+1), 0])$ and $\boldsymbol{g} = (\gamma, \gamma - \mu, (-\infty, 0])$ we have natural inclusions

$$\boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}_{k+1}; \boldsymbol{v}) \subset \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}_k; \boldsymbol{v})$$

for all $k \in \mathbb{N}$, and we then define $\boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ to be the intersection of these spaces over $k \in \mathbb{N}$. Similarly, we define the subspaces with subscript $M + G$ $(G)$ for $\boldsymbol{g}$ with an infinite weight strip.

*Remark* 2.12. For every $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ there is an $\mathcal{A}_0 \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ such that $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$, where $\mathcal{A}_0$ has the form

$$\mathcal{A}_0 = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \mathrm{Op}(\boldsymbol{a}_0) \begin{pmatrix} \tilde{\omega} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \chi \mathcal{B}_0 \tilde{\chi} & 0 \\ 0 & 0 \end{pmatrix}$$

such that $\mathrm{Op}(\boldsymbol{a}_0)$ is properly supported with respect to $y$-variables, cf. Remark 2.6, and $\mathcal{B}_0$ properly supported in all variables (in the sense of the algebra of pseudo-differential boundary value problems). The change from $\mathcal{B}$ to $\mathcal{B}_0$ is standard, while $\boldsymbol{a}_0(y, y', \eta)$ can be taken to be $\varkappa(y, y')\boldsymbol{a}(y, \eta)$ for a properly supported element $\varkappa \in C^\infty(\Omega \times \Omega)$ that equals 1 in a neighbourhood of $\mathrm{diag}(\Omega \times \Omega)$. Let us call a representative $\mathcal{A}_0$ of $\mathcal{A}$ mod $\mathcal{V}^{-\infty,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ properly supported, if it has these properties (this notion will be employed below for purely technical reasons; it could be avoided completely).

Let us now introduce the principal symbols

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_\wedge(\mathcal{A})) \tag{103}$$

of operators $\mathcal{A} \in \boldsymbol{V}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ (in the interpretation of DN-orders for $\sigma_\partial$ and $\sigma_\wedge$). Writing $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2,3}$ from (102) we get the homogeneous principal interior symbol

$$\sigma_\psi(\mathcal{A}) := \sigma_\psi(\mathcal{A}_{11}). \tag{104}$$

Moreover, (101) gives us the DN-homogeneous principal boundary symbol

$$\sigma_\partial(\mathcal{A}) := \sigma_\partial\big((\mathcal{A}_{ij})_{i,j=1,2}\big). \tag{105}$$

Variables and covariables are $(r, y, \varrho, \eta)$. (105) is a pair of families of continuous operators $\sigma_\partial(\mathcal{A}) = (\sigma_\partial(\mathcal{A})_-, \sigma_\partial(\mathcal{A})_+)$,

$$\sigma_\partial(\mathcal{A})_\pm(r, y, \varrho, \eta) : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-(\iota_\pm)} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_+(\iota_\pm)}, \end{array} \tag{106}$$

for $s > d - \frac{1}{2}$, and homogeneity means

$$\begin{aligned} &\sigma_\partial(\mathcal{A})_\pm(r, y, \lambda\varrho, \lambda\eta) = \\ &= \lambda^\mu \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \sigma_\partial(\mathcal{A})_\pm(r, y, \varrho, \eta) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix}^{-1} \end{aligned} \tag{107}$$

for all $(r, y) \in (\partial I)^\wedge \times \Omega,\ (\varrho, \eta) \in \mathbb{R}^{1+q} \backslash 0,\ \lambda \in \mathbb{R}_+$. Finally, we set

$$\sigma_\wedge(\mathcal{A}) := \sigma_\wedge(\boldsymbol{a}), \tag{108}$$

evaluated in the sense (95). Variables and covariables are $(y, \eta)$. (108) is a familiy of continuous operators

$$\sigma_\wedge(\mathcal{A})(y, \eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}((I, \partial I)^\wedge; \boldsymbol{n}) \\ \oplus \\ \mathbb{C}^{L_-} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}((I, \partial I)^\wedge; \boldsymbol{m}) \\ \oplus \\ \mathbb{C}^{L_+} \end{array} \tag{109}$$

for $s > d - \frac{1}{2}$, and homogeneity means

$$\begin{aligned} &\sigma_\wedge(\mathcal{A})(y, \lambda\eta) = \\ &= \lambda^\mu \begin{pmatrix} \kappa_\lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{\frac{1}{2}}\kappa_\lambda^{(0)} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \sigma_\wedge(\mathcal{A})(y, \eta) \begin{pmatrix} \kappa_\lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{\frac{1}{2}}\kappa_\lambda^{(0)} & 0 \\ 0 & 0 & \lambda \end{pmatrix}^{-1} \end{aligned} \tag{110}$$

for all $(y, \eta) \in \Omega \times (\mathbb{R}^q \backslash 0), \lambda \in \mathbb{R}_+$. We call (109) the (DN-) homogeneous principal crack symbol of the operator $\mathcal{A}$.

**Theorem 2.13.** *Every* $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ *for* $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta),\ \Theta = (-(k+1), 0],\ k \in \mathbb{N} \cup \{\infty\}$ *and* $\boldsymbol{v} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+)$ *induces continuous operators*

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}((I, \partial I)^\wedge \times \Omega; \boldsymbol{n}) \\ \oplus \\ H^{s-1}_{\mathrm{comp}}(\Omega, \mathbb{C}^{L_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathrm{loc}(y)}((I, \partial I)^\wedge \times \Omega; \boldsymbol{m}) \\ \oplus \\ H^{s-\mu-1}_{\mathrm{loc}}(\Omega, \mathbb{C}^{L_+}) \end{array} \tag{111}$$

*for all* $s > d - \frac{1}{2}$. *Moreover, for every* $P \in \mathrm{As}((I, \partial I), (\gamma, \Theta); \boldsymbol{n})$ *there is a* $Q \in \mathrm{As}((I, \partial I), (\gamma - \mu, \Theta); \boldsymbol{m})$ *such that* (111) *restricts to a continuous operator*

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}_{P,\mathrm{comp}(y)}((I, \partial I)^\wedge \times \Omega; \boldsymbol{n}) \\ \oplus \\ H^{s-1}_{\mathrm{comp}}(\Omega, \mathbb{C}^{L_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}_{Q,\mathrm{loc}(y)}((I, \partial I)^\wedge \times \Omega; \boldsymbol{m}) \\ \oplus \\ H^{s-\mu-1}_{\mathrm{loc}}(\Omega, \mathbb{C}^{L_+}) \end{array} \tag{112}$$

*for all* $s > d - \frac{1}{2}$.

The proof of Theorem 2.13 can be given for the summands in the representation (100) separately. Smoothing operators have by definition the asserted mapping property by definition. The result in general is a consequence of the fact that operators $\mathrm{Op}(\boldsymbol{a})$ like in (100) for amplitude functions $\boldsymbol{a}(y,\eta)$ as in Theorem 2.5 or Definition 2.1 induce corresponding continuous mappings, cf. formula (33). This yields, in particular, the following remark.

*Remark* 2.14. Every $\mathcal{G} \in \boldsymbol{\mathcal{V}}^{\mu,d}_{M+G}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ induces continuous operators

$$\mathcal{G}: \begin{array}{c} \mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}((I,\partial I)^\wedge \times \Omega; \boldsymbol{n}) \\ \oplus \\ H^{s-1}_{\mathrm{comp}}(\Omega, \mathbb{C}^{L_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}_{P,\mathrm{loc}(y)}((I,\partial I)^\wedge \times \Omega; \boldsymbol{m}) \\ \oplus \\ H^{s-\mu-1}_{\mathrm{loc}}(\Omega, \mathbb{C}^{L_+}) \end{array}$$

for all $s > d - \frac{1}{2}$, where $P \in \mathrm{As}((I,\partial I),(\gamma-\mu,\Theta);\boldsymbol{m})$ is some $\mathcal{G}$-dependent asymptotic type.

*Remark* 2.15. Let $\varphi = (\varphi_1, \varphi_-, \varphi_+, \varphi')$, $\varphi_1 \in C^\infty(\overline{\mathbb{R}}_+ \times I \times \Omega)$, $\varphi_\pm \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega)$, $\varphi' \in C^\infty(\Omega)$, and let $M_\varphi = \mathrm{diag}(M_{\varphi_1}, M_{\varphi_-}, M_{\varphi_+}, M_{\varphi'})$ denote the corresponding operator of multiplication. Then $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ implies $M_\varphi \mathcal{A}$, $\mathcal{A} M_\varphi \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$. Moreover, if $\varphi$ and $\tilde{\varphi}$ are such tuples of functions, where the components of $\tilde{\varphi}$ equal 0 on the supports of the components of $\varphi$, we have $M_\varphi \mathcal{A} M_{\tilde{\varphi}} \in \boldsymbol{\mathcal{V}}^{-\infty,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$.

This is a simple consequence of standard oscillatory integral arguments.

Choose any $\varphi = (\varphi_1, \varphi_-, \varphi_+, \varphi')$ such that $\varphi_1 \in C_0^\infty(\overline{\mathbb{R}}_+ \times I \times \Omega)$, $\varphi_\pm \in C_0^\infty(\overline{\mathbb{R}}_+ \times \Omega)$, $\varphi' \in C_0^\infty(\Omega)$.

**Theorem 2.16.** $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ *for* $\boldsymbol{g} = (\gamma - \nu, \gamma - \nu - \mu, \Theta)$, $\Theta$ *finite or infinite,* $\boldsymbol{v} = (\boldsymbol{n}_0, \boldsymbol{m}; L_0, L_+)$ *and* $\mathcal{B} \in \boldsymbol{\mathcal{V}}^{\nu,e}(I^\wedge \times \Omega, \boldsymbol{f}; \boldsymbol{u})$ *for* $\boldsymbol{f} = (\gamma, \gamma-\nu, \Theta)$, $\boldsymbol{u} = (\boldsymbol{n}, \boldsymbol{n}_0; L_-, L_0)$ *implies* $\mathcal{A} M_\varphi \mathcal{B} \in \boldsymbol{\mathcal{V}}^{\mu+\nu,h}(I^\wedge \times \Omega, \boldsymbol{g} \circ \boldsymbol{f}; \boldsymbol{v} \circ \boldsymbol{u})$ *for* $h = \max(\nu + d, e)$, $\boldsymbol{g} \circ \boldsymbol{f} = (\gamma, \gamma - \nu - \mu, \Theta)$, $\boldsymbol{v} \circ \boldsymbol{u} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+)$, *and we have* $\sigma(\mathcal{A} M_\varphi \mathcal{B}) = \sigma(\mathcal{A}) \sigma(M_\varphi \mathcal{B})$ (*with componentwise multiplication*).

The main point of the proof of to characterise the composition $\mathrm{Op}(\boldsymbol{a}) \circ M_\varphi \mathrm{Op}(\boldsymbol{b})$ for operator-valued amplitude functions $\boldsymbol{a}(y,\eta), \boldsymbol{b}(y,\eta)$ like in Definition 2.11. The remaining contributions are either smoothing or are understood in the framework of pseudo- differential boundary value problems with the transmission property, cf. Theorem 1.17. To treat $\mathrm{Op}(\boldsymbol{a}) M_\varphi \mathrm{Op}(\boldsymbol{b})$ we can use Remark 2.15 that allows us to reduce the consideration to global amplitude functions, i.e., for $\Omega = \mathbb{R}^q$, with compact support with respect to $y$. In this part of the proof the main aspect is the desired composition behaviour of upper left corners with holomorphic Mellin symbols. This is well- understood by a corresponding abstract machinery, cf. Gil, Seiler, and Schulze [7], or Krainer [16], for a more refined technique. Considering the composition rule for symbols, the only new point is the behaviour for the $\sigma_\wedge$-components, but this is analogous to a corresponding result for the edge theory, cf. [27], Theorem 3.4.39. Another very efficient way to argue for the multiplicativity of edge symbols is a formula from Schulze and Tarkhanov [37], Section 3.4, which can easily be adapted to the present situation.

*Remark* 2.17. If $\mathcal{A}$ or $\mathcal{B}$ in Theorem 2.16 belongs to the corresponding subspace with subscript $M+G$ ($G$), then the same is true of $\mathcal{A}M_\varphi\mathcal{B}$.

This is a consequence of the fact that the pointwise compositions of amplitude functions have such a behaviour, known from the corresponding parameter- dependent theory of boundary value problems on a cone.

