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ON BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER

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Let $-\infty < a < b < +\infty$, $I = [a, b]$, $p : C(I; \mathbb{R}^n) \to L(I; \mathbb{R}^n)$ and $\ell : C(I; \mathbb{R}^n) \to \mathbb{R}^n$ be linear bounded operators, $q \in L(I; \mathbb{R}^n)$ and $c_0 \in \mathbb{R}^n$. On the basis of the results from [1], in the present paper we establish new sufficient conditions for unique solvability of the boundary value problem

$$\frac{dx(t)}{dt} = \varepsilon p(x)(t) + q(t), \tag{1}$$

$$\ell(x) = c_0, \tag{2}$$

where $\varepsilon > 0$ is a small parameter.

Throughout the paper, the following notation will be used.

$\mathbb{R} =]-\infty, \infty[$, $\mathbb{R}_+ = [0, \infty[$;

χ_I is the characteristic function of the interval I, i.e.,

$$\chi_I(t) = \begin{cases} 1 & \text{for } t \in I \\ 0 & \text{for } t \notin I \end{cases};$$

$\mathbb{R}^n$ is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with the elements $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$\mathbb{R}^{n\times n}$ is the space of $n \times n$-matrices $X = (x_{ik})_{i,k=1}^n$ with the elements $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$ and the norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

if $x = (x_i)_{i=1}^n$, $y = (y_i)_{i=1}^n \in \mathbb{R}^n$ and $X = (x_{ik})_{i,k=1}^n$, $Y = (y_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$, then

$$x \le y \Longleftrightarrow x_i \le y_i \ \ (i = 1, \dots, n) \ \text{ and } \ X \le Y \Longleftrightarrow x_{ik} \le y_{ik} \ \ (i, k = 1, \dots, n),$$
$$|x| = (|x_i|)_{i=1}^n, \ \ |X| = (|x_{ik}|)_{i,k=1}^n;$$

$\det(X)$ is the determinant of the matrix X;

X^{-1} is the inverse matrix to X;

$r(X)$ is the spectral radius of the matrix X;

E is the unit matrix;

Θ is the zero matrix;

$C(I; \mathbb{R}^n)$ is the space of continuous vector functions $x : I \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\big\{\|x(t)\| : \ t \in I\big\};$$

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if $x = (x_i)_{i=1}^n \in C(I;\mathbb{R}^n)$, then

$$|x|_C = \big(\|x_i\|_C\big)_{i=1}^n;$$

$L(I;\mathbb{R}^n)$ is the space of integrable vector functions $x : I \to \mathbb{R}^n$ with the norm

$$\|x\|_L = \int_a^b \|x(t)\|\,dt;$$

$L(I;\mathbb{R}^{n\times n})$ is the space of integrable matrix functions $X : I \to \mathbb{R}^{n\times n}$;

if $Z \in C(I;\mathbb{R}^{n\times n})$ is a matrix function with the columns $z_1,\dots,z_n$ and $g : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ is a linear operator, then $g(Z)$ stands for the matrix function with columns $g(z_1),\dots,g(z_n)$.

Below we will assume that $p : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ is a strongly bounded operator, i.e., there exists $\eta \in L(I;\mathbb{R}_+)$ such that

$$\|p(x)(t)\| \le \eta(t)\|x\|_C \ \text{ for } \ t \in I, \ \ x \in C(I;\mathbb{R}^n).$$

Theorem 1. *Let*

$$\mathcal{P}_0(t) = E, \quad \mathcal{P}_k(t) = \int_a^t p(\mathcal{P}_{k-1})(s)\,ds \ \textit{ for } \ t \in I \ \ (k = 1,2,\dots) \tag{3}$$

and there exist a nonnegative integer k_0 such that

$$\det\big(l(\mathcal{P}_{k_0})\big) \ne 0 \tag{4}$$

and, in case $k_0 \ge 1$,

$$l(\mathcal{P}_k) = \Theta \ \ (k = 0,\dots,k_0-1). \tag{5}$$

Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0,\varepsilon_0[$ the problem (1), (2) has a unique solution.

Proof. Note in the first place that since the operators p and l are bounded, there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that

$$\int_a^b |p(x)(t)|\,dt \le B|x|_C, \quad |l(x)| \le B|x|_C \ \text{ for } \ x \in C(I;\mathbb{R}^n). \tag{6}$$

For any $\varepsilon > 0$ and $x \in C(I;\mathbb{R}^n)$ set

$$p_\varepsilon(x)(t) = \varepsilon p(x)(t),$$

$$p_\varepsilon^0(x) = x(t), \quad p_\varepsilon^k(x)(t) = \int_a^t p_\varepsilon(p_\varepsilon^{k-1}(x))(s)\,ds \ \ (k = 1,2,\dots),$$

$$\Lambda_{k,\varepsilon} = l\big(p_\varepsilon^0(E) + \dots + p_\varepsilon^{k-1}(E)\big) \ \ (k = 1,2,\dots).$$

