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ADJOINT VARIABLES: A COMMON WAY TO FIRST INTEGRALS AND INVERSE PROBLEMS

Abstract. We obtain a type of first integrals for ODE by using a generalization of the method of adjoint variables to higher-order systems. A generalization of the harmonic oscillator and a classical spinning particle is completely discussed in order to solve the associated inverse problem.

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Introduction

Until now, the method of *adjoint* (or *added*) *variables* was used only for systems of differential equations which involve the derivatives of first and second order. So, for second order ODE see [6], [7] and for second order PDE see [1], [2]. A geometrical framework of this method for first and second-order ODE can be found in [8]. For other applications about this approach see [10, chapter 3].

The aim of the present paper is to extend this method to higher-order ODE. This generalization appears in the first section via the usual notion of adjoint for a linear operator. While the first section deals with implicit systems, in the next section systems in kinematical form are considered. The results of that section are generalizations of similar results from [6]. The linear case, and more particularly, the linear autonomous case are also studied. Another important part of the paper is Section 4 in which the inverse problem yields the Helmholtz equations as characterization of self-adjoint systems. As an example, a system generalizing both the harmonic oscillator and a spinning particle is given. Using the obtained first integrals the corresponding inverse problem is solved. A short version of this paper is [3].

1. The adjoint variables approach

Let us consider a general system of kth order ODE, with k and n two natural numbers:

$$F^j(t, x, x_1, \dots, x_k) = 0, \quad 1 \leq j \leq n, \tag{1.1}$$

with the unknow function $x = (x^i)_{1 \leq i \leq n}$ and

$$x^i_\alpha = \frac{d^\alpha x^i}{dt^\alpha}, \quad 1 \leq \alpha \leq k. \tag{1.2}$$

A *first integral* (or *conservation law* or *conserved quantity*) for the system (1.1) is a function $\mathcal{F} = \mathcal{F}(t, x, x_1, \dots, x_{k-1})$ satisfying:

$$\frac{d\mathcal{F}}{dt} = 0, \mathrm{mod}(1.1), \tag{1.3}$$

where mod(1.1) means "along the solutions of (1.1)" and $\frac{d}{dt}$ denotes the total differentiation with respect to t: $\frac{d}{dt} = \frac{\partial}{\partial t} + x^i_1 \frac{\partial}{\partial x^i} + \dots + x^i_k \frac{\partial}{\partial x^i_{k-1}}$.

For $\xi = (\xi^i)_{1 \leq i \leq n}$ define a point-transformation by

$$\tilde{x}^i = x^i + \varepsilon \xi^i \tag{1.4}$$

with ε a real number with sufficiently small absolute value. The Fréchet derivative of F^j is:

$$dF^j(x)(\xi) = \lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon}(\tilde{F}^j - F^j), \tag{1.5}$$

where $\tilde{F}^j = F^j(t, \tilde{x}, \dots, \tilde{x}_k)$. Then

$$dF^j(x)(\xi) = \sum_{\alpha=1}^{k+1} \frac{\partial F^j}{\partial x^i_{k-\alpha+1}} \frac{d^{k-\alpha+1}\xi^i}{dt^{k-\alpha+1}}. \tag{1.6}$$

We search for (1.1) a first integral in the form

$$\mathcal{F} = \sum_{\alpha=1}^{k} (-1)^{\alpha+1} T_{(\alpha)i} \frac{d^{k-\alpha}\xi^i}{dt^{k-\alpha}} - K \tag{1.7}$$

with $\frac{d^0 f}{dt^0} = f$ and $K = K(t, x, \dots, x_{k-1})$. We require that:

$$\frac{d\mathcal{F}}{dt} = \mu_j \sum_{\alpha=1}^{k+1} \frac{\partial F^j}{\partial x^i_{k-\alpha+1}} \frac{d^{k-\alpha+1}\xi^i}{dt^{k-\alpha+1}} - \frac{dK}{dt} \tag{1.8}$$

with $\mu = (\mu_j)_{1 \leq j \leq n}$.