Let us emphasize that, due to Definition 2.2 (i), our operators are degenerate in a specific way near the crack boundary. In fact, the conditions encode what is called edge-degeneracy, where $\Omega$ is the edge of $I^\wedge \times \Omega$ or of $(\partial I)^\wedge \times \Omega$. Compared with the original problems, that have been formulated in forms of operators $A$ with smooth coefficients, cf. formula (2), the edge-degenerate behaviour is much more general. A similar remark holds about the boundary conditions $T_\pm$, cf. the assumptions on the operators (3). The edge-degenerate form appeared by substituting polar coordinates, and one may expect that our operator algebra contains a relevant subalgebra, where the interior and boundary symbols are regular in a similar sense as the original symbols in problems of the type (1). In a sense, this is true, indeed, cf. Section 3.1 below. Nevertheless, the framework of edge-degenerate symbols and operators is absolutely natural and necessary. The reason is that the calculus (e.g., in the frame of parametrix constructions) also produces (classical) pseudo-differential operators on the $\pm$-sides $S_\pm$ of the crack (their homogeneous principal symbols are just the right lower corners in the block matrices (110)). These have not the transmission property with respect to the crack boundary, even if we consider the subalgebra generated by differential crack problems (1) and parametrices of elliptic elements. Now these operators on $S_\pm$ are typically to be studied as edge operators (where the edge in this case is the boundary of the crack), cf. the monograph [25]. In the full calculus the corresponding degeneracy also entails (e.g., under compositions) edge-degenerate symbols in other entries of the block matrices, and, as it turns out, the more special calculus is not easier than the full one.

From the conditions in Definition 2.2 we see that

$$\sigma_{\psi,F}(\mathcal{A})(r,\phi,y,\varrho,\vartheta,\eta) := r^\mu \sigma_\psi(\mathcal{A})(r,\phi,y,r^{-1}\varrho,\vartheta,r^{-1}\eta) \qquad (113)$$

is smooth up to $r=0$. For similar reasons

$$\sigma_{\partial,F}(\mathcal{A})_\pm(r,y,\varrho,\eta) := r^\mu \sigma_\partial(\mathcal{A})_\pm(r,y,r^{-1}\varrho,r^{-1}\eta) \qquad (114)$$

is smooth up to $r=0$. Also (114) represents families of continuous operators (107) with an evident analogue of the homogeneity (108).

**2.4. Ellipticity, parametrices, and asymptotics of solutions.** We now turn to the ellipticity of local crack operators. Intuitively, ellipticity is the bijectivity of all components of the principal symbol. However, in the present situation we do not exclude the (in fact typical) edge-degenerate behaviour of our operators. In this context it is necessary to employ a corresponding adequate terminology.

**Defintion 2.18.** An operator $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ for $(\mu, d) \in \mathbb{Z} \times \mathbb{N}, \boldsymbol{g} = (\gamma, \gamma - \mu, \Theta),\ \boldsymbol{v} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+),\ \boldsymbol{n} = (m, N_-(\iota_-), N_-(\iota_+)),\ \boldsymbol{m} = (m, N_+(\iota_-), N_+(\iota_+))$ is called elliptic if

(i) $\sigma_{\psi,F}(\mathcal{A})(r, \phi, y, \varrho, \vartheta, \eta) \neq 0$ for all $(r, \phi, y, \varrho, \vartheta, \eta) \in \overline{\mathbb{R}}_+ \times I \times \Omega \times (\mathbb{R}^{2+q} \setminus 0)$,

(ii)

$$\sigma_{\partial,F}(\mathcal{A})_\pm(r, y, \varrho, \eta) : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-(\iota_\pm)} \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_+(\iota_\pm)} \end{matrix}$$

are isomorphisms for all $(r, y, \varrho, \eta) \in \overline{\mathbb{R}}_+ \times \Omega \times (\mathbb{R}^{1+q} \setminus 0)$ (for some $s = s_0 > \max(\mu, d) - \frac{1}{2}$), both for the $+$ and $-$ sign,

(iii)

$$\sigma_\wedge(\mathcal{A})(y, \eta) : \begin{matrix} \mathcal{K}^{s,\gamma}((I, \partial I)^\wedge; \boldsymbol{n}) \\ \oplus \\ \mathbb{C}^{L_-} \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}^{s-\mu,\gamma-\mu}((I, \partial I)^\wedge; \boldsymbol{m}) \\ \oplus \\ \mathbb{C}^{L_+} \end{matrix}$$

are isomorphisms for all $(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus 0)$ (for some $s = s_0 > \max(\mu, d) - \frac{1}{2}$).

*Remark* 2.19. Condition (ii) or (iii) imply corresponding bijectivities for all $s > \max(\mu, d) - \frac{1}{2}$.

Observe that condition (iii) shows that the $2 \times 2$-upper left corner of $\sigma_\wedge(\mathcal{A})(y, \eta)$

$$\sigma_\wedge((\mathcal{A}_{ij})_{i,j=1,2})(y, \eta) : \mathcal{K}^{s,\gamma}((I, \partial I)^\wedge; \boldsymbol{n}) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}((I, \partial I)^\wedge; \boldsymbol{m}) \quad (115)$$

is a family of Fredholm operators, parametrised by $(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus 0)$. For every fixed $(y, \eta)$ the operators belong to the cone algebra of boundary value problems (with the transmission property) on the infinite stretched cone $I^\wedge$ with boundary $(\partial I)^\wedge = \mathbb{R}_+ \cup_{\mathrm{d}} \mathbb{R}_+$. As such they are necessarily elliptic with respect to the symbol hierarchy of the cone theory which consists of several components $(\sigma_{\psi,\mathrm{cone}}, \sigma_{\partial,\mathrm{cone}}, \sigma_M, \sigma_{\mathrm{exit,cone}})$, namely the (Fuchs-type) principal interior and boundary symbols, the conormal symbol $\sigma_M$ and the tuple of exit interior and boundary symbols in the sense of Kapanadze and Schulze [14]. The exit components are automatically elliptic as a consequence of conditions (i), (ii). Moreover, the Fredholm property of (115) entails the ellipticity with respect to $(\sigma_{\psi,\mathrm{cone}}, \sigma_{\partial,\mathrm{cone}}, \sigma_M)$, where that of $\sigma_{\psi,\mathrm{cone}}$ and $\sigma_{\partial,\mathrm{cone}}$ is again automatic from conditions (i), (ii). The conormal symbol is of particular interest. It consists of an operator function

$$\sigma_M(\mathcal{A})(y,z) : \begin{array}{c} H^s(I,\mathbb{C}^m) \\ \oplus \\ \mathbb{C}^{N_-(\iota_-)+N_-(\iota_+)} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(I,\mathbb{C}^m) \\ \oplus \\ \mathbb{C}^{N_+(\iota_-)+N_+(\iota_+)} \end{array} \tag{116}$$

$s > \max(\mu, d) - \frac{1}{2}$ that is a family of isomorphisms for all $y \in \Omega$, $z \in \Gamma_{1-\gamma}(= \{z : \mathrm{Re}\, z = 1-\gamma\})$. By construction we have $\sigma_M(\mathcal{A})(y,z) \in C^\infty(\Omega, \boldsymbol{\mathcal{M}}_R^{\mu,d}(I;\boldsymbol{w}))$, $\boldsymbol{w} = (\boldsymbol{n},\boldsymbol{m})$, for some (continuous) Mellin asymptotic type $R$ with $\Gamma_{1-\gamma} \cap$ carrier$R = \emptyset$.

**Defintion 2.20.** Given $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ for $(\mu, d) \in \mathbb{Z} \times \mathbb{N}$, $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, $\boldsymbol{v} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+)$, an operator $\mathcal{P} \in \boldsymbol{\mathcal{V}}^{-\mu,e}(I^\wedge \times \Omega, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ for a certain $e \in \mathbb{N}$ and $\boldsymbol{g}^{-1} = (\gamma-\mu, \gamma, \Theta)$, $\boldsymbol{v}^{-1} = (\boldsymbol{m}, \boldsymbol{n}; L_+, L_-)$, is called a parametrix of $\mathcal{A}$, if for arbitrary $\varphi, \psi \in C_0^\infty(\overline{\mathbb{R}}_+ \times I \times \Omega) \times C_0^\infty(\overline{\mathbb{R}}_+ \times \Omega) \times C_0^\infty(\overline{\mathbb{R}}_+ \times \Omega) \times C_0^\infty(\Omega)$, $\varphi = (\varphi_1, \varphi_-, \varphi_+, \varphi')$, $\psi = (\psi_1, \psi_-, \psi_+, \psi')$ such that $\psi_1 \equiv 1$ on $\mathrm{supp}\,\varphi_1$, $\psi_\pm \equiv 1$ on $\mathrm{supp}\,\varphi_\pm$, $\psi' \equiv 1$ on $\mathrm{supp}\,\varphi'$, we have

$$M_\varphi \mathcal{P} M_\psi \mathcal{A} - M_\varphi \mathcal{I} \in \boldsymbol{\mathcal{V}}^{-\infty,d_l}(I^\wedge \times \Omega, \boldsymbol{g}_l; \boldsymbol{v}_l),$$
$$M_\psi \mathcal{A} M_\varphi \mathcal{P} - M_\varphi \mathcal{I} \in \boldsymbol{\mathcal{V}}^{-\infty,d_r}(I^\wedge \times \Omega, \boldsymbol{g}_r; \boldsymbol{v}_r)$$

for certain types $d_l, d_r \in \mathbb{N}$, where $\boldsymbol{g}_l = (\gamma, \gamma, \Theta)$, $\boldsymbol{v}_l = (\boldsymbol{n}, \boldsymbol{n}; L_-, L_-)$ and $\boldsymbol{g}_r = (\gamma-\mu, \gamma-\mu, \Theta)$, $\boldsymbol{v}_l = (\boldsymbol{m}, \boldsymbol{m}; L_+, L_+)$.

*Remark* 2.21. If $\mathcal{P}$ is a parametrix of $\mathcal{A}$, Theorem 2.16 gives us $\sigma(\mathcal{P}) = \sigma(\mathcal{A})^{-1}$ with the componentwise inversion.

**Theorem 2.22.** *An elliptic operator* $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ *has a parametrix* $\mathcal{P} \in \boldsymbol{\mathcal{V}}^{-\mu,(d-\mu)^+}(I^\wedge \times \Omega, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$, *where the types of the remainders in Definition* 2.20 *are* $d_l = \max(\mu, d)$ *and* $d_r = (d-\mu)^+$ *(recall that* $\nu^+ = \max(\nu, 0)$*).*

The proof is a consequence of the fact that for the tuple of symbols $\sigma(\mathcal{A})^{-1}$ there exists an operator $\mathcal{P}_0 \in \boldsymbol{\mathcal{V}}^{-\mu,(d-\mu)^+}(I^\wedge \times \Omega, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ such that $\sigma(\mathcal{A})^{-1} = \sigma(\mathcal{P}_0)$. The idea for the construction of $\mathcal{P}_0$ is analogous to that for pseudo-differential boundary value problems. Compared with that case, the new point is to include the edge symbol component. First we form an operator $\mathcal{Q} \in \boldsymbol{\mathcal{V}}^{-\mu,(d-\mu)^+}(I^\wedge \times \Omega, \boldsymbol{g}^{-1}; \boldsymbol{w}^{-1})$ with $\sigma_\psi^{-1}(\mathcal{A}) = \sigma_\psi(\mathcal{Q}), \sigma_\partial^{-1}(\mathcal{A}) = \sigma_\partial(\mathcal{Q})$; such a $\mathcal{Q}$ can be found by the machinery of boundary value problems (see, for instance, Kapanadze and Schulze [14], Section 2.6). Then

$$\sigma_\wedge(\mathcal{Q})(y,\eta) : \mathcal{K}^{s-\mu,\gamma-\mu}((I,\partial I)^\wedge; \boldsymbol{m}) \longrightarrow \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge; \boldsymbol{n})$$

is a Fredholm family that can be filled up to a family of isomorphisms. Similarly to a corresponding argument in boundary value problems, without loss of generality we may assume that this family has the form

$$\begin{array}{ccc} \mathcal{K}^{s-\mu,\gamma-\mu}((I,\partial I)^\wedge;\boldsymbol{m}) & & \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}) \\ \oplus & \longrightarrow & \oplus \\ \mathbb{C}^{L_+} & & \mathbb{C}^{L_-} \end{array}$$

i.e., with the dimensions $L_-$ and $L_+$ from the given operator $\mathcal{A}$. This is already the edge symbol of some elliptic operator $\mathcal{Q}_0 \in \boldsymbol{\mathcal{V}}^{-\mu,(d-\mu)^+}(I^\wedge \times \Omega, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$, and it has the property $\sigma_\wedge(\mathcal{Q}_0)(y,\eta)\sigma_\wedge(\mathcal{A})(y,\eta) = 1 + g_{(0)}(y,\eta)$ for some edge symbol $g_{(0)}(y,\eta)$ of an operator of Green type. Since $1 + g_{(0)}(y,\eta)$ is invertible in our class, we can pass to the composition $(1 + g_{(0)}(y,\eta))\sigma_\wedge(\mathcal{Q}_0)(y,\eta)$ which is the edge symbol of the desired $\mathcal{P}_0$. A standard argument allows us to pass to a properly supported operator $\mathcal{P}_0$, cf. Remark 2.12 . A formal Neumann series argument then yields $\mathcal{P}$ itself, more precisely $\mathcal{P} \sim \{\sum_{j=0}^{\infty}(-1)^j(\mathcal{P}_0\mathcal{A} - \mathcal{I})^j\}\mathcal{P}_0$.

