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POSITIVE SOLUTIONS FOR THE BOUNDARY VALUE PROBLEM $(|u''|^{p-2}u'')'' - \lambda q(t)f(u(t)) = 0$

Abstract. This paper considers the boundary value problem:

$$\begin{cases} \left(|u''|^{p-2}\, u''\right)'' - \lambda q(t) f(u\,(t)) = 0, \text{ in } (0,1)\,, \\ u(0) = u(1) = u''\,(0) = u''\,(1) = 0, \end{cases}$$

with $\lambda > 0$. The value of λ is chosen so that the boundary value problem has a positive solution. Moreover, we derive an explicit interval for λ such that for any λ in this interval, the existence of a positive solution to the boundary value problem is guaranteed. In addition we also discuss the existence of two positive solutions for λ in an appropriate interval.

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რეზიუმე. ნაშრომში განხილულია სასაზღვრო ამოცანა

$$\begin{cases} \left(|u''|^{p-2}\, u''\right)'' - \lambda q(t) f(u\,(t)) = 0, \quad (0,1) - \text{ზე}, \\ u(0) = u(1) = u''\,(0) = u''\,(1) = 0, \end{cases}$$

სადაც $\lambda > 0$. λ-ს მნიშვნელობა ისეა შერჩეული, რომ სასაზღვრო ამოცანას დადებითი ამონახსნი ჰქონდეს. გარდა ამისა, ცხადი სახით მითითებულია λ-ს იმ მნიშვნელობების ინტერვალი რომელთათვისაც სასაზღვრო ამოცანის დადებითი ამონახსნის არსებობა გარანტირებულია. ჩვენ აგრეთვე განვიხილავთ ორი დადებითი ამონახსნის არსებობის საკითხს.

1. Introduction

This paper studies the two-point boundary value problem

$$\begin{cases} (\varphi_p(u''))'' - \lambda q(t) f(u(t)) = 0, & t \in (0,1) \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-1} s$, and $p > 1$.

Equations of the above form (1.1) occur in beam theory [1], for example,

1. a beam with small deformations (also called geometric linearity);
2. a beam of a material which satisfies a nonlinear power-like stress -strain law;
3. a beam with two-sided links (for example, springs) which satisfies a nonlinear power-like elasticity law.

The best known setting is the boundary value problem when $p = 2$, namely

$$u^{(4)} - q(t) f(u(t)) = 0, \quad t \in (0,1).$$

This equation arises when one describes deformations of an elastic beam. Usually both ends are simply supported, or one end is simply supported and the other end is clamped by sliding clamps. Also vanishing moments and shear forces at the beam ends are frequently included in the boundary conditions; see Gupta [2] and Yang [5]. One derivation of this fourth order equation plus the two-point boundary conditions occurs when the method of lines is used in the description over regions of certain partial differential equations describing the deflection of an elastic beam.

Closely related to the results of this paper is the recent work by Ma and Wang [4]. In [4] the authors establish the existence of at least one positive solution of the above fourth order equation for $p = 2$ satisfying the boundary value conditions, when the nonlinearity f is superlinear and sublinear.

Singular nonlinear two-point boundary value problem arise naturally in applications and usually, only positive solutions are meaningful. By a positive solution of (1.1), we mean a function $u \in C^2([0,1], R)$ with $\varphi_p(u'') \in C^2((0,1), R)$ satisfying (1.1), and with u nonnegative and not identically zero on $[0,1]$. If, for a particular λ the boundary value problem (1.1) has a positive solution u, then λ is called an *eigenvalue* and u a corresponding *eigenfunction* of (1.1). The set of eigenvalues of (1.1) will be denoted by E, i.e.,

$$E = \{\lambda > 0 \mid (1.1) - (1.2) \text{ has a positive solution } \}.$$

In section 2, some preliminary results are presented. Section 3 presents explicit eigenvalue intervals in terms of f_0 and f_∞, where

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x^{p-1}} \text{ and } f_\infty = \lim_{x \to \infty} \frac{f(x)}{x^{p-1}}.$$

Finally, we state a fixed point theorem due to Krasnosel'skii which will be needed in this paper.

