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FORMULAS OF VARIATION OF SOLUTION FOR QUASI-LINEAR CONTROLLED NEUTRAL DIFFERENTIAL EQUATIONS

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Let $J = [a, b]$ be a finite interval, $O \subset R^n$, $G \subset R^r$ be open sets. Let the function $f : J \times O^s \times G \to R^n$ satisfy the following conditions: for almost all $t \in J$ the function $f(t, \cdot) : O^s \times G \to R^n$ is continuously differentiable; for any $(x_1, \dots, x_s, u) \in O^s \times G$ the functions $f(t, x_1, \dots, x_s, u)$, $f_{x_i}(\cdot)$, $i = 1, \dots, s$, $f_u(\cdot)$ are measurable on J; for arbitrary compacts $K \subset O$, $N \subset G$ there exists a function $m_{K,N}(\cdot) \in L(J, R_+)$, $R_+ = [0, \infty)$, such that for any $(x_1, \dots, x_s, u) \in K^s \times N$ and for almost all $t \in J$, the following inequality is fulfilled

$$| f(t, x_1, \dots, x_s, u) | + \sum_{i=1}^{s} | f_{x_i}(\cdot) | + | f_u(\cdot) | \leq m_{K,N}(t).$$

Let the scalar functions $\tau_i(t)$, $i = 1, \dots, s$, $t \in R$, and $\eta_j(t)$, $j = 1, \dots, k$, be absolutely continuous and continuously differentiable, respectively, and satisfying the conditions: $\tau_i(t) \leq t$, $\dot{\tau}_i(t) > 0$, $i = 1, \dots, s$, $\eta_j(t) < t$, $\dot{\eta}_j(t) > 0$, $j = 1, \dots, k$. Let Φ be the set of continuously differentiable functions $\varphi : J_1 = [\tau, b] \to O$, $\tau = \min\{\eta_1(a), \dots, \eta_k(a), \tau_1(a), \dots, \tau_s(a)\}$, $\| \varphi \| = \sup\{| \varphi(a) | + | \dot{\varphi}(t) | : t \in J\}$. Ω be the set of measurable functions $u : J \to G$, satisfying the condition $cl\{u(t) : t \in J\}$ is a compact lying in G, $\| u \| = \sup\{| u(t) | : t \in J\}$; $A_i(t)$, $t \in J$, $i = 1, \dots, k$, be continuous matrix functions with dimensions $n \times n$.

To every element $\mu = (t_0, x_0, \varphi, u) \in E = J \times 0 \times \Phi \times \Omega$ let us correspond the differential equation

$$\dot{x}(t) = \sum_{j=1}^{k} A_j(t) \dot{x}(\eta_j(t)) + f(t, x(\tau_1(t)), \dots, x(\tau_s(t)), u(t)), \tag{1}$$

with discontinuous initial condition

$$x(t) = \varphi(t), \quad t \in [\tau, t_0), \quad x(t_0) = x_0. \tag{2}$$

Definition 1. *Let $\mu = (t_0, x_0, \varphi, u) \in E$, $t_0 < b$. The function $x(t) = x(t; \mu) \in O$, $t \in [\tau, t_1]$, $t_1 \in (t_0, b]$ is said to be a solution corresponding to the element $\mu \in E$, defined on the interval $[\tau, t_1]$, if on the interval $[\tau, t_0]$ the function $x(t)$ satisfies the condition* (2)*, while on the interval $[t_0, t_0]$ it is absolutely continuous and almost everywhere satisfies the equation* (1).

Let us introduce the set of variation:

$$V = \{\delta\mu = (\delta t_0, \delta x_0, \delta\varphi, \delta u) \in E - \tilde{\mu} : \ | \delta t_0 | \leq c, \ \ | \delta x_0 | \leq c, \ \ \| \delta\varphi \| \leq c, \ \ \| \delta u \| \leq c\},$$

where $\tilde{\mu} \in E$ is a fixed element, $c > 0$ is a fixed number.