**Theorem 2.23.** *Let $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ be elliptic. Then*

$$u \in \mathcal{W}^{r,\gamma}_{\mathrm{comp}(y)}((I,\partial I)^\wedge \times \Omega; \boldsymbol{n}) \oplus H^{r-1}_{\mathrm{comp}}(\Omega, \mathbb{C}^{L_-})$$

*for some $r > \max(\mu, d) - \frac{1}{2}$ and*

$$\mathcal{A}u \in \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathrm{loc}(y)}((I,\partial I)^\wedge \times \Omega; \boldsymbol{m}) \oplus H^{s-\mu-1}_{\mathrm{loc}}(\Omega, \mathbb{C}^{L_+})$$

*for some $s$ with $s > \max(\mu, d) - \frac{1}{2}$ implies*

$$u \in \mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}((I,\partial I)^\wedge \times \Omega; \boldsymbol{n}) \oplus H^{s-1}_{\mathrm{comp}}(\Omega, \mathbb{C}^{L_-}).$$

*In addition, $\mathcal{A}u \in \mathcal{W}^{s-\mu,\gamma-\mu}_{Q,\mathrm{loc}(y)}((I,\partial I)^\wedge \times \Omega; \boldsymbol{m}) \oplus H^{s-\mu-1}_{\mathrm{loc}}(\Omega, \mathbb{C}^{L_+})$ for some continuous asymptotic type $Q \in \mathrm{As}((I,\partial I), (\gamma - \mu, \Theta); \boldsymbol{m})$ implies $u \in \mathcal{W}^{s,\gamma}_{P,\mathrm{comp}(y)}((I,\partial I)^\wedge \times \Omega; \boldsymbol{n}) \oplus H^{s-1}_{\mathrm{comp}}(\Omega, \mathbb{C}^{L_-})$ for some resulting $P \in \mathrm{As}((I,\partial I), (\gamma, \Theta); \boldsymbol{n})$.*

The result of Theorem 2.23 can be interpreted as elliptic regularity. Using Theorem 2.22 and Remark 2.12 there is a properly supported parametrix $\mathcal{P}$ of $\mathcal{A}$. Then, multiplying $\mathcal{A}u = f$ from the left by $\mathcal{P}$ we get

$$\mathcal{P}\mathcal{A}u = (1 + \mathcal{G})u = \mathcal{P}f \tag{117}$$

for some $\mathcal{G} \in \boldsymbol{\mathcal{V}}^{-\infty,d_l}(I^\wedge \times \Omega, \boldsymbol{g}_l; \boldsymbol{v}_l)$. By virtue of Theorem 2.13 the element $\mathcal{P}f$ belongs to the desired space (with or without asymptotics, according to the assumption about $f$), while $\mathcal{G}u$ is smooth with asymptotics, cf. Section 2.2. In other words, (117) yields the assertion.

**2.5. Invariance and global calculus.** We now return to the global situation, i.e., crack problems, in a bounded domain $G$ with an embedded compact crack $S$ of codimension 1. More generally, we may replace $\overline{G}$ by a smooth compact manifold $M$ with boundary and the crack by a oriented, smooth and compact submanifold $S \subset \text{int}\, M$ with boundary $Y$. We define $M_{\text{crack}}$ to be $(M \backslash S) \cup \text{int}\, S_- \cup \text{int}\, S_+$, i.e., a (non-compact) smooth manifold having a smooth boundary with the components $\text{int}\, S_-, \text{int}\, S_+$ and $\partial M$. Similarly to Section 1.5 we have the space of pseudo-differential boundary value problems $\boldsymbol{\mathcal{B}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{b})$ of order $\mu \in \mathbb{Z}$, type $d$, and the dimension data $\boldsymbol{b} = (k, m; N_-(\iota_-), N_-(\iota_+); N_+(\iota_-), N_+(\iota_+); b_-, b_+)$ (with $(k, m)$ belonging to the interior operator, $(N_-(\iota_-), N_+(\iota_-))$ to $\text{int}\, S_-$, $(N_-(\iota_+), N_+(\iota_+))$ to $\text{int}\, S_+$ and $(b_-, b_+)$ to $\partial M$). Concerning the constructions in a neighbourhood of $S$ we choose a tubular neighbourhood of $Y = \partial S$ of the form $B \times Y$, where $B$ is the unit disk in $\mathbb{R}^2$. Thus, in the discussion of invariance properties of the local (wedge) Sobolev spaces and of our operators, we may assume that transition maps $B \times \Omega \to B \times \tilde{\Omega}$ for different charts $\chi : U \to \Omega$, $\tilde{\chi} : \tilde{U} \to \tilde{\Omega}$ on $Y$ to open sets in $\mathbb{R}^q$ are independent of $(x_1, x_2) \in B$. It is known from the general calculus of pseudo-differential operators with operator-valued symbols, cf. Schulze [33], Section 3.2.2 and 3.2.5 or [27], Section 3.4.4, that both the spaces and subspaces with asymptotics (for the case of closed compact bases of model cones) as well as the operator-valued symbols behave invariant under corresponding transition maps. This can easily be adapted to the present situation. In other words, using standard procedures with partitions of unity we can construct global crack operators.

The definition of the space $\boldsymbol{\mathcal{V}}^{-\infty,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ of global smoothing operators (with continuous asymptotics near $Y$) relies on chosen weight data $\boldsymbol{g} = (\gamma, \delta, \Theta)$, $\Theta = (\vartheta, 0]$, and on the dimensions contained in $\boldsymbol{v} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+)$ (cf. the notation in Definition 2.20) as well as on the contributions $(b_-, b_+)$ from $\partial M$. In other words, we set

$$\boldsymbol{w} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+; b_-, b_+). \tag{118}$$

Let us first introduce the corresponding spaces: Define

$$C^\infty(M_{\text{crack}}, \mathbb{C}^k)$$

to be the subspace of all $u \in C^\infty(M \backslash S, \mathbb{C}^k)$ that are $C^\infty$ up to $\text{int}\, S$ from both sides, further

$$C_0^\infty(M_{\text{crack}}, \mathbb{C}^k) := \{u \in C^\infty(M_{\text{crack}}, \mathbb{C}^k) : \text{supp}\, u \cap Y = \emptyset\}.$$

Let $\boldsymbol{b}_- := (\boldsymbol{n}, L_-, b_-)$ for $\boldsymbol{n} = (k, N_-(\iota_-), N_-(\iota_+))$, and set

$$C_0^\infty(M_{\text{crack}}, \boldsymbol{b}_-) = C_0^\infty(M_{\text{crack}}, \mathbb{C}^k) \oplus C_0^\infty(\text{int}\, S_-, \mathbb{C}^{N_-(\iota_-)}) \oplus \\ \oplus C_0^\infty(\text{int}\, S_+, \mathbb{C}^{N_-(\iota_+)}) \oplus C^\infty(Y, \mathbb{C}^{L_-}) \oplus C^\infty(\partial M, \mathbb{C}^{b_-})$$

and for $\boldsymbol{b}_+ := (\boldsymbol{m}, L_+, b_+)$, $\boldsymbol{m} = (m, N_+(\iota_-), N_+(\iota_+))$,

$$C^\infty(M_{\text{crack}}, \boldsymbol{b}_+) = C^\infty(M_{\text{crack}}, \mathbb{C}^m) \oplus C^\infty(\text{int } S_-, \mathbb{C}^{N_+(\iota_-)}) \oplus$$
$$\oplus C^\infty(\text{int } S_+, \mathbb{C}^{N_+(\iota_+)}) \oplus C^\infty(Y, \mathbb{C}^{L_+}) \oplus C^\infty(\partial M, \mathbb{C}^{b_+}).$$

Moreover, define the weighted crack Sobolev spaces and subspaces with asymptotics, namely

$$\mathcal{W}^{s,\gamma}(M_{\text{crack}}, \boldsymbol{b}_-) = \mathcal{W}^{s,\gamma}(M_{\text{crack}}, \mathbb{C}^k) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(S_-, \mathbb{C}^{N_-(\iota_-)})$$
$$\oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(S_+, \mathbb{C}^{N_-(\iota_+)}) \oplus H^{s-1}(Y, \mathbb{C}^{L_-}) \oplus H^{s-\frac{1}{2}}(\partial M, \mathbb{C}^{b_-}), \quad (119)$$

where $\mathcal{W}^{s,\gamma}(M_{\text{crack}}, \mathbb{C}^k)$ is locally near $Y$ given by $\mathcal{W}^{s,\gamma}(I^\wedge \times \mathbb{R}^q, \mathbb{C}^k)$, locally near int $S_\pm$ by standard ($\mathbb{C}^k$-valued) Sobolev spaces (with Sobolev smoothness $s$ up to int $S_\pm$ from both sides), and locally near $\partial M$ by standard ($\mathbb{C}^k$-valued) Sobolev spaces of smoothnes $s$ up to $\partial M$. Moreover, $\mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(S_\pm, \mathbb{C}^l)$ (for $l = N_-(\iota_-)$ or $N_-(\iota_+)$) is the weighted Sobolev space of smoothness $s - \frac{1}{2}$ and weight $\gamma - \frac{1}{2}$, defined to be the subspace of all elements of $H^s_{\text{loc}}(\text{int } S_\pm, \mathbb{C}^l)$ that are locally near $Y$ given by $\mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^q, \mathbb{C}^l)$ where $\mathbb{R}_+$ plays the role of the inner normal of $Y$ in $S_\pm$, $q = \dim Y$. Given a (continuous) asymptotic type $P \in \text{As}((I, \partial I), (\gamma, \Theta), \boldsymbol{n})$, $P = (P_1, P_-, P_+)$, we set

$$\mathcal{W}^{s,\gamma}_P(M_{\text{crack}}, \boldsymbol{b}_-) = \mathcal{W}^{s,\gamma}_{P_1}(M_{\text{crack}}, \mathbb{C}^k) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{P_-}(S_-, \mathbb{C}^{N_-(\iota_-)})$$
$$\oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{P_+}(S_+, \mathbb{C}^{N_-(\iota_+)}) \oplus H^{s-1}(Y, \mathbb{C}^{L_-}) \oplus H^{s-\frac{1}{2}}(\partial M, \mathbb{C}^{b_-}).$$

In particular, similarly to (97), we can form the $C^\infty$-spaces with asymptotics

$$C^{\infty,\gamma}_P(M_{\text{crack}}, \boldsymbol{b}_-) = \mathcal{W}^{\infty,\gamma}_P(M_{\text{crack}}, \boldsymbol{b}_-).$$

The dimensions contained in $\boldsymbol{b}_-$ are arbitrary, as far as we talk about the spaces in general, so we have, in particular, corresponding versions with $\boldsymbol{b}_+$. Finally, we employ spaces with supercripts $(s^*, \gamma^*)$ in the same convention as before, concerning the first three components, $s+1$ for the forth, and $s+\frac{1}{2}$ for the last component. In the latter notation, $u$ is a tuple $(u_1, u_-, u_+, u', u'')$, according to the components of elements in $C^\infty(M_{\text{crack}}, \boldsymbol{b}_-)$. The spaces $\mathcal{W}^{s,\gamma}_P$, $C^{\infty,\gamma}_P$ and $C^\infty$ on $M_{\text{crack}}$ (with the various dimension data) are Fréchet in a canonical way, while the spaces $\mathcal{W}^{s,\gamma}_P$ on $M_{\text{crack}}$ are Banach spaces with norms that may be generated by suitable Hilbert space scalar products. Let $(.,.)_{\boldsymbol{b}_-}$ denote the scalar product of the space

$$\mathcal{W}^{0,0}(M_{\text{crack}}, \mathbb{C}^k) \oplus \mathcal{W}^{0,0}(S_-, \mathbb{C}^{N_-(\iota_-)}) \oplus \mathcal{W}^{0,0}(S_+, \mathbb{C}^{N_+(\iota_+)})$$
$$\oplus L^2(Y, \mathbb{C}^{L_-}) \oplus L^2(\partial M, \mathbb{C}^{b_-}).$$

Now $\boldsymbol{\mathcal{V}}^{-\infty,0}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ for $\boldsymbol{g} = (\gamma, \delta, \Theta)$, $\boldsymbol{w} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+; b_-, b_+)$, is the space of all continuous operators $\mathcal{C} : C^\infty_0(M_{\text{crack}}, \boldsymbol{b}_-) \to C^\infty(M_{\text{crack}}, \boldsymbol{b}_+)$

such that there are asymptotic types $P \in \mathrm{As}((I, \partial I), (\delta, \Theta); \boldsymbol{m})$, $Q \in \mathrm{As}((I, \partial I), ((-\gamma)^*, \Theta); \boldsymbol{n})$ such that

$$\mathcal{C} : \mathcal{W}^{s,\gamma}(M_{\mathrm{crack}}, \boldsymbol{b}_-) \to C_P^{\infty,\delta}(M_{\mathrm{crack}}, \boldsymbol{b}_+)$$

and

$$\mathcal{C}^* : \mathcal{W}^{s^*,(-\delta)^*}(M_{\mathrm{crack}}, \boldsymbol{b}_+) \to C_Q^{\infty,(-\gamma)^*}(M_{\mathrm{crack}}, \boldsymbol{b}_-)$$

are continuous for all $s > -\frac{1}{2}$, with $\mathcal{C}^*$ being taken to be the formal adjoint in sense

$$(\mathcal{C}u, v)_{\boldsymbol{b}_+} = (u, \mathcal{C}^* v)_{\boldsymbol{b}_-}$$

for all $u \in C_0^\infty(M_{\mathrm{crack}}, \boldsymbol{b}_-)$, $v \in C_0^\infty(M_{\mathrm{crack}}, \boldsymbol{b}_+)$. Moreover, an operator $\mathcal{C}$ is said to belong to $\boldsymbol{\mathcal{V}}^{-\infty,d}(M_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{w})$, the space of smoothing operators of type $d \in \mathbb{N}$, if it has the form

$$\mathcal{C} = \mathcal{C}_0 + \mathrm{diag}\,(D^j, 0, 0, 0, 0),$$

where $D$ is any differential operator of first order in $M \backslash S$ with $C^\infty$ cofficients up to $\partial M$ and up to int $S_\pm$ (from both sides) generated by a vector field that is transversal to $\partial M$ and to int $S_\pm$ and localises near $Y$ to $\partial_\phi$ in coordinates from $I^\wedge \times \Omega$, cf. the corresponding definition in Section 2.2.