Theorem 1.1. *Let X be a Banach space, and let $K(\subset X)$ be a cone. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to K$$

be a completely continuous operator such that, either

(a) $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$, or

(b) $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$.

Then, T has a fixed point in $K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

2. Preliminary Results

Throughout this paper, it is assumed that $f : [0, \infty) \to [0, \infty)$ is continuous and that the following condition is satisfied:

$(H1)$ $q \in C(0,1)$ is nonnegative with $\int_0^1 t(1-t)\,q(t)\,dt < \infty$ and there exist a natural number $m \geq 3$ and $c \in \left(\frac{1}{m}, \frac{m-1}{m}\right)$ with $q(c) > 0$.

Let $X = (C[0,1], \|\cdot\|)$ (here $\|u\| = \sup_{t\in[0,1]} |u(t)|$, $u \in C[0,1]$) be our Banach space and

$$K = \left\{ u \in C[0,1] \ \mid \ u(t) \geq 0 \text{ for } t \in [0,1] \text{ and } \min_{t\in[\frac{1}{m},1-\frac{1}{m}]} u(t) \geq \frac{1}{m} \| u \| \right\},$$

$$K_r = \{u \in K : \|u\| < r\} \text{ and } \partial K_r = \{u \in K : \|u\| = r\}.$$

Let $G(t,s)$ be the Green's function for

$$\begin{cases} u'' = 0, \ \ 0 \leq t \leq 1, \\ u(0) = u(1) = 0. \end{cases}$$

Then

$$G(t,s) = \begin{cases} (1-t)\,s, \ \ 0 \leq s \leq t \leq 1, \\ (1-s)\,t, \ \ 0 \leq t \leq s \leq 1. \end{cases}$$

Let

$$A_1 = \max_{0\leq t\leq 1} \int_0^1 G(t,s)\,\varphi_p^{-1}\left(\int_0^1 G(s,x)\,q(x)\,dx\right) ds,$$

and

$$A_2 = \min_{1/m\leq t\leq 1-1/m} \int_0^1 G(t,s)\,\varphi_p^{-1}\left(\int_{1/m}^{1-1/m} G(s,x)\,q(x)\,dx\right) ds.$$

where $\varphi_p^{-1}(s) = |s|^{\frac{1}{p-1}} s$ is an inverse function of φ_p. It is easy to see that $0 < A_1 < \infty$ and $0 < A_2 < \infty$.

Define $T, H : X \to X$ by

$$(Tu)(s) = \int_0^1 \lambda G(x,s)\,q(x)\,f(u(x))\,dx \tag{2.1}$$

and

$$(Hu)(t) = \int_0^1 G(t,s)\,\varphi_p^{-1}(u(s))\,ds. \tag{2.2}$$

Lemma 2.1. $H(T(K)) \subset K$.

Proof. A direct calculation shows that

$$G(t,s) \le G(s,s), \text{ for } 0 \le t \le 1 \text{ and } 0 \le s \le 1, \tag{2.3}$$

and

$$G(t,s) \ge \frac{1}{m} G(s,s), \text{ for } \frac{1}{m} \le t \le 1 - \frac{1}{m} \text{ and } 0 \le s \le 1. \tag{2.4}$$

From (2.3), we obtain

$$\begin{aligned}(H(Tu))(t) &= \int_0^1 G(t,x)\, \varphi_p^{-1}\left(\int_0^1 \lambda G(x,s)q(s)f(u(s))ds\right) dx \\ &\le \int_0^1 G(x,x)\, \varphi_p^{-1}\left(\int_0^1 \lambda G(x,s)q(s)f(u(s))ds\right) dx.\end{aligned}$$

Thus

$$\|H(Tu)\| \le \int_0^1 G(x,x)\, \varphi_p^{-1}\left(\int_0^1 \lambda G(x,s)q(s)f(u(s))ds\right) dx.$$