From (1.7) and (1.8) it results by comparing the coefficients of $\frac{d^{k-\alpha+1}\xi^i}{dt^{k-\alpha+1}}$:

$$T_{(\alpha)i} = \frac{d}{dt} T_{(\alpha-1)i} + (-1)^{\alpha+1} \mu_j \frac{\partial F^j}{\partial x^i_{k-\alpha+1}}, \quad 1 \leq \alpha \leq k, \quad T_{(0)i} = 0, \tag{1.9}$$

and we obtain the main result of the paper:

Theorem. *If the functions $\xi = (\xi^i)$, $\mu = (\mu_j)$ of (t, x) satisfy the system:*

$$\frac{d}{dt} T_{(k)i} + (-1)^k \mu_j \frac{\partial F^j}{\partial x^i} = 0, \tag{1.10a}$$

$$\mu_j \sum_{\alpha=1}^{k+1} \frac{\partial F^j}{\partial x^i_{k-\alpha+1}} \frac{d^{k-\alpha+1}\xi^i}{dt^{k-\alpha+1}} = \frac{dK}{dt} \tag{1.10b}$$

along the solutions of (1.1) *with* $K = K(t, x, \dots, x_{k-1})$ *a given function, then* $\mathcal{F}$ *given by* (1.7) *is a first integral for* (1.1).

Remarks. (i) If F^j does not depend on x, then (1.10a) becomes $T_{(k)i} =$ const.

(ii) The relation (1.10b) is exactly:

$$\mu_j dF^j(x)(\xi) = \frac{dK}{dt} \tag{1.11}$$

and then the equations (1.9) + (1.10a) are:

$$(dF(x))^*_i(\mu) = 0, \tag{1.12}$$

where $(dF(x))_i^*(\mu)$ is *the adjoint* to the linear operator $dF(x)$:

$$(-1)^k(dF(x))_i^*(\mu) = \frac{d^k}{dt^k}\left(\frac{\partial F^j}{\partial x_k^i}\mu_j\right) - \frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial F^j}{\partial x_{k-1}^i}\mu_j\right) + \cdots + \\ +(-1)^{k-1}\frac{d}{dt}\left(\frac{\partial F^j}{\partial x_1^i}\mu_j\right) + (-1)^k\frac{\partial F^j}{\partial x^i}\mu_j. \tag{1.13}$$

(ii) For $k = 2$, i.e., for system

$$F^j(t, x, \dot{x}, \ddot{x}) = 0$$

the equations (1.10) read ([6, p. 2])

$$\frac{d}{dt}\Big[\frac{d}{dt}\Big(\mu_j\frac{\partial F^j}{\partial \ddot{x}^i}\Big) - \mu_j\frac{\partial F^j}{\partial \dot{x}^i}\Big] + \mu_j\frac{\partial F^j}{\partial x^i} = 0,$$

$$\mu_j\Big(\frac{\partial F^j}{\partial \ddot{x}^i}\frac{d^2\xi^i}{dt^2} + \frac{\partial F^j}{\partial \dot{x}^i}\frac{d\xi^i}{dt} + \frac{\partial F^j}{\partial x^i}\xi^i\Big) = \frac{dK}{dt}$$

and

$$\mathcal{F} = \mu_j\frac{\partial F^j}{\partial \ddot{x}^i}\frac{d\xi^i}{dt} - \Big[\frac{d}{dt}\Big(\mu_j\frac{\partial F^j}{\partial \ddot{x}^i}\Big) - \mu_j\frac{\partial F^j}{\partial \dot{x}^i}\Big]\xi^i - K.$$

2. Systems in Kinematical Form

Since the expression of the adjoint operator is more complicated comparing with the expression of the direct operator, an interesting question appears: Under what conditions on the system (1.1) there exists a system $G^j = 0$ such that the adjoint $(dF(x))_i^*(\mu)$ is exactly $dG^j(x)(\mu)$?