*Remark* 2.24. Let $\mathcal{C} \in \mathcal{V}^{-\infty,d}(M_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{w})$ for $\boldsymbol{g} = (\gamma, \gamma, \Theta)$, $\boldsymbol{w} = (\boldsymbol{n}, \boldsymbol{n}; L_-, L_-; b_-, b_-)$ and assume that

$$\mathcal{I} + \mathcal{C} : \mathcal{W}^{s,\gamma}(M_{\mathrm{crack}}, \boldsymbol{b}_-) \to \mathcal{W}^{s,\gamma}(M_{\mathrm{crack}}, \boldsymbol{b}_-) \tag{120}$$

is an invertible operator for some $s = s_0 > d - \frac{1}{2}$ ($\mathcal{I}$ is the identity operator in the corresponding space). Then (120) is invertible for all $s > d - \frac{1}{2}$, and there is a $\mathcal{G} \in \mathcal{V}^{-\infty,d}(M_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{w})$ such that $(\mathcal{I} + \mathcal{C})^{-1} = \mathcal{I} + \mathcal{G}$.

The elements of $\boldsymbol{\mathcal{V}}^{-\infty,d}(M_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{w})$ with the dimension data (118) as well as those of the general crack operator spaces of arbitrary orders $\mu$ (to be defined below) are block matrices

$$\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2,3,4,5}, \tag{121}$$

according to the meaning of the components of $\boldsymbol{w}$. They will represent continuous operators

$$\mathcal{A} : \bigoplus_{i=1}^{5} E_i \longrightarrow \bigoplus_{j=1}^{5} F_j,$$

where the subcripts $1, 2, 3, 4$ and $5$ denote spaces of distributions on int $M \backslash S$, int $S_-$, int $S_+$, $Y$ and $\partial M$, respectively. Let us form the submatrix $\mathcal{A}_{\mathrm{crack}}$ of all entries $\mathcal{A}_{ij}$ for $i \neq 5$ or $j \neq 5$ and the submatrix $\mathcal{B}$ of all entries $\mathcal{A}_{ij}$ for $i \neq 4$ or $j \neq 4$.

**Defintion 2.25.** Given $(\mu, d) \in \mathbb{Z}\times\mathbb{N}$, $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$ and dimension data (118), the space $\boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ is defined to be the set of all operators

$$\mathcal{A} + \mathcal{C}, \qquad \mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2,3,4,5},$$

such that

(i) $\mathcal{A}_{11} \in L^{\mu}_{\text{cl}}(\text{int}\, M\backslash S) \otimes \mathbb{C}^m \otimes \mathbb{C}^k$,

(ii) the matrix $\mathcal{B}$ belongs to $\boldsymbol{\mathcal{B}}^{\mu,d}(M_{\text{crack}}; \boldsymbol{b})$,

(iii) locally near $Y$ the matrix $\mathcal{A}_{\text{crack}}$ is a crack operator in the sense of Definition 2.11, i.e., if we consider any neighbourhood of a point $y \in Y$, modelled by $I^{\wedge}\times\Omega$, and if further $\varphi = (\varphi_1, \varphi_-, \varphi_+, \varphi')$ and $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_-, \tilde{\varphi}_+, \tilde{\varphi}')$ are $C^\infty$ functions in that neighbourhood, compactly supported like in Theorem 2.16, we have $M_\varphi \mathcal{A}_{\text{crack}} M_{\tilde{\varphi}} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^{\wedge} \times \Omega, \boldsymbol{g}; \boldsymbol{v})$,

(iv) $\mathcal{C} \in \boldsymbol{\mathcal{V}}^{-\infty,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$.

Let $\boldsymbol{\mathcal{V}}^{\mu,d}_{M+G}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ ($\boldsymbol{\mathcal{V}}^{\mu,d}_{G}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$) denote the subspace of all $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ such that (in the notation of Definition 2.25) $\mathcal{B} \in \mathcal{B}^{-\infty,d}(M_{\text{crack}}; \boldsymbol{b})$ and $M_\varphi \mathcal{A}_{\text{crack}} M_{\tilde{\varphi}} \in \boldsymbol{\mathcal{V}}^{\mu,d}_{M+G}(I^{\wedge}\times\Omega, \boldsymbol{g}; \boldsymbol{v})$ ($\boldsymbol{\mathcal{V}}^{\mu,d}_{G}(I^{\wedge}\times\Omega, \boldsymbol{g}; \boldsymbol{v})$) for all $\varphi$ and $\tilde{\varphi}$.

Let us now define the global principal symbol structure of operators

$$\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w}).$$

We set

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_\wedge(\mathcal{A})),$$

called the principal symbol of $\mathcal{A}$ (in DN-orders), where (in the notation of Definition 2.25)

(i) $\sigma_\psi(\mathcal{A}) := \sigma_\psi(\mathcal{B}) = \sigma_\psi(\mathcal{A}_{11})$ is the homogeneous principal interior symbol of order $\mu$ of the upper left corner of $\mathcal{A}$,

(ii) $\sigma_\partial(\mathcal{A}) := (\sigma_\partial(\mathcal{A})_-, \sigma_\partial(\mathcal{A})_+, \sigma_\partial(\mathcal{A})_0)$, where $\sigma_\partial(\mathcal{A})_\pm := \sigma_\partial(\mathcal{B})_\pm$ is the homogeneous principal boundary symbol of $\mathcal{B}$ with respect to int $S_\pm$, and $\sigma_\partial(\mathcal{A})_0$ is the principal boundary symbol of $\mathcal{B}$ with respect to $\partial M$ (always in DN-orders),

(iii) $\sigma_\wedge(\mathcal{A}) := \sigma_\wedge(\mathcal{A}_{\text{crack}})$ is the homogeneous principal crack symbol, locally defined by (109) through $M_\varphi \mathcal{A}_{\text{crack}} M_{\tilde{\varphi}}$, where $\varphi$ and $\tilde{\varphi}$ are chosen to be functions that equal 1 in a neighbourhood of a point $y \in \Omega$ where we just evaluate the symbol (which is an invariant definition, independent of the choice of $\varphi$ and $\tilde{\varphi}$).

The homogeneous principal interior symbol

$$\sigma_\psi(\mathcal{A})(x, \xi) : \mathbb{C}^k \longrightarrow \mathbb{C}^m, \tag{122}$$

is defined for all $(x, \xi) \in T^*(\text{int } M\backslash S)\backslash 0$, and it has smooth boundary values near int $S_\pm$ (i.e., from both sides of $S\backslash Y$). Moreover, locally near any $y \in Y$ it is edge-degenerate in stretched coordinetes $(r, \phi, y) \in I^{\wedge} \times \Omega$ i.e., that it has the form

$$r^{-\mu}\sigma_{\psi,F}(\mathcal{A})(r, \phi, y, r\varrho, \vartheta, r\eta) \tag{123}$$

for a function $\sigma_{\psi,F}(\mathcal{A})(r,\phi,y,\tilde{\varrho},\vartheta,\tilde{\eta}) \in C^\infty(T^*(\overline{\mathbb{R}}_+ \times I \times \Omega)\setminus 0)$ that is homogeneous of order $\mu$ in $(\tilde{\varrho},\vartheta,\tilde{\eta}) \neq 0$, cf. formula (113). The components of the boundary symbol near int $S_\pm$ are families of mappings

$$\sigma_\partial(\mathcal{A})_\pm(x',\xi') : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_-(\iota_\pm)} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N_+(\iota_\pm)} \end{array}, \tag{124}$$

$(x',\xi') \in T^*(\text{int } S_\pm)\setminus 0, s > d - \frac{1}{2}$; locally near any $y \in Y$ they are edge-degenerate in stretched coordinates $(r,y) \in (\partial I)^\wedge \times \Omega$ i.e., that they have the form

$$r^{-\mu}\sigma_{\partial,F}(\mathcal{A})_\pm(r,y,r\varrho,r\eta)$$

for families $\sigma_{\partial,F}(\mathcal{A})_\pm(r,y,\tilde{\varrho},\tilde{\eta})$ of analogous structure, now defined for $(r,y,\tilde{\varrho},\tilde{\eta}) \in \overline{\mathbb{R}}_+ \times \Omega \times (\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}\setminus 0)$ and DN-homogeneous of order $\mu$ in $(\tilde{\varrho},\tilde{\eta}) \neq 0$, cf. formula (114). The third component of the boundary symbol is the usual one:

$$\sigma_\partial(\mathcal{A})_0(x',\xi') : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{b_-} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{b_+} \end{array} \tag{125}$$

$(x',\xi') \in T^*(\partial M)\setminus 0, s > d - \frac{1}{2}$, cf. formula (54). Finally, the crack symbol is a family of mappings

$$\sigma_\wedge(\mathcal{A})(y,\eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge;\boldsymbol{n}) \\ \oplus \\ \mathbb{C}^{L_-} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}((I,\partial I)^\wedge;\boldsymbol{m}) \\ \oplus \\ \mathbb{C}^{L_+} \end{array}, \tag{126}$$

$(y,\eta) \in T^*Y\setminus 0, s > d - \frac{1}{2}$, where the entries are DN-homogeneous in the sense of relation (110).

*Remark* 2.26. We could easily define our crack operator spaces $\boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}},\boldsymbol{g};\boldsymbol{w})$ in the context of maps between spaces of distributional sections of vector bundles $E,F \in \text{Vect}(M)$, $G^\pm_{(\iota_\pm)} \in \text{Vect}(S_\pm)$, $J^\pm \in \text{Vect}(\partial M)$ and $V^\pm \in \text{Vect}(Y)$. In this case $\boldsymbol{w}$ would be a corresponding tuple $\boldsymbol{w} = (E,F;G^-_{(\iota_-)},G^+_{(\iota_-)};G^-_{(\iota_+)},G^+_{(\iota_+)};V^-,V^+;J^-,J^+)$. The symbols themselves would be bundle homomorphisms between corresponding infinite-dimensional fibres (except for $\sigma_\psi$), cf. formulas (52) and (54). We do not elaborate the corresponding formalism in detail (which is straightforward anyway) but return to the case of trivial bundles on $M$, $S_\pm$, $Y$ and $\partial M$.

In the following assertions $\boldsymbol{w}$ is given by (118) and

$$\boldsymbol{b}_- = (\boldsymbol{n};L_-,b_-), \qquad \boldsymbol{b}_+ = (\boldsymbol{m};L_+,b_+)$$

for $\boldsymbol{n} = (k,N_-(\iota_-),N_-(\iota_+))$, $\boldsymbol{m} = (m,N_+(\iota_-),N_+(\iota_+))$. The following result is a direct consequence of Theorem 2.13:

**Theorem 2.27.** *Every* $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{w})$, $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, *induces continuous operators*

$$\mathcal{A} : \mathcal{W}^{s,\gamma}(M_{\mathrm{crack}}; \boldsymbol{b}_-) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(M_{\mathrm{crack}}; \boldsymbol{b}_+) \tag{127}$$

*and*

$$\mathcal{A} : \mathcal{W}_P^{s,\gamma}(M_{\mathrm{crack}}; \boldsymbol{b}_-) \longrightarrow \mathcal{W}_Q^{s-\mu,\gamma-\mu}(M_{\mathrm{crack}}; \boldsymbol{b}_+) \tag{128}$$

*for all* $s > d - \frac{1}{2}$ *and every* $P \in \mathrm{As}((I, \partial I), (\gamma, \Theta); \boldsymbol{n})$ *for some resulting* $Q \in \mathrm{As}((I, \partial I),$
$(\gamma - \mu, \Theta); \boldsymbol{m})$.