Let $\tilde{x}(t)$ be a solution corresponding to the element $\tilde{\mu} = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}, \tilde{u}) \in E$, defined on the interval $[\tau, \tilde{t_1}]$, $\tilde{t_i} \in (a, b)$, $i = 0, 1$. There exist numbers $\varepsilon_1 > 0$, $\delta_1 > 0$, such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$ to the element $\tilde{\mu} + \varepsilon\delta\mu \in E$ there corresponds a solution $x(t; \tilde{\mu} + \varepsilon\delta\mu)$ defined on $[\tau, \tilde{t_1} + \delta_1]$.

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Due to uniqueness, the solution $x(t;\tilde{\mu})$ is a continuation of the solution $\tilde{x}(t)$ to the interval $[\tau,\tilde{t}_1+\delta_1]$. Therefore the solution $\tilde{x}(t)$ is assumed to be defined on the interval $[\tau,\tilde{t}_1+\delta_1]$.

Let us define the increment of the solution $\tilde{x}(t)=x(t;\tilde{\mu})$

$$\Delta x(t;\varepsilon\delta\mu)=x(t;\tilde{\mu}+\varepsilon\delta\mu)-\tilde{x}(t),\quad (t,\varepsilon,\delta\mu)\in[\tau,\tilde{t}_1+\delta_1]\times[0,\varepsilon_1]\times V.$$

In order to formulate the main results, we will need the following notation:

$$\sigma_i=(\tilde{t_0},\underbrace{\tilde{x_0},\dots,\tilde{x_0}}_{i},\underbrace{\tilde{\varphi}(\tilde{t_0}),\dots,\tilde{\varphi}(\tilde{t_0})}_{(p-i)},\tilde{\varphi}(\tau_{p+1}(\tilde{t_0})),\dots,\tilde{\varphi}(\tau_s(\tilde{t_0}))),\ i=0,\dots,p;$$

$$\sigma_i=(\gamma_i,\tilde{x}(\tau_1(\gamma_i)),\dots,\tilde{x}(\tau_{i-1}(\gamma_i)),\tilde{x}_0,\tilde{\varphi}(\tau_{i+1}(\gamma_i)),\dots,\tilde{\varphi}(\tau_s(\gamma_i)),$$

$$\sigma_i^0=(\gamma_i,\tilde{x}(\tau_1(\gamma_i)),\dots,\tilde{x}(\tau_{i-1}(\gamma_i)),\tilde{\varphi}(\tilde{t}_0),\tilde{\varphi}(\tau_{i+1}(\gamma_i)),\dots,\tilde{\varphi}(\tau_s(\gamma_i))),$$

$$i=p+1,\dots,s;\quad \gamma_i=\gamma_i(\tilde{t_0}),\ \ \rho_i=\rho_i(\tilde{t_0}),\ \ \gamma_i(t)=\tau_i^{-1}(t),\ \ \rho_i(t)=\eta_i^{-1}(t);$$

$$\omega=(t,x_1,\dots,x_s),\ \tilde{f}[\omega]=f(\omega,\tilde{u}(t)),\ \tilde{f_{x_i}}[t]=f(t,\tilde{x}(\tau_1(t)),\dots,\tilde{x}(\tau_s(t)),\tilde{u}(t)).$$

Theorem 1. *Let the following conditions be fulfilled:*
1) $\gamma_i=\tilde{t}_0,\ \ i=1,\dots,p,\ \ \gamma_{p+1}<\cdots<\gamma_s<\tilde{t}_1,\rho_j(\tilde{t}_0)<\tilde{t}_1,\ \ j=1,\dots,k;$
2) *there exists a number* $\delta>0$ *such that*

$$\gamma_1(t)\le\cdots\le\gamma_p(t),\quad t\in(\tilde{t_0}-\delta,\tilde{t_0}];$$

3) *there exist the finite limits:* $\dot{\gamma}_i^-=\dot{\gamma}_i(\tilde{t}_0-),\ \ i=1,\dots,s,$

$$\lim_{\omega\to\sigma_i}\tilde{f}[\omega]=f_i^-,\quad \omega\in(\tilde{t}_0-\delta,\tilde{t}_0]\times O^s,\ \ i=0,\dots,p\ ,$$

$$\lim_{(\omega_1,\omega_2)\to(\sigma_i,\sigma_i^0)}\Big[\tilde{f}[\omega_1]-\tilde{f}[\omega_2]\Big]=f_i^-,\quad \omega_1,\omega_2\in(\gamma_i-\delta,\gamma_i]\times O^s,\ \ i=p+1,\dots,s.$$