Finally notice

$$\begin{aligned}&\min_{t\in[\frac{1}{m},1-\frac{1}{m}]} (H(Tu))(t) \\ &= \min_{t\in[\frac{1}{m},1-\frac{1}{m}]} \int_0^1 G(t,x)\, \varphi_p^{-1}\left(\int_0^1 \lambda G(x,s)q(s)f(u(s))ds\right) dx \\ &\ge \frac{1}{m}\int_0^1 G(x,x)\, \varphi_p^{-1}\left(\int_0^1 \lambda G(x,s)q(s)f(u(s))ds\right) dx \\ &\ge \frac{1}{m}\|HTu\|.\end{aligned}$$

□

Remark 1. We can easily prove that $T(K) \subset K$ and that $H(K) \subset K$.

Lemma 2.2. *$HT : K \to K$ is completely continuous.*

Proof. We first prove $T : K \to K$ is completely continuous.

For $n = 1, 2, \ldots$, let $q_n(t) = \min\{q(t), n\}$, and $e_n = \{t \in [0,1] \mid q(t) \ge n\}$. Let

$$(T_n u)(s) = \int_0^1 \lambda G(s,x)\, q_n(x)\, f(u(x))\, dx.$$

It is easy to see [4] that $T_n : K \to K$ is completely continuous, for $n \in N = \{1, 2, \cdots\}$. By $(H1)$, we have

$$\lim_{n\to\infty} \int_{e_n} t(1-t)\, q(t) = 0.$$

For $\forall R > 0$ and $\forall u \in \overline{K}_R$, with $M = \max_{0\le x\le R} f(x)$, we have

$$\begin{aligned}\|Tu - T_n u\| &= \max_{0\le t\le 1} |(Tu)(t) - (T_n u)(t)| \\ &= \lambda \max_{0\le t\le 1} \int_{e_n} G(x,t)\,(q(x) - n)\, f(u(x))\, dx\end{aligned}$$

$$\leq \lambda M \max_{0\leq t\leq 1} \int_{e_n} G(x,t)\, q(x)\, dx$$
$$\leq \lambda M \int_{e_n} x(1-x)\, q(x)\, dx \to 0,\ (n\to\infty),$$

so $\sup\left\{\|Tu - T_n u\| : u \in \overline{K}_R\right\} \to 0$, as $n \to \infty$. Therefore, $T : K \to K$ is completely continuous. Also it is easy to prove that $H : K \to K$ is completely continuous. Consequently, $HT : K \to K$ is completely continuous. □

3. Eigenvalue Intervals

Theorem 3.1. *Suppose that* $(H1)$ *holds. Then,* (1.1) *has at least one positive solution for each*

$$\lambda \in \left(\frac{\varphi_p(m)}{f_\infty \varphi_p(A_2)}, \frac{1}{f_0 \varphi_p(A_1)}\right); \tag{3.1}$$

here m *is chosen as in* $(H1)$.

Proof. Let λ satisfy (3.1) and let $\varepsilon > 0$ be such that

$$\frac{\varphi_p(m)}{(f_\infty - \varepsilon)\varphi_p(A_2)} \leq \lambda \leq \frac{1}{(f_0+\varepsilon)\varphi_p(A_1)} \tag{3.2}$$

Next, we pick $r > 0$ so that

$$f(x) \leq (f_0+\varepsilon)x^{p-1},\ 0 < x \leq r. \tag{3.3}$$

Let $u \in \partial K_r$. We find that for $t \in [0,1]$,

$$\begin{aligned}
(H(Tu))(t) &= \int_0^1 G(t,x)\, \varphi_p^{-1}\left(\int_0^1 \lambda G(x,s) q(s) f(u(s)) ds\right) dx \\
&\leq \int_0^1 G(t,x)\, \varphi_p^{-1}\left(\int_0^1 \lambda G(x,s) q(s) (f_0+\varepsilon) u^{p-1}(s) ds\right) dx \\
&\leq \lambda^{\frac{1}{p-1}} (f_0+\varepsilon)^{\frac{1}{p-1}} r \int_0^1 G(t,x)\, \varphi_p^{-1}\left(\int_0^1 G(x,s) q(s) ds\right) dx \\
&\leq r A_1 \lambda^{\frac{1}{p-1}} (f_0+\varepsilon)^{\frac{1}{p-1}} \\
&\leq r = \|u\|.
\end{aligned}$$

Hence,

$$\|HTu\| \leq \|u\|,\ \text{for } u \in \partial K_r. \tag{3.4}$$

If we set $\Omega_1 = \{u \in X : \|u\| < r\}$, then (3.4) holds for $u \in K \cap \partial\Omega_1$.