**Theorem 2.28.** *Let* $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{w})$ *and suppose* $\sigma(\mathcal{A}) = 0$. *Then* $\mathcal{A}$ *is compact as an operator* (127) *for every* $s > d - \frac{1}{2}$.

The proof follows from the fact $\sigma(\mathcal{A}) = 0$ implies the continuity of $\mathcal{A}$ as an operator

$$\mathcal{A} : \mathcal{W}^{s,\gamma}(M_{\mathrm{crack}}; \boldsymbol{b}_-) \longrightarrow \mathcal{W}^{s-\mu+1,\gamma-\mu+\varepsilon}(M_{\mathrm{crack}}; \boldsymbol{b}_+) \tag{129}$$

$s > d - \frac{1}{2}$, for some $\varepsilon > 0$. The improvement of smoothness by 1 outside $Y$ is a consequence of $\sigma_\psi(\mathcal{A}) = 0$, $\sigma_\partial(\mathcal{A})_\pm = 0$ and $\sigma_\partial(\mathcal{A})_0 = 0$. It remains to observe that vanishing of these components together with $\sigma_\wedge(\mathcal{A}) = 0$ gives rise to the relation

$$\boldsymbol{a}(y, \eta) \in S^{\mu-1}(\Omega \times \mathbb{R}^q; E, F)$$

for $E = \mathcal{K}^{s,\gamma}((I, \partial I)^\wedge; \boldsymbol{n}) \oplus \mathbb{C}^{L_-}$ and $F = \mathcal{K}^{s-\mu+1,\gamma-\mu+\varepsilon}((I, \partial I)^\wedge; \boldsymbol{m}) \oplus \mathbb{C}^{L_+}$ for some $\varepsilon > 0$. In fact, the improvement of the smoothness is rather obvious, while the improved weight by an $\varepsilon > 0$ follows from the fact that Green symbols automatically map into spaces with a better weight. Combining (129) with the compactness of the embedding $\mathcal{W}^{s-\mu+1,\gamma-\mu+\varepsilon}(M_{\mathrm{crack}}; \boldsymbol{b}_+) \hookrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(M_{\mathrm{crack}}; \boldsymbol{b}_+)$ we get the assertion.

**Theorem 2.29.** $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\mathrm{crack}}, \boldsymbol{g}_1; \boldsymbol{w}_1)$ *for* $\boldsymbol{g}_1 = (\gamma - \nu, \gamma - \nu - \mu, \Theta)$, $\boldsymbol{w}_1 = (\boldsymbol{n}_0, \boldsymbol{m}; L_0, L_+; b_0, b_+)$ *and* $\mathcal{B} \in \boldsymbol{\mathcal{V}}^{\nu,e}(M_{\mathrm{crack}}, \boldsymbol{g}_2; \boldsymbol{w}_2)$ *for* $\boldsymbol{g}_2 = (\gamma, \gamma - \nu, \Theta)$, $\boldsymbol{w}_2 = (\boldsymbol{n}, \boldsymbol{n}_0; L_-, L_0; b_-, b_0)$ *implies* $\mathcal{A}\mathcal{B} \in \boldsymbol{\mathcal{V}}^{\mu+\nu,h}(M_{\mathrm{crack}}, \boldsymbol{g}_1 \circ \boldsymbol{g}_2; \boldsymbol{w}_1 \circ \boldsymbol{w}_2)$ *for* $h = \max(\nu + d, e)$, $\boldsymbol{g}_1 \circ \boldsymbol{g}_2 = (\gamma, \gamma - \nu - \mu, \Theta)$ *and* $\boldsymbol{w}_1 \circ \boldsymbol{w}_2 = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+; b_-, b_+)$, *and we have* $\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ (*with componentwise multiplication*). *Moreover, if* $\mathcal{A}$ *or* $\mathcal{B}$ *belongs to the corresponding subspace with subscript* $M + G$ ($G$), *then the same is true of the composition.*

Theorem 2.29 is an easy consequence of Theorem 2.16 and Remark 2.3. We now pass to the ellipticity of global crack operators.

**Defintion 2.30.** An operator $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ (in the notation of Theorem 2.27 for $k = m$) is called elliptic if

(i) (122) is an isomorphism for all $(x, \xi) \in T^*(\text{int}\, M \backslash S) \backslash 0$ and the associated Fuchs type interior symbol $\sigma_{\psi,F}(\mathcal{A})$ (in local coordinates near $Y$) satisfy condition (i) in Definition 2.18,

(ii) the boundary symbols (124) are ismorphisms for all $(x', \xi') \in T^*(\text{int}\, S_\pm) \backslash 0$, $s > \max(\mu, d) - \frac{1}{2}$, and the associated Fuchs type boundary symbols $\sigma_{\partial,F}(\mathcal{A})_\pm$ (in local coordinates near $Y$) satisfy condition (ii) in Definition 2.18,

(iii) the boundary symbol (125) is an isomorphism for all $(x', \xi') \in T^*(\partial M) \backslash 0$, $s > \max(\mu, d) - \frac{1}{2}$,

(iv) the crack symbol (126) is an isomorphism for all $(y, \eta) \in T^*Y \backslash 0$, $s > \max(\mu, d) - \frac{1}{2}$.

*Remark* 2.31. Condition (iv) in Definition 2.30 is an analogue of the Shapiro-Lopatinskij condition of the theory of boundary value problems, here, with respect to the crack boundary $Y$. In Section 3.4 below we give some further remarks on the role of the homogeneous principal edge symbol $\sigma_\wedge(\mathcal{A})$, including the subordinate nature of the principal conormal symbol

$$\sigma_M(\mathcal{A})(y, z) : \begin{array}{c} H^s(I, \mathbb{C}^m) \\ \oplus \\ \mathbb{C}^{N_-(\iota_-)+N_-(\iota_+)} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(I, \mathbb{C}^m) \\ \oplus \\ \mathbb{C}^{N_+(\iota_-)+N_+(\iota_+)} \end{array},$$

$s > \max(\mu, d) - \frac{1}{2}$, that is necessarily a family of ismorphisms for all $(y, z) \in Y \times \Gamma_{1-\gamma}$, as soon as $\sigma_\wedge(\mathcal{A})(y, \eta)$ satisfies condition (iv) of Definition 2.30.

**Defintion 2.32.** Given $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ for $(\mu, d) \in \mathbb{Z} \times \mathbb{N}$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, $\boldsymbol{w} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+; b_-, b_+)$, an operator $\mathcal{P} \in \boldsymbol{\mathcal{V}}^{-\mu,e}(M_{\text{crack}}, \boldsymbol{g}^{-1}; \boldsymbol{w}^{-1})$ for a certain $e \in \mathbb{N}$ and $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$, $\boldsymbol{w}^{-1} = (\boldsymbol{m}, \boldsymbol{n}; L_+, L_-; b_+, b_-)$ is called a parametrix of $\mathcal{A}$ if

$$\mathcal{P}\mathcal{A} - \mathcal{I} \in \boldsymbol{\mathcal{V}}^{-\infty,d_l}(M_{\text{crack}}, \boldsymbol{g}_l; \boldsymbol{w}_l), \quad \mathcal{A}\mathcal{P} - \mathcal{I} \in \boldsymbol{\mathcal{V}}^{-\infty,d_r}(M_{\text{crack}}, \boldsymbol{g}_r; \boldsymbol{w}_r) \quad (130)$$

for certain types $d_l, d_r \in \mathbb{N}$, where $\boldsymbol{g}_l = (\gamma, \gamma, \Theta)$, $\boldsymbol{w}_l = (\boldsymbol{n}, \boldsymbol{n}; L_-, L_-; b_-, b_-)$ and $\boldsymbol{g}_r = (\gamma - \mu, \gamma - \mu, \Theta)$, $\boldsymbol{w}_r = (\boldsymbol{m}, \boldsymbol{m}; L_+, L_+; b_+, b_+)$.

*Remark* 2.33. If $\mathcal{P}$ is a parametrix of $\mathcal{A}$, a consequence of Theorem 2.29 is $\sigma(\mathcal{P}) = \sigma(\mathcal{A})^{-1}$ with the componentwise inversion.

**Theorem 2.34.** *An elliptic operator* $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ (*in the notation of Theorem* 2.27) *has a parametrix* $\mathcal{P} \in \boldsymbol{\mathcal{V}}^{-\mu,(d-\mu)^+}(M_{\text{crack}}, \boldsymbol{g}^{-1}; \boldsymbol{w}^{-1})$, *where the types of the remainders are* $d_l = \max(\mu, d)$ *and* $d_r = (d - \mu)^+$.

To prove Theorem 2.34 is suffices to apply the local parametrix construction of Theorem 2.22, combined with a parametrix near $\partial M$ in the sense of a DN-analogue of the second part of Theorem 1.18, and to get a global parametrix in an obvious manner by using a partition of unity.

**Theorem 2.35.** *Let* $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ *(in the notation of Theorem* 2.27*) be elliptic. Then*

$$\mathcal{A} : \mathcal{W}^{s,\gamma}(M_{\text{crack}}; \boldsymbol{b}_-) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(M_{\text{crack}}; \boldsymbol{b}_+) \tag{131}$$

*is a Fredholm operator for all* $s > \max(\mu, d) - \frac{1}{2}$*. In addition,* $u \in \mathcal{W}^{r,\gamma}(M_{\text{crack}}; \boldsymbol{b}_-)$ *for some* $r > \max(\mu, d) - \frac{1}{2}$ *and* $\mathcal{A}u \in \mathcal{W}^{s-\mu,\gamma-\mu}(M_{\text{crack}}; \boldsymbol{b}_+)$ *for some* $s > \max(\mu, d) - \frac{1}{2}$ *implies* $u \in \mathcal{W}^{s,\gamma}(M_{\text{crack}}; \boldsymbol{b}_-)$.

*Moreover,* $\mathcal{A}u \in \mathcal{W}_Q^{s-\mu,\gamma-\mu}(M_{\text{crack}}; \boldsymbol{b}_+)$ *for some continuous asymptotic type* $Q \in \operatorname{As}((I, \partial I), (\gamma - \mu, \Theta); \boldsymbol{m})$ *yields* $u \in \mathcal{W}_P^{s,\gamma}(M_{\text{crack}}; \boldsymbol{b}_-)$ *for some resulting* $P \in \operatorname{As}((I, \partial I),$
$(\gamma, \Theta); \boldsymbol{n})$.

The Fredholm property of $\mathcal{A}$ follows from the fact that $\mathcal{A}$ has a parametrix $\mathcal{P}$, cf. Theorem 2.34; then $\sigma(\mathcal{P}) = \sigma(\mathcal{A})^{-1}$. Hence the remainders in (130) are compact, cf. Theorem 2.28. The scheme of the proof of the second part of the theorem is the same as that for Theorem 2.23.

*Remark* 2.36. It can be proved that when an elliptic operator $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ defines an isomorphism (131) for some $s = s_0 > \max(\mu, d) - \frac{1}{2}$, then it is an isomorphism for all $s > \max(\mu, d) - \frac{1}{2}$, and we have $\mathcal{A}^{-1} \in \boldsymbol{\mathcal{V}}^{-\mu,(d-\mu)^+}(M_{\text{crack}}, \boldsymbol{g}^{-1}; \boldsymbol{w}^{-1})$.

In fact, the ellipticity gives us a parametrix that can be used to show that kernel and cokernel of the Fredholm operator are independent of $s$. If $\mathcal{A}$ is an isomorphism, there is a parametrix $\mathcal{P}$ that is also an isomorphism (this can be achieved by adding a suitable smoothing operator to some arbitrary parametrix). Then, setting $\mathcal{C} = \mathcal{P}\mathcal{A} - \mathcal{I}$ we see that also $\mathcal{I} + \mathcal{C}$ is invertible; however $(\mathcal{I} + \mathcal{C})^{-1}$ is of analogous structure, cf. Remark 2.24, i.e., $(\mathcal{I} + \mathcal{C})^{-1}\mathcal{P} = \mathcal{A}^{-1}$ belongs to our crack algebra.

*Remark* 2.37. The result on asymptotics of solutions in Theorem 2.35 can be further specified by a more concrete description of the correspondence $Q \to P$ (that is mainly caused by the structure of $\sigma_M(\mathcal{A})^{-1}(y, \eta)$, the inverse of the principal conormal symbol, cf. Remark 3.6 below) and by more concrete computations of coefficients. The latter ones are influenced both by local and global effects. The functional analytic characterisation of (local) coefficients and singular functions of continuous (and discrete) asymptotics for the case of a closed cone base from Schulze [33], Section 3.2.5 or [27], Section 3.1.5 can easily be generalised to the present situation, where the cone base is the interval $I$.