By comparing the coefficients of $\frac{d^{k-1}\mu_j}{dt^{k-1}}$, we obtain

$$\frac{\partial G^j}{\partial x_{k-1}^i} = (-1)^k k\frac{d}{dt}\Big(\frac{\partial F^j}{\partial x_k^i}\Big) + (-1)^{k-1}\frac{\partial F^j}{\partial x_{k-1}^i},$$

that is,

$$\frac{d}{dt}\Big(\frac{\partial F^j}{\partial x_k^i}\Big) = \frac{1}{k}\Big[\frac{\partial F^j}{\partial x_{k-1}^i} + (-1)^k\frac{\partial G^j}{\partial x_{k-1}^i}\Big].$$

But the right hand side of the last relation contains at most x_k^i and then $\frac{\partial F^j}{\partial x_k^i}$ must depend on $(t, x, \ldots, x_{k-1})$ i.e. we need:

Definition 1. The system (1.1) is said to be *in principal form* if

$$F^j = A_i^j(t, x, \ldots, x_{k-1})x_k^i + B^j(t, x, \ldots, x_{k-1}).$$

The dependence of F^j by x_k implies that the matrix $A = \left(A_i^j\right)$ is nondegenerate, i. e. $\det A \neq 0$. Since the systems $F^j = 0$ and $A^{-1} \cdot F^j = 0$ have the same solutions, we need

Definition 2. The system (1.1) is said to be *in kinematical form* if

$$F^j = x^j_k - f^j(t, x, x_1, \dots, x_{k-1}). \tag{2.1}$$

Then, along solutions of (1.1) the total differentiation operator $\frac{d}{dt}$ is reduced to the vector field $\frac{D}{Dt} = \frac{\partial}{\partial t} + x^i_1 \frac{\partial}{\partial x^i} + \cdots + x^i_{k-1}\frac{\partial}{\partial x^i_{k-2}} + f^i \frac{\partial}{\partial x^i_{k-1}}$ and the theorem becomes

Proposition 1. *If ξ and μ satisfy*

$$\frac{D}{Dt} T_{(k)i} = (-1)^k \mu_j \frac{\partial f^j}{\partial x^i}, \tag{2.2a}$$

$$\mu_j\Big(\frac{D^k\xi^j}{Dt^k} - \sum_{\alpha=2}^{k+1} \frac{\partial f^j}{\partial x^i_{k-\alpha+1}} \frac{D^{k-\alpha+1}\xi^j}{Dt^{k-\alpha+1}}\Big) = \frac{DK}{Dt}, \tag{2.2b}$$

where

(i) $K = K(t, x, \dots, x_{k-1})$ *is a given function,*

(ii) $T_{(1)i} = \mu_i$ *and*

$$T_{(\alpha)i} = \frac{D}{Dt} T_{(\alpha-1)i} + (-1)^\alpha \mu_j \frac{\partial f^j}{\partial x^i_{k-\alpha+1}}, \quad 2 \le \alpha \le k, \tag{2.2a'}$$

then $\mathcal{F}$ given by (1.5) *is a first integral for* (2.1).

Using the relations

$$\Big[\frac{\partial}{\partial x^i_\alpha}, \frac{D}{Dt}\Big] = \frac{\partial f^j}{\partial x^i_\alpha}\frac{\partial}{\partial x^i_{k-1}} + \frac{\partial}{\partial x^i_{\alpha-1}}, \quad 0 \le \alpha \le k-1, \tag{2.6}$$

it results in

Proposition 2. *If $\mathcal{F}_1, \mathcal{F}_2$ are first integrals of* (2.1), *then*

$$\mu_i = \mathcal{F}_1 \frac{\partial \mathcal{F}_2}{\partial x^i_{k-1}} \tag{2.7}$$

satisfy (2.2a) + (2.2a').