Let $R_1 > 0$ be such that

$$f(x) \geq (f_\infty - \varepsilon)\, x^{p-1}, \quad x \geq R_1. \tag{3.5}$$

Let $u \in K$ be such that $\|u\| = R := \max\{2r, mR_1\}$; here m is chosen as in $(H1)$. Then, for $t \in \left[\frac{1}{m}, \frac{m-1}{m}\right]$,

$$u(t) \geq \frac{1}{m}\|u\| \geq \frac{1}{m}\cdot mR_1 = R_1,$$

which in view of (3.5) leads to

$$f(u(t)) \geq (f_\infty - \varepsilon)\, u^{p-1}(t),\ t \in \left[\frac{1}{m}, \frac{m-1}{m}\right]. \tag{3.6}$$

Consequently (here $c \in \left[\frac{1}{m}, 1-\frac{1}{m}\right]$ is chosen as in $(H1)$),

$$\begin{aligned}
(H(Tu))(c) &= \int_0^1 G(c,x)\,\varphi_p^{-1}\left(\int_0^1 \lambda G(x,s)q(s)f(u(s))ds\right)dx \\
&\geq \int_0^1 G(c,x)\,\varphi_p^{-1}\left(\int_{\frac{1}{m}}^{\frac{m-1}{m}} \lambda G(x,s)q(s)\,(f_\infty-\varepsilon)\,u^{p-1}(s)ds\right)dx \\
&\geq \lambda^{\frac{1}{p-1}}(f_0-\varepsilon)^{\frac{1}{p-1}} \cdot \frac{R}{m} \cdot \int_0^1 G(c,x)\,\varphi_p^{-1}\left(\int_{\frac{1}{m}}^{\frac{m-1}{m}} G(x,s)q(s)ds\right)dx \\
&\geq \frac{R}{m} \cdot A_2 \lambda^{\frac{1}{p-1}}(f_0-\varepsilon)^{\frac{1}{p-1}} \\
&\geq R = \|u\|.
\end{aligned}$$

Therefore,

$$\|HTu\| \geq \|u\|,\ \text{for } u \in \partial K_R. \tag{3.7}$$

If we set $\Omega_2 = \{u \in X : \|u\| < R\}$, then (3.7) holds for $u \in K \cap \partial\Omega_2$.

Now (3.4), (3.7), and Theorem 1.1 guarantee that HT has a fixed point $u \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$ with $r \leq \|u\| \leq R$. Clearly, this u is a positive solution of (1.1). □

Theorem 3.2. *Suppose that* $(H1)$ *holds. Then* (1.1) *has at least one positive solution for each*

$$\lambda \in \left(\frac{\varphi_p(m)}{f_0 \varphi_p(A_2)}, \frac{1}{f_\infty \varphi_p(A_1)}\right); \tag{3.8}$$

here m *is chosen as in* $(H1)$.

Proof. Let λ satisfy (3.8) and let $\varepsilon > 0$ be such that

$$\frac{\varphi_p(m)}{(f_0-\varepsilon)\varphi_p(A_2)} \leq \lambda \leq \frac{1}{(f_\infty+\varepsilon)\varphi_p(A_1)}. \tag{3.9}$$