## 3. Examples and Remarks

**3.1. Regular symbols.** The operator algebra on $M_{\text{crack}}$ in our notation is the union of all $\boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ over $(\mu, d) \in \mathbb{Z} \times \mathbb{N}$, weight data $\boldsymbol{g}$ and dimension data $\boldsymbol{w}$. Algebra operations, e.g., compositions, are admitted whenever weight and dimension data of the first factor are compatible with those of the second one, cf. Theorem 2.29, and we then have a corresponding rule for the principal symbols. Locally we can also compose ("complete") left symbols, e.g., left interior symbols by the Leibniz multiplication. Also globally it is possible to establish a complete algebra of symbols consisting of the system of local representatives; then a complete symbol in that sense determines an operator up to $\boldsymbol{\mathcal{V}}^{-\infty,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$. We do not elaborate this aspect in detail here. More information may be found in Kapanadze and Schulze [12]. Concerning complete symbols we may ask specific properties that remain preserved under the algebra operations. The operators in basic models of crack theory, cf. (4), have interior symbols that are regular in $M$, i.e., they are classical and smooth. These properties survive under algebra operations and parametrix constructions in the elliptic case. (Recall that in Definition 2.25 the interior symbols are admitted to be edge-degenerate near $Y$ and discontinuous near $\operatorname{int} S$ (i.e., they may have different boundary values from both sides of the crack); as such they are much more general than to be smooth in the above-mentiond sense). In other words, by requiring regular interior symbols we get a subspace of $\boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})$ which constitutes a subalgebra of the general crack operator algebra. For instance, elliptic operators connected with the Lamé system and their parametrices belong to that subalgebra. In the applications we have a similar regularity of the trace conditions from both sides of the crack, e.g., when they consist of Dirichlet or Neumann conditions. Such trace conditions are not edge-degenerate but have (say for the case of differential boundary conditions, cf. formula (3)) smooth coefficients up to $Y = \partial S$. Let us call such trace operators regular. The operators with regular interior and trace (and potential, etc.) symbols along int $S_\pm$ in this sense form subspaces

$$\boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})_{\text{reg}} \subset \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w}) \tag{132}$$

which constitute a subalgebra of the crack operator algebra.

*Remark* 3.1. For $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})_{\text{reg}}$ we have a homogeneous principal symbol

$$\sigma_\psi(\mathcal{A})(x, \xi) : \mathbb{C}^k \longrightarrow \mathbb{C}^m \tag{133}$$

of order $\mu$ in the standard sense, $(x, \xi) \in T^*M \backslash 0$. Similarly, the boundary symbols (124) along int $S_\pm$ are smooth with respect to $x'$ up to $Y$.

Ellipticity conditions (concerning $k = m$) in the regular case with respect to $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})_\pm$ are to be posed in the sense that (133) is an isomorphism for all $(x, \xi) \in T^*M \backslash 0$ and that the boundary symbols (124) are isomorphisms for all $(x', \xi') \in T^*S_\pm \backslash 0$. This entails conditions (i) and (ii) of Definition 2.25. Then, if $\mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\text{crack}}, \boldsymbol{g}; \boldsymbol{w})_{\text{reg}}$ is elliptic with respect to all components of the principal symbol, every parametrix $\mathcal{P}$ belongs to the space $\boldsymbol{\mathcal{V}}^{-\mu,(d-\mu)^+}(M_{\text{crack}}, \boldsymbol{g}^{-1}; \boldsymbol{w}^{-1})_{\text{reg}}$, and the relations $\sigma_\psi(\mathcal{P}) = \sigma_\psi(\mathcal{A})^{-1}$ and $\sigma_\partial(\mathcal{P})_\pm = \sigma_\partial(\mathcal{A})_\pm^{-1}$ are valid in the regular sense, i.e., on $T^*M \backslash 0$ and $T^*S_\pm \backslash 0$, respectively.

**3.2. Reductions of orders.** The examples of crack problems for differential operators in Sections 1.1 and 1.2 show that our order convention in the general psedo-differential approach contains a simplification that is only justified, if we apply suitable reductions of orders along the boundary that do not affect the basic results. Let us first emphazise that reductions of orders could be avoided completely, cf. Remark 3.3 below. To treat problems with arbitrary orders in the boundary data it suffices to slightly modify the DN-order formalism. The chosen orders in our psedo-differential machinery are all the same except for a shift by $\frac{1}{2}$ of smoothness and weight indices, caused by the fact that the boundary is of codimension 1 (recall that this is an essential difference to [26]). To reach arbitrary orders with respect to the boundary conditions we have to compose the corresponding operators from both sides with suitable diagonal matrices of elliptic wedge operators on the boundary along $(\partial I)^\wedge \times \Omega$ in the local calculus and along $S_\pm$ in the global situation. The existence of such reductions of orders is by no means evident, but there is a corresponding general theorem.

Let us first note that the general crack operator algebra contains many interesting subalgebras, according to the $5 \times 5$-block matrix structure $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,\dots,5}$. For instance, we have the $3 \times 3$-block matrix subalgebra, consisting of all elements $\mathcal{A}$ where $\mathcal{A}_{ij} = 0$ whenever $i$ or $j$ equals 1 or 5. What we obtain in this way (say, for the global situation) is the algebra $\boldsymbol{\mathcal{Y}}^\mu(S_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{b})$ of $3\times 3$-block matrices of operators along the crack $S$, where subscript "crack" means that we talk about the space

$$S_{\mathrm{crack}} := (S_- \cup_{\mathrm{d}} S_+)/\sim$$

with $\cup_{\mathrm{d}}$ being the disjoint union of both sides and $/\sim$ the quotient space under the identification of corresponding points from the two copies of the boundary $Y$. In this way $S_{\mathrm{crack}}$ is related to $2S$, the double of $S$, though we do not observe some $C^\infty$ structure on $2S$ but take $S_{\mathrm{crack}}$ to be embedded like a "sandwhich" in a neighbouring manifold. The weight data $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$ are as before, while $\boldsymbol{b}$ are the dimension data, inherited from the general crack algebra, namely

$$\boldsymbol{b} = (N_-(\iota_-), N_+(\iota_-); N_-(\iota_+), N_+(\iota_+); L_-, L_+).$$

The operators $\mathcal{B} \in \boldsymbol{\mathcal{Y}}^\mu(S_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{b})$ are continuous in the sense

$$\mathcal{B}: \begin{array}{c} \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(S_-, \mathbb{C}^{N_-(\iota_-)}) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(S_+, \mathbb{C}^{N_-(\iota_+)}) \\ \oplus \\ H^{s-1}(Y, \mathbb{C}^{L_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}(S_-, \mathbb{C}^{N_+(\iota_-)}) \\ \oplus \\ \mathcal{W}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}(S_+, \mathbb{C}^{N_+(\iota_+)}) \\ \oplus \\ H^{s-\mu-1}(Y, \mathbb{C}^{L_+}) \end{array}, \quad (134)$$

for all $s \in \mathbb{R}$. Notice that there is an analogoue of this operator space for $S_-$ and $S_+$ separately, that is, we have the spaces $\boldsymbol{\mathcal{Y}}^\mu(S_-, \boldsymbol{g}; (N_-(\iota_-), N_+(\iota_-);$

$L_-, L_+)$) (and the same for $S_+$) for arbitrary $N_-(\iota_-), N_+(\iota_-), L_-, L_+ \in \mathbb{N}$, consisting of operators

$$\mathcal{B}: \begin{array}{c} \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(S_-, \mathbb{C}^{N_-(\iota_-)}) \\ \oplus \\ H^{s-1}(Y, \mathbb{C}^{L_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}(S_-, \mathbb{C}^{N_+(\iota_-)}) \\ \oplus \\ H^{s-\mu-1}(Y, \mathbb{C}^{L_+}) \end{array},$$

$s \in \mathbb{N}$, that are nothing else than psedo-differential boundary value problems on $S_-$ with respect to the boundary $Y$, where the interior operators are edge-degenerate (the boundary is regarded as an edge and the inner normal $\mathbb{R}_+$ as the model cone of the wedge). Clearly, we may write $S = S_-$ (or $= S_+$) in this case. Notice that this theory contains the theory of boundary value problems for psedo-differential operators on $S$ without the transmission property (to be denoted by $\boldsymbol{\mathcal{Y}}^\mu(S, \boldsymbol{g}; \{\text{dimension data}\})_{\text{reg}}$ in our scheme of notation), cf. the monograph [25], or Harutjunjan, Schulze and Witt [9].

The following result (in the more general context of edge-degenerate operators with non-trivial model cones) may be found in Behm [2].

**Theorem 3.2.** *For every $\mu, \gamma \in \mathbb{R}$ there exists an elliptic operator $R^\mu \in \boldsymbol{\mathcal{Y}}^\mu(S, \boldsymbol{g}; (1, 1; 0, 0))$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, that induces isomorphisms*

$$R^\mu : \mathcal{W}^{s,\gamma}(S) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(S)$$

*for all $s \in \mathbb{R}$, where*

$$(R^\mu)^{-1} \in \boldsymbol{\mathcal{Y}}^{-\mu}(S, \boldsymbol{g}^{-1}; (1, 1; 0, 0)).$$

The first two components of spaces in formula (134) are direct sums of corresponding "scalar" spaces $\mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(S_\pm)$ and $\mathcal{W}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}(S_\pm)$, respectively. Applying Theorem 3.2 we find diagonal block matrices of order reducing isomorphisms

$$R_1 : \begin{array}{c} \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(S_-, \mathbb{C}^{N_-(\iota_-)}) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(S_+, \mathbb{C}^{N_-(\iota_+)}) \end{array} \longrightarrow \begin{array}{c} \bigoplus_{l=1}^{N_-(\iota_-)} \mathcal{W}^{s-n_{-,l}-\frac{1}{2},\gamma-n_{-,l}-\frac{1}{2}}(S_-) \\ \oplus \\ \bigoplus_{l=1}^{N_-(\iota_+)} \mathcal{W}^{s-n_{+,l}-\frac{1}{2},\gamma-n_{+,l}-\frac{1}{2}}(S_+) \end{array}$$

and

$$R_2 : \begin{array}{c} \mathcal{W}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}(S_-, \mathbb{C}^{N_+(\iota_-)}) \\ \oplus \\ \mathcal{W}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}(S_+, \mathbb{C}^{N_+(\iota_+)}) \end{array} \longrightarrow \begin{array}{c} \bigoplus_{j=1}^{N_+(\iota_-)} \mathcal{W}^{s-m_{-,j}-\frac{1}{2},\gamma-m_{-,j}-\frac{1}{2}}(S_-) \\ \oplus \\ \bigoplus_{j=1}^{N_+(\iota_+)} \mathcal{W}^{s-m_{+,j}-\frac{1}{2},\gamma-m_{+,j}-\frac{1}{2}}(S_+) \end{array}$$

for all $s \in \mathbb{R}$. To reach crack Sobolev spaces with "realistic" orders, suggested by formula (31), we transform the Sobolev spaces (119) to

$$\mathcal{R}_1 \mathcal{W}^{s,\gamma}(M_{\text{crack}}, \boldsymbol{b}_-) \tag{135}$$

for $\mathcal{R}_1 = \mathrm{diag}(1, R_1, 1, 1)$, where 1 are the identity maps in $\mathcal{W}^{s,\gamma}(M_{\mathrm{crack}}, \mathbb{C}^k)$, $H^s(Y, \mathbb{C}^{L_-})$ and $H^{s-\frac{1}{2}}(\partial M, \mathbb{C}^{b_-})$, respectively. Analogously, we can form

$$\mathcal{R}_2 \mathcal{W}^{s-\mu,\gamma-2}(M_{\mathrm{crack}}, \boldsymbol{b}_+) \tag{136}$$

for $\mathcal{R}_2 = \mathrm{diag}(1, R_2, 1, 1)$. This gives us an associated space of crack operators

$$\boldsymbol{\mathcal{V}}^{\boldsymbol{\mu},d}(M_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{v}) := \{\boldsymbol{\mathcal{A}} := \mathcal{R}_2 \mathcal{A} \mathcal{R}_1^{-1} : \mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{v})\} \tag{137}$$

that are continuous as maps

$$\boldsymbol{\mathcal{A}} : \mathcal{R}_1 \mathcal{W}^{s,\gamma}(M_{\mathrm{crack}}, \boldsymbol{b}_-) \longrightarrow \mathcal{R}_2 \mathcal{W}^{s-\mu,\gamma-\mu}(M_{\mathrm{crack}}, \boldsymbol{b}_+)$$

(as well as between corresponding subspaces with asymptotics). The operators in (137) are block-matrices with analogues of Douglis-Nirenberg orders, and our notation means that $\boldsymbol{\mu}$ represents the corresponding order information :

$$\boldsymbol{\mu} = (\mu; (-n_{\pm,l})_{l=1,\dots,N_-(\iota_\pm)}, (m_{\pm,j})_{j=1,\dots,N_+(\iota_\pm)})$$

Similarly to (132) we set

$$\boldsymbol{\mathcal{V}}^{\boldsymbol{\mu},d}(M_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{v})_{\mathrm{reg}} := \{\mathcal{R}_2 \mathcal{A} \mathcal{R}_1^{-1} : \mathcal{A} \in \boldsymbol{\mathcal{V}}^{\mu,d}(M_{\mathrm{crack}}, \boldsymbol{g}; \boldsymbol{v})_{\mathrm{reg}}\}.$$

*Remark* 3.3. Our calculus of crack psedo-differential operators could have been established directly from the very beginning with such orders and a definition of smoothness and weight indices like in (135) and (136), without referring to reductions of orders. The generalisation is straightforward.