Proof. By induction we get

$$T_{(\alpha)i} = (-1)^{\alpha+1}\mathcal{F}_1 \frac{\partial \mathcal{F}_2}{\partial x^i_{k-\alpha}}, \quad 0 \le \alpha \le k. \tag{2.8}$$

Indeed, for $\alpha = 1$ we have (2.7) from $T_{(1)i} = \mu_i$ (cf. Proposition 1) and (2.7). Let us suppose that we have (2.8) for a fixed α. Then, using (2.2a'),

$$T_{(\alpha+1)i} = (-1)^{\alpha+1}\mathcal{F}_1 \frac{D}{Dt}\frac{\partial \mathcal{F}_2}{\partial x^i_{k-\alpha}} + (-1)^{\alpha+1}\mu_j \frac{\partial f^j}{\partial x^i_{k-\alpha}} \overset{(2.6)}{=}$$

$$\overset{(2.6)}{=} (-1)^{\alpha+2}\mathcal{F}_1 \frac{\partial \mathcal{F}_2}{\partial x^i_{k-(\alpha+1)}},$$

which is what we require. Then $T_{(k)i} = (-1)^{k+1}\mathcal{F}_1 \frac{\partial \mathcal{F}_2}{\partial x^i}$, which gives

$$\frac{D}{Dt}T_{(k)i} = (-1)^{k+1}\mathcal{F}_1 \frac{D}{Dt}\frac{\partial \mathcal{F}_2}{\partial x^i} \overset{(2.6)}{=} (-1)^{k+2}\mathcal{F}_1 \frac{\partial f^j}{\partial x^i}\frac{\partial \mathcal{F}_2}{\partial x^j_{k-1}} = (-1)^k \mu_j \frac{\partial f^j}{\partial x^i},$$

hence the conclusion of the proposition. □

In particuler, by putting $\mathcal{F}_1 = 1$ we obtain

Proposition 3. *A first integral* Ω *of* (2.1) *gives rise to a new one:*

$$\mathcal{F}_\Omega = \sum_{\alpha=1}^{k} \frac{\partial \Omega}{\partial x^i_{k-\alpha}} \frac{D^{k-\alpha}\xi^i}{Dt^{k-\alpha}} - K, \tag{2.9}$$

where $\xi = \xi(t,x)$ *satisfy:*

$$\frac{\partial \Omega}{\partial x^j_{k-1}}\Big(\frac{D^k \xi^j}{Dt^k} - \sum_{\alpha=2}^{k+1} \frac{\partial f^j}{\partial x^i_{k-\alpha+1}} \frac{D^{k-\alpha+1}\xi^i}{Dt^{k-\alpha+1}}\Big) = \frac{DK}{Dt}. \tag{2.10}$$

Let us consider the particular case where the functions (f^j) does not depend on time, that is, the considered system is *autonomous* (or time-independent). Then a straightforward calculus gives that $\xi^i = x^i_1$ (although we require only $\xi = \xi(t,x)$!) satisfy (2.2b) with $K = 0$. Hence

Proposition 4. *If* $f^j = f^j(x, \dots, x_{k-1})$ *and* $\mu = \mu(t,x)$ *satisfy* (2.2a) + (2.2a')*, then:*

$$\mathcal{F}_\mu = \mu_i f^i + \sum_{\alpha=2}^{k} (-1)^{\alpha+1} T_{(\alpha)i} x^i_{k-\alpha+1} \tag{2.11}$$

is a first integral for (2.1).

3. The Linear Case

Let us suppose that F^j is linear:

$$F^j(x) = A^j_{(0)i} x^i_k + \ldots + A^j_{(k)i} x^i \tag{3.1}$$

with $A^j_{(\alpha)i} = A^j_{(\alpha)i}(t)$, $1 \le i,j \le n$, $0 \le \alpha \le k$. Therefore

$$\frac{\partial F^j}{\partial x^i_{k-\alpha}} = A^j_{(\alpha)i} \tag{3.2}$$

and then $dF^j(x)(\xi) = F^j(\xi)$. One obtains

$$(-1)^k((dF(x))^*_i(\mu) = (-1)^k F^*_i(\mu) = \frac{d^k}{dt^k}(A^j_{(0)i}\mu_j)6 - \frac{d^{k-1}}{dt^{k-1}}(A^j_{(1)i}\mu_j) + \cdots + \\ + (-1)^{k-1}\frac{d}{dt}(A^j_{(k-1)i}\mu_j) + (-1)^k A^j_{(k)i}\mu_j, \tag{3.3}$$

and then, for linear systems the theorem becomes

Proposition 5. *If ξ and μ satisfy*

$$F_i^*(\mu) = 0, \tag{3.4a}$$

$$\mu_j F^j(\xi) = \frac{dK}{dt}, \tag{3.4b}$$

then $\mathcal{F}$ given by (1.7) *is a first integral for* (3.1). *For $K = 0$, it follows from* (3.4*b*) *the solution $\xi = x$.*