Then

$$p_\varepsilon^k(x)(t) = \varepsilon^k p^k(x) \ \ (k = 0,1,\dots), \tag{7}$$

where $p^k : C(I;\mathbb{R}^n) \to C(I;\mathbb{R}^n)$ $(k = 0,1,\dots)$ is a sequence of operators such that

$$p^0(x)(t) = x(t), \quad p^k(x)(t) = \int_a^t p\big(p^{k-1}(x)\big)(s)\,ds \ \ (k = 1,2,\dots). \tag{8}$$

On the other hand, by (3)–(5) we have

$$\Lambda_{k_0+1,\varepsilon} = \varepsilon^{k_0} l(\mathcal{P}_{k_0}), \quad \det(\Lambda_{k_0+1,\varepsilon}) \ne 0. \tag{9}$$

Assuming that

$$p_\varepsilon^{k_0+1,1}(x)(t) = p_\varepsilon^1(x)(t) - \Lambda_{k_0+1,\varepsilon}^{-1} l(p_\varepsilon^{k_0+1}(x)),$$

by virtue of conditions (6)–(8) we find

$$|p_\varepsilon^{k_0+1,1}(x)|_C \le A_\varepsilon |x|_C \ \text{ for } \ x \in C(I;\mathbb{R}^n), \tag{10}$$

where

$$A_\varepsilon = \varepsilon A, \quad A = B + |\Lambda_{k_0+1}^{-1}|B^{k_0+2}.$$

Clearly, if

$$\varepsilon_0 = 1/r(A),$$

then

$$r(A_\varepsilon) < 1 \ \text{ for } \ \varepsilon \in]0, \varepsilon_0[\,. \tag{11}$$

However, by Theorem 1.2 from [1], the conditions (9)–(11) guarantee the unique solvability of the problem (1), (2) for arbitrary $\varepsilon \in]0, \varepsilon_0[\,$. □

A particular case of (2) is the boundary condition

$$\sum_{j=1}^{\nu} A_j x(t_j) = c_0, \tag{12}$$

where $t_j \in I$ and $A_j \in \mathbb{R}^{n\times n}$ $(j = 1, \dots, \nu)$.

The proven theorem immediately implies

Corollary 1. *Let either*

$$\det\Big(\sum_{j=1}^{\nu} A_j\Big) \neq 0,$$

or

$$\sum_{j=1}^{\nu} A_j = \Theta \ \textit{ and } \ \det\Big(\sum_{j=1}^{\nu} A_j \int_a^{t_j} p(E)(s)\, ds\Big) \neq 0.$$

Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0[$ problem (1), (12) *has a unique solution.*

Now we consider the differential system

$$\frac{dx(t)}{dt} = \varepsilon \mathcal{P}(t) x(\tau(t)) + q_0(t) \tag{13}$$

with the boundary conditions

$$x(t) = u(t) \ \text{ for } \ t \notin I, \quad l(x) = c_0 \tag{14}$$

or

$$x(t) = u(t) \ \text{ for } \ t \notin I, \ \ \sum_{j=1}^{\nu} A_j x(t_j) = c_0, \tag{15}$$

where $\mathcal{P} \in L(I; \mathbb{R}^{n\times n})$, $q_0 \in L(I; \mathbb{R}^n)$, $\tau : I \to \mathbb{R}$ is a measurable function and $u : \mathbb{R} \to \mathbb{R}^n$ is a continuous bounded vector function.

If we assume that

$$\tau_0(t) = \begin{cases} a & \text{for } \ t < a \\ \tau(t) & \text{for } \ a \le \tau(t) \le b \\ b & \text{for } \ \tau(t) > b \end{cases},$$

$$p(x)(t) = \chi_I(\tau(t)) \mathcal{P}(t) x(\tau_0(t)),$$

$$q(t) = \varepsilon\big(1 - \chi_I(\tau(t))\big) \mathcal{P}(t) u(\tau(t)) + q_0(t),$$

then the problem (13), (14) reduces to the problem (1), (2). Hence for the problem (13), (14) Theorem 1 can be formulated in the form of

Theorem 2. *Let*

$$\mathcal{P}_0(t) = E, \quad \mathcal{P}_k(t) = \int_a^t \chi_I(\tau(s)) \mathcal{P}(s) \mathcal{P}_{k-1}(\tau_0(s))\, ds \ \ (k = 1, 2, \dots)$$

and there exist a nonnegative integer k_0 such that

$$\det\big(l(\mathcal{P}_{k_0})\big) \neq 0$$

and, in case $k_0 \ge 1$,

$$l(\mathcal{P}_k) = \Theta \ \ (k = 0, \dots, k_0 - 1).$$

Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0[$ the problem (13), (14) *has a unique solution.*

Corollary 2. *Let either*

$$\det\Big(\sum_{j=1}^{\nu} A_j\Big) \neq 0$$

or

$$\sum_{j=1}^{\nu} A_j = \Theta \ \ \textit{and} \ \ \det\Big(\sum_{j=1}^{\nu} A_j \int_a^{t_j} \chi_I(\tau(s))\mathcal{P}(s)\,ds\Big) \neq 0.$$

Then there exists $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in]0, \varepsilon_0[$ *the problem* (13), (15) *has a unique solution.*

In the case $\tau(t) \equiv t$, Theorem 2 implies Corollary 3.1 from [2].

References

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