We could also refer to the "true" orders of boundary operators with respect to $\partial M$; then also this part of the operators (right lower block matrix corners) would be described by Douglis-Nirenberg orders. However, for the psedo- differential effects this is not essential (and, in fact, not customary); in particular, asymptotics near the crack remain untouched by such a modification.

*Remark* 3.4. To establish asymptotics of solutions to a concrete elliptic problem it may be advisable to employ the theory of operators (137), not the simpler one in the sense of Definition 2.25, because the reductions of orders within our classes formally contribute asymptotic information (though their Mellin symbols and Green ingredients can be chosen to be holomorphic and flat, respectively). In the discussion of examples below we tacitly use a variant of our theory, directly defined in the sense of Remark 3.3.

**3.3. Conormal symbols.** The considerations in this paper on asymptotics of solutions mainly concerned continuous asymptotics near the crack boundary. In applications and examples we may expect pointwise discrete asymptotics though, as is well-known by many concrete investigations, they are a rather subtle and individual information. It makes sense to try to formulate a subalgebra of the general crack algebra that encodes such so-called variable discrete asymptotics in terms of a suitable generalisation of the approach of [31], [32] for the case of boundary value problems. The necessary background in variable (in general branching) discrete asymptotic types is contained in a paper of Schulze and Witt [38]; unfortunately, the formalities are voluminous, and a program to adapt them for the crack algebra requires a separate exposition. On the other hand it is reasonable to study the case of constant discrete asymptotics.

To illustrate the idea we first discuss some examples. Let us consider the Lamé system in a domain $G$ in $\mathbb{R}^3_x,\ x=(x_1,x_2,x_3)$, and assume that the crack $S$ is a subset in the half plane $H:=\{x\in\mathbb{R}^3 : x_1\geq 0, x_2=0, x_3\in\mathbb{R}\}$, that the origin belongs to $Y=\partial S$ and that there is an $\varepsilon>0$ such that $S\cap H=\{x\in\mathbb{R}^3 : (x_1^2+x_3^2)^{\frac{1}{2}}<\varepsilon, x_2=0\}$. We want to calculate the various symbol levels when we pose, for instance, two- sided Dirichlet conditions (the corresponding expressions for two-sided Neumann conditions are completely analogous). The system has constant coefficients and is of the form $A(D_x)u=f$ for a matrix $A=(A_{ij})_{i,j=1,2,3}$ of second order differential operators. They have to be expressed in polar coordinates in the $(x_1,x_2)$-plane, and the crack intersects that plane in $\overline{\mathbb{R}}_+=\{x_1\geq 0, x_2=0\}$. Setting $y=x_3$ we can apply formula (5) and get

$$A=r^{-2}\sum_{k+|\beta|\leq 2}a_{k\beta}(r)\left(-r\frac{\partial}{\partial r}\right)^k(rD_y)^\beta$$

with coefficients $a_{k\beta}(r)\in C^\infty(\overline{\mathbb{R}}_+,\mathrm{Diff}^{2-(k+|\beta|)}_{3\times 3}(I)),\ I=[0,2\pi]$. For two-sided Dirichlet conditions we simply have to set

$$T_\pm=\mathrm{r}'_\pm\,.$$

In other words, we can apply (13) to our case, where $l_-=l_+=1$, and $\Omega=(-\varepsilon,\varepsilon)$ with the above-mentioned $\varepsilon>0$. The crack boundary value problem in this case induces a continuous operator

$$\mathcal{A}:\mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(I^\wedge\times\Omega,\mathbb{C}^3)\longrightarrow\begin{array}{c}\mathcal{W}^{s-2,\gamma-2}_{\mathrm{loc}(y)}(I^\wedge\times\Omega,\mathbb{C}^3)\\ \oplus\\ \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{\mathrm{loc}(y)}(\mathbb{R}_+\times\Omega,\mathbb{C}^3)\\ \oplus\\ \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{\mathrm{loc}(y)}(\mathbb{R}_+\times\Omega,\mathbb{C}^3)\end{array}\tag{138}$$

for every $s>\frac{1}{2}$ and every $\gamma\in\mathbb{R}$. The operator (138) belongs to $\boldsymbol{\mathcal{V}}^{\mathbf{2},1}(I^\wedge\times\Omega,(\gamma,\gamma-2,(-\infty,0]);\boldsymbol{v})_{\mathrm{reg}}$ for $\boldsymbol{v}=(\boldsymbol{n},\boldsymbol{m})$, $\boldsymbol{n}=(3,0,0)$, $\boldsymbol{m}=(3,3,3)$ (cf., in particular, the notation of the preceding section). The shape of $\sigma_\psi(\mathcal{A})$ is obvious : We have

$$\sigma_\psi(\mathcal{A})(\xi) : \mathbb{C}^3 \longrightarrow \mathbb{C}^3,$$

$\xi \neq 0$. For the boundary symbols with respect to int $S_\pm$ we have

$$\sigma_\partial(\mathcal{A})_\pm(\varrho,\eta) : H^s(\mathbb{R}_+,\mathbb{C}^3) \longrightarrow \begin{array}{c} H^{s-2}(\mathbb{R}_+,\mathbb{C}^3) \\ \oplus \\ \mathbb{C}^3 \end{array},$$

$s > \frac{1}{2}, (\varrho,\eta) \neq 0$, and the crack symbol equals

$$\sigma_\wedge(\mathcal{A})(\eta) = \begin{pmatrix} \sigma_\wedge(A)(\eta) \\ \mathrm{r}'_- \\ \mathrm{r}'_+ \end{pmatrix} : \mathcal{K}^{s,\gamma}(I^\wedge,\mathbb{C}^3) \longrightarrow \begin{array}{c} \mathcal{K}^{s-2,\gamma-2}(I^\wedge,\mathbb{C}^3) \\ \oplus \\ \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+,\mathbb{C}^3) \\ \oplus \\ \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+,\mathbb{C}^3) \end{array} \tag{139}$$

$\eta \neq 0$, where

$$\sigma_\wedge(A)(\eta) = r^{-2} \sum_{k+|\beta|\leq 2} a_{k\beta}(0) \left(-r\frac{\partial}{\partial r}\right)^k (r\eta)^\beta.$$

This is an $\eta$-dependent family of boundary value problems in the infinite cone $I^\wedge$ with boundary $(\partial I)^\wedge$. This cone can be regarded as a manifold with boundary and conical exits to infinity; in particular, (139) fits into the calculus of Kapanadze and Schulze [12], concerning the aspects at infinity. Combining this with a corresponding information from elliptic boundary value problems near conical singularities we see that (139) is a Fredholm operator (for any $s > \frac{3}{2}$) if and only if the principal conormal symbol

$$\sigma_M(\mathcal{A})(z) = \begin{pmatrix} \sum_{k=0}^2 a_{k0}(0)z^k \\ \mathrm{r}'_- \\ \mathrm{r}'_+ \end{pmatrix} : H^s(I,\mathbb{C}^3) \longrightarrow \begin{array}{c} H^{s-2}(I,\mathbb{C}^3) \\ \oplus \\ \mathbb{C}^3 \\ \oplus \\ \mathbb{C}^3 \end{array}$$

is an ismorphism for all $z \in \Gamma_{1-\gamma}$ (recall that $\Gamma_\beta = \{z : \mathrm{Re}\ z = \beta\}$).

More generally, for the local differential crack problems (13) we have

$$\sigma_\psi(\mathcal{A})(x,\xi) = \sigma_\psi(A)(x,\xi) : \mathbb{C}^N \longrightarrow \mathbb{C}^N \tag{140}$$

(cf. formulas (2) and (31)) for all $(x,\xi), \xi \neq 0$, furthermore

$$\sigma_\partial(\mathcal{A})_\pm(r,y,\varrho,\eta) : H^s(\mathbb{R}_+,\mathbb{C}^N) \longrightarrow \begin{array}{c} H^{s-m}(\mathbb{R}_+,\mathbb{C}^N) \\ \oplus \\ \mathbb{C}^{N_+(\iota_-)} \\ \oplus \\ \mathbb{C}^{N_+(\iota_+)} \end{array}, \tag{141}$$

$s > \max(m_{\pm,j}) + \frac{1}{2}$, for all $(r, y, \varrho, \eta), (\varrho, \eta) \neq 0$, $N_+(\iota_\pm) = \sum_{j=1}^{l_\pm} M_{\pm,j}$ (cf. formula (6)), and

$$\sigma_\wedge(\mathcal{A})(y,\eta) : \mathcal{K}^{s,\gamma}(I^\wedge, \mathbb{C}^N) \longrightarrow \begin{array}{c} \mathcal{K}^{s-m,\gamma-m}(I^\wedge, \mathbb{C}^N) \\ \oplus \\ \oplus_{j=0}^{l_-} \mathcal{K}^{s-m_{-,j}-\frac{1}{2},\gamma-m_{-,j}-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{M_{-,j}}), \\ \oplus \\ \oplus_{j=0}^{l_+} \mathcal{K}^{s-m_{+,j}-\frac{1}{2},\gamma-m_{+,j}-\frac{1}{2}}(\mathbb{R}_+, \mathbb{C}^{M_{+,j}}) \end{array} \tag{142}$$

$s > \max(m_{\pm,j}) + \frac{1}{2}$, for all $(y, \eta), \eta \neq 0$,

$$\sigma_\wedge(\mathcal{A})(y,\eta) := \begin{pmatrix} \sigma_\wedge(A)(y,\eta) \\ \sigma_\wedge(T_-)(y,\eta) \\ \sigma_\wedge(T_+)(y,\eta) \end{pmatrix}, \tag{143}$$

$$\sigma_\wedge(A)(y,\eta) = r^{-m} \sum_{k+|\beta|\leq m} a_{k\beta}(0,y) \left(-r\frac{\partial}{\partial r}\right)^k (r\eta)^\beta,$$

$$\sigma_\wedge(T_\pm)(y,\eta) = \mathrm{r}'_\pm\, r^{-m_{\pm,j}} \sum_{k+|\beta|\leq m_{\pm,j}} b_{\pm,j;k\beta}(0,y) \left(-r\frac{\partial}{\partial r}\right)^k (r\eta)^\beta,$$

(cf. formulas (5) and (6)). The principal conormal symbol is the familiy of operators

$$\sigma_M(\mathcal{A})(y,z) : H^s(I, \mathbb{C}^N) \longrightarrow \begin{array}{c} H^{s-m}(I, \mathbb{C}^N) \\ \oplus \\ \mathbb{C}^{N_+(\iota_-)} \\ \oplus \\ \mathbb{C}^{N_+(\iota_+)} \end{array}, \tag{144}$$

$s > \max(m_{\pm,j}) + \frac{1}{2}$, for $y \in \Omega, z \in \Gamma_{1-\gamma}$, given by the column matrix

$$\sigma_M(\mathcal{A})(y,z) = \begin{pmatrix} \sum_{k=0}^{m} a_{k0}(0,y) z^k \\ \left(\mathrm{r}'_- \sum_{k=0}^{m_{-,j}} b_{-,j;k0}(0,y) z^k\right)_{j=1,\ldots,l_-} \\ \left(\mathrm{r}'_+ \sum_{k=0}^{m_{+,j}} b_{+,j;k0}(0,y) z^k\right)_{j=1,\ldots,l_+} \end{pmatrix}. \tag{145}$$

Also in the general situation, $\sigma_\wedge(\mathcal{A})(y,\eta)$ fits into the theory of Kapanadze and Schulze [14] for every fixed $y \in \Omega, \eta \neq 0$, concerning the conical exit of $I^\wedge$ to infinity, while $\sigma_\wedge(\mathcal{A})(y,\eta)$ for fixed $y \in \Omega, \eta \neq 0$ near the tip of the cone belongs to the classical cone theory of boundary value problems (cf. Kondrat'ev [15]).