Choose $r > 0$ so that

$$f(x) \geq (f_0-\varepsilon)\, x^{p-1}, \quad 0 < x \leq r. \tag{3.10}$$

Now, let $u \in K$ be such that $\|u\| = r$. Then, $u(t) \geq \frac{1}{m}\|u\|$ for $t \in \left[\frac{1}{m}, \frac{m-1}{m}\right]$. Then (3.10) guarantees (here c is as in $(H1)$),

$$\begin{aligned}
(H(Tu))(c) &= \int_0^1 G(c,x)\,\varphi_p^{-1}\left(\int_0^1 \lambda G(x,s)q(s)f(u(s))ds\right)dx \\
&\geq \int_0^1 G(c,x)\,\varphi_p^{-1}\left(\int_{\frac{1}{m}}^{\frac{m-1}{m}} \lambda G(x,s)q(s)\,(f_\infty-\varepsilon)\,u^{p-1}(s)ds\right)dx
\end{aligned}$$

$$\begin{aligned}
&\geq \lambda^{\frac{1}{p-1}}(f_0-\varepsilon)^{\frac{1}{p-1}}\cdot\frac{r}{m}\cdot\int_0^1 G(c,x)\,\varphi_p^{-1}\left(\int_{\frac{1}{m}}^{\frac{m-1}{m}} G(x,s)q(s)ds\right)dx\\
&\geq \frac{r}{m}\cdot A_2\lambda^{\frac{1}{p-1}}(f_0-\varepsilon)^{\frac{1}{p-1}}\\
&\geq r=\|u\|.
\end{aligned}$$

Therefore,

$$\|HTu\|\geq\|u\|,\ \text{for } u\in\partial K_r. \tag{3.11}$$

Next, we may choose $R_2>0$ such that

$$f(x)\leq(f_\infty+\varepsilon)x^{p-1},\ x\geq R_2. \tag{3.12}$$

Here there are two cases to consider, namely, f bounded and f unbounded.

Case 1. Suppose that f is bounded. Then, there exists some $M>0$ with

$$f(x)\leq M,\ x\in(0,\infty). \tag{3.13}$$

We define

$$R=\max\left\{2r,(\lambda M)^{\frac{1}{p-1}}A_1\right\}.$$

Let $y\in K$ be such that $\|y\|=R$. For $t\in[0,1]$, from (3.12) we have

$$\begin{aligned}
(H(Tu))(t)&=\int_0^1 G(t,x)\,\varphi_p^{-1}\left(\int_0^1\lambda G(x,s)q(s)f(u(s))ds\right)dx\\
&\leq\int_0^1 G(t,x)\,\varphi_p^{-1}\left(\int_0^1\lambda G(x,s)q(s)Mds\right)dx\\
&=\lambda^{\frac{1}{p-1}}M^{\frac{1}{p-1}}\int_0^1 G(t,x)\,\varphi_p^{-1}\left(\int_0^1 G(x,s)q(s)ds\right)dx\\
&\leq A_1\lambda^{\frac{1}{p-1}}M^{\frac{1}{p-1}}\\
&\leq R=\|u\|.
\end{aligned}$$

Hence,

$$\|HTu\|\leq\|u\|,\ \text{for } u\in\partial K_R. \tag{3.14}$$

Case 2. Suppose that f is unbounded. Then, there exists $R\geq\max\{2r,R_2\}$ such that

$$f(x)\leq f(R),\ 0<x\leq R. \tag{3.15}$$

Let $y\in K$ be such that $\|y\|=R$. Then, (3.15) yields for $t\in[0,1]$ that

$$\begin{aligned}
(H(Tu))(t)&=\int_0^1 G(t,x)\,\varphi_p^{-1}\left(\int_0^1\lambda G(x,s)q(s)f(u(s))ds\right)dx\\
&\leq\int_0^1 G(t,x)\,\varphi_p^{-1}\left(\int_0^1\lambda G(x,s)q(s)f(R)\,ds\right)dx\\
&=\lambda^{\frac{1}{p-1}}f(R)^{\frac{1}{p-1}}\int_0^1 G(t,x)\,\varphi_p^{-1}\left(\int_0^1 G(x,s)q(s)ds\right)dx\\
&\leq A_1\lambda^{\frac{1}{p-1}}(f_\infty+\varepsilon)^{\frac{1}{p-1}}R
\end{aligned}$$

$$\leq R = \|u\| .$$

Thus (3.14) is true also in this case.