More specifically, if the coefficients $A^j_{(\alpha)i}$ are constants, that is, in the case of linear autonomous systems, we have

Corollary. *If μ satisfy:*

$$A^j_{(0)i}\frac{d^k\mu_j}{dt^k} - A^j_{(1)i}\frac{d^{k-1}\mu_j}{dt^{k-1}} + \ldots + (-1)^{k-1}A^j_{(k-1)i}\frac{d\mu_j}{dt} + \\ +(-1)^k A^j_{(k)i}\mu_j = 0, \tag{3.5}$$

then $\mathcal{F}$ given by

$$\mathcal{F} = \sum_{\alpha=1}^{k}(-1)^{\alpha+1}T_{(\alpha)i}x^i_{k-\alpha} \tag{3.6}$$

is a first integral for (3.1).

4. Self-Adjoint Systems and Helmholtz Equations

Recall that the vector operator $F = (F^j)$ is called *self-adjoint* if the Fréchet derivative of F is self-adjoint and this property is a necessary and sufficient condition for the system $F^j = 0$ to be obtained by a variational principle, i.e., for a Lagrangian L to exist such that the given system is exactly the Euler-Lagrange system for L.

Comparing the relations (1.6) and (1.13), we see that the given system is self-adjoint if and only if

$$(-1)^k\frac{\partial F^i}{\partial x^j_{k-\alpha}} = (-1)^\alpha\frac{\partial F^j}{\partial x^i_{k-\alpha}} + \binom{k}{\alpha}\frac{d^\alpha}{dt^\alpha}\Big(\frac{\partial F^j}{\partial x^i_k}\Big) - \\ - \binom{k-1}{\alpha-1}\frac{d^{\alpha-1}}{dt^{\alpha-1}}\Big(\frac{\partial F^j}{\partial x^i_{k-1}}\Big) + \cdots \tag{4.1}$$

for $0 \leq \alpha \leq k$. For example, if $k = 2$, we obtain the classical *Helmholtz equations* [4]

$$\begin{cases} \frac{\partial F^i}{\partial x^j_2} = \frac{\partial F^j}{\partial x^i_2} \\ \frac{\partial F^i}{\partial x^j_1} + \frac{\partial F^j}{\partial x^i_1} = 2\frac{d}{dt}\Big(\frac{\partial F^j}{\partial x^i_2}\Big) \\ \frac{\partial F^i}{\partial x^j} - \frac{\partial F^j}{\partial x^i} = \frac{1}{2}\frac{d}{dt}\Big(\frac{\partial F^i}{\partial x^j_1} - \frac{\partial F^j}{\partial x^i_1}\Big) \end{cases}. \tag{4.2}$$

The reader is invited to compare our equations (4.1) with the relations (5.5) from [11, p. 1186].

5. A Generalization of the Harmonic oscillator and a Spinning Particle

Let us consider the system of order $k = 2m$:

$$F^i = x^i_{2m} + x^i_{2m-2} = 0, 1 \leq i \leq 3, \tag{5.1}$$

which is a linear system and so the above corollary works. For $m = 1$ this system describes the harmonic oscillator and for $m = 2$ a spinning particle([5]). The associated system (3.5) is

$$\frac{d^{2m}\mu_i}{dt^{2m}} + \frac{d^{2m-2}\mu_i}{dt^{2m-2}} = 0, \tag{5.2}$$

and we have two solutions

$$\mu_i = \cos t, \tag{5.3a}$$

$$\mu_i = \sin t \tag{5.3b}$$

with the corresponding first integrals (3.6)

$$\mathcal{F}^i_{\cos t} = x^i_{2k-1} \sin t - x^i_{2k-2} \cos t, \tag{5.4a}$$

$$\mathcal{F}^i_{\sin t} = x^i_{2k-1} \cos t + x^i_{2k-2} \sin t. \tag{5.4b}$$

Note that from Proposition 3 we have $\mathcal{F}_{\mathcal{F}_{\cos t}} = \mathcal{F}_{\cos t}$ and $\mathcal{F}_{\mathcal{F}_{\sin t}} = \mathcal{F}_{\sin t}$.