*Remark* 3.5. Let $A$ be elliptic, i.e., $\sigma_\psi(A)(x,\xi)$ is an isomorphism for all $x$ and $\xi \neq 0$, and let the operators $T_\pm$ satisfy the Shapiro-Lopatinskij condition (in the regular sense, i.e., up to $\partial S$, cf. the notation in Section 3.1). Then for every $y \in \Omega$ there is a countable set $D(y) \subset \mathbb{C}$ with $D(y) \cap \{z : c \leq \text{Re } z \leq c'\}$ finite for every $c \leq c'$, such that

(i) (142) is a Fredholm operator for every $\gamma \in \mathbb{R}$ such that $\Gamma_{1-\gamma} \cap D(y) = \emptyset, s > m - \frac{1}{2}$,

(ii) (144) is a family of isomorphisms for all $z \in \Gamma_{1-\gamma}$ for every $\gamma \in \mathbb{R}$ with $\Gamma_{1-\gamma} \cap D(y) = \emptyset, s > m - \frac{1}{2}$.

*Remark* 3.6. There is, in fact, more precise information about the index of (142); in particular there are relative index theorems when we take another $\gamma$ such that $\Gamma_{1-\gamma} \cap D(y) = \emptyset$. This is connected with the fact that $\sigma_M^{-1}(\mathcal{A})(y,z)$ can be extended to a meromorphic Fredholm function with poles of finite orders at the points $z \in D(y)$ and with Laurent coefficients at $(z-p)^{-(k+1)}, k \in \mathbb{N}$, that are smoothing operators (in Boutet de Monvel's theory on the interval $I$) of finite-rank.

The theory on such meromorphic operator functions in general may be found in Gohberg and Sigal [8]; it has been specified by Schulze and Tarkhanov [36] for the psedo-differential set-up on a closed compact manifold, and it can easily be adapted to boundary value problems with the transmission property which is the framework for the present application.

*Remark* 3.7. If we consider a general psedo-differential crack problem of our theory where the involved conormal symbols are all meromorphic (with corresponding finite- rank Laurent cofficients which is a special case for the continuous asymptotics) the above assertions on $\sigma_\wedge(\mathcal{A})(y,\eta)$ and $\sigma_\partial(\mathcal{A})(y,\eta)$ remain true also in this situation. In particular, calculating $\sigma_\partial(\mathcal{A})^{-1}(y,\eta)$, again within a corresponding space of meromorphic operator functions, we get the typical contributions to the asymptotics of solutions from the poles of $\sigma_\partial(\mathcal{A})^{-1}(y,\eta)$ by the same scheme as in classical problems for differential operators.

By definition the conormal symbols in the crack theory are nothing else than families of elliptic (psedo-differential) boundary value problems on an interval. As such they are much simpler than those in general edge problems when the model cone is of higher dimension. Specific results on asymptotics of solutions to elliptic crack problems require the evaluation of corresponding poles and Laurent coefficients explicitly. If they depend on $y$ (the variable on the crack boundary), a general answer seems possible only in the framework of variable discrete asymptotics. We do not discuss these questions in detail here.

Let us finally note that there are simple examples, where the asymptotics are discrete and independent of $y$. For instance, if the crack is a disk in $\mathbb{R}^3$, centred at the origin, and if both the operator $A$ and the two-sided boundary conditions $T_\pm$ are rotation symmetric, then, the conormal symbol is independent of $y$; hence also the asymptotics of solutions are independent of $y$. This is the case, for instance, for the Lamé system with Dirichlet or Neumann conditions on both sides of the crack.

**3.4. The nature of elliptic crack conditions.** Our calculus of crack problems, locally represented by the operator spaces $\boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{v} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+)$, and globally by $\boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{w})$, $\boldsymbol{w} = (\boldsymbol{n}, \boldsymbol{m}; L_-, L_+; b_-, b_+)$, contains from the very beginning the concept of additional (trace and potential) conditions along the boundary of the crack. To give more information about these so-called crack conditions we consider, for instance, the local spaces that contain the main contributions. The idea of crack conditions is completely analogous to that of boundary conditions in boundary value problems. In the crack situation this means the following. If an operator $\mathcal{B} \in \boldsymbol{\mathcal{V}}^{\mu,d}(I^\wedge \times \Omega, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{v} = (\boldsymbol{n}, \boldsymbol{m})$ is elliptic with respect to $\sigma_\psi$ and $\sigma_{\partial,\pm}$, then, for the weight $\gamma$ involved in $\boldsymbol{g}$, the principal crack symbol

$$\sigma_\wedge(\mathcal{B})(y,\eta) : \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge; \boldsymbol{n}) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}((I,\partial I)^\wedge; \boldsymbol{m}) \tag{146}$$

is a family of Fredholm operators for $(y,\eta) \in T^*\Omega\backslash 0, s > \max(\mu, d) - \frac{1}{2}$, provided the principal conormal symbol

$$\sigma_M(\mathcal{B})(y,z) : \begin{array}{c} H^s(I,\mathbb{C}^m) \\ \oplus \\ \mathbb{C}^{N_-(\iota_-)+N_-(\iota_+)} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(I,\mathbb{C}^m) \\ \oplus \\ \mathbb{C}^{N_+(\iota_-)+N_+(\iota_+)} \end{array} \tag{147}$$

is a familiy of isomorphisms for all $z \in \Gamma_{1-\gamma}$, $s > \max(\mu, d) - \frac{1}{2}$. The latter property automatically holds for all $\tilde{\gamma} \in \mathbb{R}$ with $|\gamma - \tilde{\gamma}| < \varepsilon$ for some $\varepsilon > 0$. On the other hand, if $\Gamma_{1-\gamma}$ contains non-bijectivity points of $\sigma_M(\mathcal{B})$, we can pass to an alternative weight $\tilde{\gamma}$ in a neighbourhood of the given $\gamma$ without destroying the continuity of Green operators, such that (147) is bijective for all $z \in \Gamma_{1-\tilde{\gamma}}$. Clearly, all this is only true for $y$ in a neighbourhood of a given $y_0 \in \Omega$. The theory of elliptic operators globally along $\Omega$ needs the existence of a weight $\gamma$ such that (147) is bijective for all $y \in \Omega$. We require the latter property in our calculus; in other words, under this assumption (146) is a family of Fredholm operators, DN-homogeneous with respect to $\eta \neq 0$. Now the point is that ellipticity of a full "crack problem", associated with $\mathcal{B}$, requires more, namely the existence of additional conditions along $\Omega$, that means the existence of a $3 \times 3$- block matrix

$$\sigma_\wedge(\mathcal{A})(y,\eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge; \boldsymbol{n}) \\ \oplus \\ \mathbb{C}^{L_-} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}((I,\partial I)^\wedge; \boldsymbol{m}) \\ \oplus \\ \mathbb{C}^{L_+} \end{array}$$

$(y,\eta) \in T^*\Omega\backslash 0$, with $(\mathcal{A}_{ij})_{i,j=1,2} = (\mathcal{B}_{ij})_{i,j=1,2} (= \mathcal{B})$, which fills up the Fredholm family (146) to a family of isomorphisms. In this connection we have necessarily

$$\operatorname{ind} \sigma_\wedge(\mathcal{B})(y,\eta) = L_+ - L_-; \tag{148}$$

in particular, $\dim\ker\sigma_\wedge(\mathcal{B})(y,\eta)$ and $\dim\operatorname{coker}\sigma_\wedge(\mathcal{B})(y,\eta)$ only depend on $|\eta|$. In crack problems for differential operators $A$ with differential conditions $T_\pm$ (in the notation of Section 1.1) the conormal symbol $\sigma_M(\mathcal{B})(y,z)$ (with $\mathcal{B}$ representing the problem (13)) is holomorphic, and $\mathcal{B}$ does not contain Green operators at all. We may have many admissible weights $\gamma$, i.e., where $\sigma_M(\mathcal{B})(y,z)$ is bijective for all $z \in \Gamma_{1-\gamma}$, provided the $y$-dependence of the coefficients is not too nasty. To illustrate the idea we assume that $\sigma_M(\mathcal{B})$ is independent of $y$ (cf. the examples in the preceding section). Then our countable exceptional set $D \subset \mathbb{C}$ of non-bijectivity points of $\sigma_M(\mathcal{B})(y,z)$ is also independent of $y$, and each possible choice of a weight $\gamma$ is determined by $\Gamma_{1-\gamma} \cap D = \emptyset$. Now, $\operatorname{ind}\sigma_\wedge(\mathcal{B})(y,\eta)$ depends on $\gamma$ (denote it for the moment by $\operatorname{ind}_\gamma\sigma_\wedge(\mathcal{B})(y,\eta)$), and as is well known (by many concrete investigations, and by the calculus of psedo-differential cone boundary value problems in general), there is an expression of

$$\operatorname{ind}_\gamma\sigma_\wedge(\mathcal{B})(y,\eta) - \operatorname{ind}_{\tilde{\gamma}}\sigma_\wedge(\mathcal{B})(y,\eta)$$

(a so-called relative index formula) for different choices of $\gamma, \tilde{\gamma}$ with $(\Gamma_{1-\gamma} \cup \Gamma_{1-\tilde{\gamma}}) \cap D = \emptyset$, cf. Remark 3.6. Our theory only employs the property $\Gamma_{1-\gamma} \cap D = \emptyset$ but not that $\sigma_\wedge(\mathcal{B})(y,\eta)$ is bijective for all $(y,\eta) \in T^*\Omega \setminus 0$. Then, according to the numbers $L_-, L_+$ in (148) we have to pose corresponding crack conditions (possibly $L_- + M$ potential and $L_+ + M$ trace conditions for some $M \in \mathbb{N}$ that is uniformly bounded on compact subsets of $\Omega$). This may appear "non-physical" in problems in mechanics. For the concrete analysis it is a task to determine such weights $\gamma$ for which $\sigma_\wedge(\mathcal{B})(y,\eta)$ is bijective without extra crack conditions, or to get more information on the dimensions $L_\pm = L_\pm(\gamma)$. This problem is not the main objective of our paper, though there are many examples, where this information is explicitly known. For instance, there is an order reducing device in our crack operator algebra that allows us to construct reductions of orders between our weighted Sobolev spaces. Such order reducing operators are elliptic in our calculus without additional crack conditions for prescribed weights.

*Remark* 3.8. To simplify notation in this paper we have mainly employed trivial vector bundles in the description of operators, distribution spaces and boundary and crack conditions, cf. Remark 2.26. In a general theory there may appear non-trivial bundles automatically. That is why in Section 1.5 we have formulated the general theory of boundary value problems in terms of arbitrary vector bundles. We can easily formulate all our results on crack operators in such a more general framework by replacing the dimension data by corresponding tuples of vector bundles. This is by no means superfluous. In fact, the family of Fredholm operators (146) that is DN-homogeneous with respect to $\eta$ can be reduced to a family of Fredholm operators, parametrised by the points $(y,\eta)$ of $S^*Y$, the unit cosphere bundle of the boundary of the crack. As such it has a K-theoretic index element

$$\mathrm{ind}_{S^*Y}\, \sigma_\wedge(\mathcal{B}) \in K(S^*Y),$$

cf. Atiyah and Bott [1], or Rempel and Schulze [17]. There are vector bundles $U^-$, $U^+ \in \mathrm{Vect}(S^*Y)$ such that $\mathrm{ind}_{S^*Y}\, \sigma_\wedge(\mathcal{B}) = [U^+] - [U^-]$ (our discussion so far referred to the case $U^\pm = S^*Y \times L_\pm$). It is in general not true that $\mathrm{ind}_{S^*Y}\, \sigma_\wedge(\mathcal{B})$ belongs to the pull-back $\pi_Y^* K(Y) \subset K(S^*Y)$ of $K(Y)$ under the canonical projection $\pi_Y : S^*Y \to Y$. The condition $\mathrm{ind}_{S^*Y}\, \sigma_\wedge(\mathcal{B}) \in \pi_Y^* K(Y)$ is an analogue of the Atiyah-Bott condition [1]; it is necessary and sufficient for the existence of additional trace and potential conditions along $Y$ in our framework, i.e., for the existence of elements $V^\pm \in \mathrm{Vect}(Y)$ such that $\sigma_\wedge(\mathcal{B})(y,\eta)$ can be filled up to a family of isomorphisms

$$\begin{matrix} \mathcal{K}^{s,\gamma}((I,\partial I)^\wedge; \boldsymbol{n}) & & \mathcal{K}^{s-\mu,\gamma-\mu}((I,\partial I)^\wedge; \boldsymbol{m}) \\ \oplus & \longrightarrow & \oplus \\ V_y^- & & V_y^+ \end{matrix}$$

for all $(y,\eta) \in T^*Y \backslash 0$. In the latter case our extra conditions just refer to the bundles $V^\pm$. Notice that the condition $\mathrm{ind}_{S^*Y}\, \sigma_\wedge(\mathcal{B}) \in \pi_Y^* K(Y)$ is independent of the choice of the weight $\gamma$, though $V^\pm$ may depend on $\gamma$, cf. [25], Proposition 2.1.136 for an analogous situation in boundary value problems.

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