In both cases, if we set $\Omega_2 = \{u \in K : \|u\| < R\}$, then (3.14) holds for $u \in K \cap \partial\Omega_2$.

If we set $\Omega_1 = \{u \in X : \|u\| < r\}$, then (3.11) holds for $u \in K \cap \partial\Omega_1$.

Now that we have obtained (3.11) and (3.14), it follows from Theorem 1.1 that HT has a fixed point $u \in K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ such that $r \leq \|u\| \leq R$. It is clear that u is a positive solution of (1.1).

Let

$$(L1)\ f_0 = \infty,\ (L2)\ f_\infty = \infty,\ (L3)\ f_0 = 0, \text{ and } (L4)\ f_\infty = 0. \quad \square$$

Corollary 3.1. *Suppose that* $(H1)$ *holds. In addition, assume one of the following conditions hold:* (1) $(L1)$ *and* $(L4)$*;* (2) $(L2)$ *and* $(L3)$*. Then we conclude from Theorem* 3.1 *and* 3.2 *that* $E = (0, \infty)$, *i.e.,* (1.1) *has a positive solution for any* $\lambda > 0$.

Theorem 3.3. *Suppose that* $(H1)$ *holds. In addition assume there exist two constants* $R > r > 0$, *such that*

$$\max_{0 \leq y \leq r} f(y) \leq \varphi_p(r/A_1)/\lambda, \quad \min_{\gamma R \leq y \leq R} f(y) \geq \varphi_p(R/A_2)/\lambda;$$

here $\gamma = \frac{1}{m}$, *and* m *is as in* $(H1)$. *Then,* (1.1) *has a solution* $u \in K$ *with* $r \leq \|u\| \leq R$.

Proof. For $u \in \partial K_r$, we have that $f(u(t)) \leq \varphi_p(r/A_1)/\lambda$, for $t \in [0,1]$. Then

$$\begin{aligned}
(H(Tu))(t) &= \int_0^1 G(t,x)\,\varphi_p^{-1}\left(\int_0^1 \lambda G(x,s)q(s)f(u(s))ds\right)dx \\
&\leq \int_0^1 G(t,x)\,\varphi_p^{-1}\left(\int_0^1 \lambda G(x,s)q(s)\varphi_p(r/A_1)/\lambda ds\right)dx \\
&= r/A_1 \cdot \int_0^1 G(t,x)\,\varphi_p^{-1}\left(\int_0^1 G(x,s)q(s)ds\right)dx \\
&\leq r = \|u\| .
\end{aligned}$$

As a result $\|HTu\| \leq \|u\|$, for $\forall u \in \partial K_r$.

For $u \in \partial K_R$, we have that $f((u(t)) \geq \varphi_p(R/A_2)/\lambda$, for $t \in \left[\frac{1}{m}, \frac{m-1}{m}\right]$. Then, with c as in $(H1)$,

$$\begin{aligned}
(H(Tu))(c) &= \int_0^1 G(c,x)\,\varphi_p^{-1}\left(\int_0^1 \lambda G(x,s)q(s)f(u(s))ds\right)dx \\
&\geq \int_0^1 G(c,x)\,\varphi_p^{-1}\left(\int_{\frac{1}{m}}^{\frac{m-1}{m}} \lambda G(x,s)q(s)\varphi_p(R/A_2)/\lambda ds\right)dx \\
&= R/A_2 \cdot \int_0^1 G(c,x)\,\varphi_p^{-1}\left(\int_{\frac{1}{m}}^{\frac{m-1}{m}} G(x,s)q(s)ds\right)dx \\
&\geq R = \|u\| .
\end{aligned}$$

Therefore,

$$\|HTu\| \geq \|u\|, \text{ for } u \in \partial K_R. \tag{3.16}$$

It follows from Theorem 1.1 that HT has a fixed point in $\overline{K}_{r,R}$.

Next we need the following condition:

$(H2)$ There exist a constant ρ_n with $\lim_{n\to\infty} \rho_n = 0$ and $f(\rho_n) > 0$, for $n = 1, 2, \ldots$.