The system (5.1) is self-adjoint and then we are interested in finding the associated Lagrangian. Also, from (5.1) we have the first integral

$$C^i = x^i_{2m-1} + x^i_{2m-3} \tag{5.5}$$

which, in order to eliminate the variable t, yields the first integral

$$\Psi^i = (C^i)^2 - (\mathcal{F}^i_1)^2 - (\mathcal{F}^i_2)^2. \tag{5.6}$$

A straightforward computation gives:

$$\Psi^i = (x^i_{2m-3})^2 - (x^i_{2m-2})^2 + 2x^i_{2m-3}x^i_{2m-1} \tag{5.7}$$

and then we have the first integral

$$H = \frac{1}{2}(\Psi^1 + \Psi^2 + \Psi^3) =$$

$$= \frac{1}{2}\sum_{i=1}^{3}(x_{2m-3})^2 - \frac{1}{2}\sum_{i=1}^{3}(x_{2m-2})^2 + \sum_{i=1}^{3} x^i_{2m-3}x^i_{2m-1}, \tag{5.8}$$

which is exactly the Hamiltonian for (5.1). The associated Lagrangian is

$$L = \frac{1}{2}\sum_{i=1}^{3}(x^i_{m-1})^2 - \frac{1}{2}\sum_{i=1}^{3}(x^i_m)^2, \tag{5.9}$$

a result very important from the point of view of Inverse Problem of Analytical Mechanics ([9]). Thus we solve in this way the inverse problem for the harmonic oscillator and a spinning particle.

REFERENCES

1. G. CAVIGLIA, Composite variational principles, added variables and constants of motion, *Internat. J. Theoret. Phys.* **25**(1986), 139–146.
2. G. CAVIGLIA, Composite variational principles and the determination of conservation laws. *J. Math. Phys.* **29**(1988), 812–816.
3. M. CRÂŞMĂREANU, Adjoint variables for higher-order equations. Applications to spinning particle. *Libertas Math.* **XIX**(1999), 77–81.
4. H. HELMHOLTZ, Über die Physikelische Bedeutung des Prinzips der Kleinsten Wirkung. *J. Reine. Angew. Math.* **100**(1887), 137–166.
5. M. DE LEÓN AND D. DE DIEGO, Symmetries and constants of motion for higher-order Lagrangian systems. *J. Math. Phys.* **36**(1995), 4138–4161.
6. F. MIMURA AND T. NÔNO, A method for deriving new conservation laws. *Bull. Kyushu Inst. Tech. Math. Natur. Sci.* 42(1995), 1–17.
7. F. MIMURA, T. IKEDA, AND F. FUJIWARA, A geometric derivation of new conservation laws. *Bull. Kyushu Inst. Tech. Math. Natur. Sci.* 43(1996), 37–55.
8. V. OBĂDEANU, M. CIOBANU, AND M. NEAMŢU, Dynamical systems and geometrical structures. *Seminar of Mechanics-Differentiable Dynamical Systems,* 69, *West University of Timisoara,* 2000.
9. R. M. SANTILLI, Foundations of theoretical mechanics, I: The inverse problem in Newtonian mechanics. *Texts and Monographs in Physics, Springer-Verlag, N.Y.*, 1978.
10. S. SIENIUTYCZ, Conservation laws in variational thermo-Hydrodynamics. *Kluwer Academic Publishers, Mathematics and Its Applications,* No. 279, 1994.
11. A. VANDERBAUWHEDE, Potential operators and the inverse problem of classical mechanics. *Hadronic J.* **1**(1979), 1177–1197.

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