Let

$$\lambda^* = \sup_{r>0} \frac{\varphi_p\left(r/A_1\right)}{\max_{0\leq y\leq r} f(y)}$$

We easily obtain that $0 < \lambda^* \leq \infty$ using $(H2)$. □

Theorem 3.4. *Suppose* $(H1), (H2), (L1)$ *and* $(L2)$ *hold. Then* (1.1) *has at least two nontrivial positive solutions for all* $\lambda \in (0, \lambda^*)$.

Proof. Define $h(r) = \frac{\varphi_p(r/A_1)}{\max_{0\leq x\leq r} f(x)}$. Using conditions $(H2)$, $(L1)$ and $(L2)$, we easily obtain that $h : (0, \infty) \to (0, \infty)$ is continuous and $\lim_{r\to 0} h\left(r\right) = \lim_{r\to\infty} h\left(r\right) = 0$. There exists $r_0 \in (0, +\infty)$ such that $h\left(r_0\right) = \sup_{r>0} h\left(r\right) = \lambda^*$. For $\lambda \in (0, \lambda^*)$, there exist constants c_1, c_2 $(0 < c_1 < r_0 < c_2 < \infty)$, such that $h\left(c_1\right) = h\left(c_2\right) = \lambda$.

As a result

$$f(y) \leq \varphi_p(c_1/A_1)/\lambda, \text{ for } y \in [0, c_1],$$

and

$$f(y) \leq \varphi_p(c_2/A_1)/\lambda, \text{ for } y \in [0, c_2].$$

On the other hand, using conditions $(L1)$ and $(L2)$, there exist constants d_1, d_2 $(0 < d_1 < c_1 < c_2 < d_2 < \infty)$ such that

$$\frac{f(y)}{y^{p-1}} \geq \frac{1}{\lambda}\varphi_p\left(\frac{1}{\gamma A_2}\right), \; y \in (0, d_1) \cup (\gamma d_2, +\infty),$$

and so,

$$\min_{\gamma d_1\leq y\leq d_1} f(y) \geq \varphi_p(d_1/A_2)/\lambda, \; \min_{\gamma d_2\leq y\leq d_2} f(y) \geq \varphi_p(d_2/A_2)/\lambda.$$ □

Theorem 3.5. *Suppose* $(H1)$ *and* $(H2)$ *hold. Assume either* $(L1)$ *or* $(L2)$ *hold. Then* (1.1) *has at least one positive solution for all* $\lambda \in (0, \lambda^*)$.

Next we need the following condition:

$(H3)$ $\min_{r>0} \sup_{\gamma r\leq y\leq r} f(y) > 0$, *here* $\gamma = \frac{1}{m}$.

Let

$$\lambda^{**} = \inf_{r>0} \frac{\varphi_p\left(r/A_2\right)}{\min_{\gamma r\leq y\leq r} f(y)}.$$

We easily obtain that $0 \leq \lambda^{**} < \infty$ *using* $(H3)$.

Theorem 3.6. *Suppose* $(H1)$, $(H3)$, $(L3)$ *and* $(L4)$ *hold. Then* (1.1) *has at least two nontrivial positive solutions for all* $\lambda \in (\lambda^{**}, \infty)$.

Proof. Define $p(r) = \frac{\varphi_p(r/A_2)}{\min_{\gamma r \le x \le r} f(x)}$. Using conditions $(H3)$, $(L3)$ and $(L4)$, we easily obtain that $p : (0, \infty) \to (0, \infty)$ is continuous and $\lim_{r\to 0} p(r) = \lim_{r\to\infty} p(r) = \infty$. There exists $r_0 \in (0, +\infty)$ such that $p(r_0) = \inf_{r>0} p(r) = \lambda^{**}$. For $\lambda \in (\lambda^{**}, \infty)$, there exist constants $d_1,\ d_2\ (0 < d_1 < r_0 < d_2 < \infty)$, such that $p(d_1) = p(d_2) = \lambda$, and so

$$f(x) \ge \varphi_p(d_1/A_2)/\lambda,\ x \in [\gamma d_1, d_1],$$

and

$$f(x) \ge \varphi_p(d_2/A_2)/\lambda,\ x \in [\gamma d_2, d_2].$$

On the other hand, using condition $(L3)$ there exists a constant c_1 $(0 < c_1 < d_1)$ such that

$$\frac{f(x)}{x^{p-1}} \le \frac{1}{\lambda}\varphi_p\left(\frac{1}{A_1}\right),\ x \in (0, c_1),$$

and so

$$\max_{0\le x\le c_1} f(x) \le \varphi_p(c_1/A_1)/\lambda.$$

Using condition $(L4)$, there exists a constant c $(d_2 < c < \infty)$ such that

$$\frac{f(x)}{x^{p-1}} \le \frac{1}{\lambda}\varphi_p\left(\frac{1}{A_1}\right),\ \text{for } x \in (c, \infty).$$

Let $M = \sup_{x\in[0,c]} f(x)$, and $c_2 \ge \max\left\{c, A_1\varphi_p^{-1}(\lambda M)\right\}$. It is easily proved that

$$\max_{0\le x\le c_2} f(x) \le \varphi_p(c_2/A_1)/\lambda. \qquad \square$$

Theorem 3.7. *Suppose* $(H1)$ *and* $(H3)$ *hold. Assume either* $(L3)$ *or* $(L4)$ *hold. Then* (1.1) *has at least one positive solution for all* $\lambda \in (\lambda^{**}, \infty)$.

Corollary 3.2. *Suppose* $(H1)$, $(H2)$, $(L1)$ *and* $(L4)$ *hold. Then* (1.1) *has at least one positive solution for all* $\lambda > 0$.

Proof. We prove $\lambda^* = \infty$.

If $\sup_{x\in[0,+\infty)} f(x) = M < \infty$, then $\lambda^* = \sup_{r>0} \frac{\varphi_p(r/A_1)}{\max_{0\le x\le r} f(x)} \ge \sup_{r>0} \frac{\varphi_p(r/A_1)}{M} = \infty$. If f is unbounded, there exist a sequence $\{r_n\}$ such that $r_n \to \infty$, and $f(r_n) = \max_{0\le x\le r_n} f(x) \to \infty$. Using $(L4)$, we obtain that

$$\lambda^* \ge \varphi_p(1/A_1) \sup_n \frac{\varphi_p(r_n)}{f(r_n)} = \infty. \qquad \square$$

Corollary 3.3. *Suppose* $(H1)$, $(H3)$, $(L2)$ *and* $(L3)$ *hold. Then the problem* (1.1) *has at least one positive solution for all* $\lambda > 0$.

Proof. We first prove that $\lambda^{**} = 0$. Using $(L2)$, $f(x) \to \infty$ for $x \to \infty$, so there exist a sequence $\{r_n\}$ such that $r_n \to \infty$, and $f(\gamma r_n) = \min_{\gamma r_n \le x\le r_n} f(x)$. As a result

$$\lambda^{**} \le \varphi_p(1/A_2) \inf_n \frac{\varphi_p(r_n)}{f(\gamma r_n)} = 0.$$

Condition $(H3)$ can easily be proved. It follows from Theorem 3.5 that (1.1) has at least one positive solution for all $\lambda > 0$. $\square$

Let

$$(L5)\ f_0 = l, \quad (L6)\ f_\infty = l, \text{ here } 0 < l < \infty.$$

Corollary 3.4. *Suppose* $(H1)$ *and* $(H2)$ *hold. Also assume one of the following conditions hold:* (1) $(L1)$ *and* $(L6)$*;* (2) $(L2)$ *and* $(L5)$*. Then* (1.1) *has at least one positive solution for all* $\lambda \in \left(0, \frac{1}{l\varphi_p(A_1)}\right)$.

Corollary 3.5. *Suppose* $(H1)$ *and* $(H3)$ *hold. Also assume one of the following conditions hold:* (1) $(L3)$ *and* $(L6)$*;* (2) $(L4)$ *and* $(L5)$*. Then* (1.1) *has at least one positive solution for all* $\lambda \in \left(\frac{1}{l\varphi_p(\gamma A_2)}, \infty\right)$.

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