Then there exist numbers $\varepsilon_2>0$, $\delta_2>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde{t}_1-\delta_2,\tilde{t}_1+\delta_2]\times(0,\varepsilon_2]\times V^-,\ V^-=\{\delta\mu\in V:\delta t_0\le 0\}$ *the formula*

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\varepsilon\delta\mu)+o(t;\varepsilon\delta\mu)\qquad(3)$$

is valid, where

$$\delta x(t;\delta\mu)=\{Y(\tilde{t}_0-;t)[\dot{\tilde{\varphi}}(\tilde{t}_0)-\sum_{j=1}^{k}A_j(\tilde{t}_0)\dot{\tilde{\varphi}}(\eta_j(\tilde{t}_0))+\sum_{i=0}^{p}(\hat{\gamma}_{i+1}^--\hat{\gamma}_i^-)f_i^-]-$$

$$-\sum_{i=p+1}^{s}Y(\gamma_i-;t)f_i^-\dot{\gamma}_i^-\}\delta t_0+\beta(t;\delta\mu),$$

$$\hat{\gamma}_0^-=1,\ \hat{\gamma}_i^-=\dot{\gamma}_i^-,\ \ i=1,\dots,p,\ \hat{\gamma}_{p+1}^-=0;$$

$$\beta(t;\delta\mu)=\Phi(\tilde{t}_0;t)[\delta x_0-\dot{\tilde{\varphi}}(\tilde{t}_0)\delta t_0]+\sum_{i=p+1}^{s}\int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0}Y(\gamma_i(\xi);t)\tilde{f}_{x_i}[\gamma_i(\xi)]\delta\varphi(\xi)\dot{\gamma}_i(\xi)d\xi+$$

$$+\sum_{j=1}^{k}\int_{\rho_j(\tilde{t}_0)}^{\tilde{t}_0}Y(\rho_j(\xi);t)A_j(\rho_j(\xi))\dot{\delta}\varphi(\xi)\dot{\rho}_j(\xi)d\xi+\int_{\tilde{t}_0}^{t}Y(\xi;t)\tilde{f}_u[\xi]\delta u(\xi)d\xi;$$

$$\lim_{\varepsilon\to 0}o(t;\varepsilon\delta\mu)/\varepsilon=0,$$

uniformly with respect to $(t,\delta\mu)\in[\tilde{t}_1-\delta_2,\tilde{t}_1+\delta_2]\times V^-$; $\Phi(\xi;t),Y(\xi;t)$ *are matrix functions satisfying the system*

$$\begin{cases}\frac{\partial\Phi(\xi;t)}{\partial\xi}=-\sum\limits_{i=1}^{s}Y(\gamma_i(\xi);t)\tilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi),\\ Y(\xi;t)=\Phi(\xi;t)+\sum\limits_{j=1}^{k}Y(\rho_j(\xi);t)A_j(\rho_j(\xi))\dot{\rho}_j(\xi),\quad \xi\in[\tilde{t}_0,t];\end{cases}$$

and the condition

$$\Phi(\xi;t)=Y(\xi;t)=\begin{cases} I, & s=t,\\ \Theta, & \xi>t.\end{cases}$$

Here I is the identity matrix, Θ is the zero matrix.

Theorem 2. *Let the condition* 1) *of Theorem* 1 *and the following conditions be fulfilled:*

4) *there exists number $\delta>0$ such that*

$$\gamma_1(t)\le\cdots\le\gamma_p(t),\quad t\in[\tilde t_0,\tilde t_0+\delta);$$

5) *there exists the finite limits:* $\dot\gamma_i{}^{+}=\dot\gamma_i(\tilde t_0+)$, $i=1,\dots,s$,

$$\lim_{\omega\to\sigma_i}\tilde f[\omega]=f_i^{+},\quad \omega\in[\tilde t_0,\tilde t_0+\delta)\times O^s,\ \ i=0,\dots,p,$$

$$\lim_{(\omega_1,\omega_2)\to(\sigma_i,\sigma_i^0)}\Big[\tilde f[\omega_1]-\tilde f[\omega_2]\Big]=f_i^{+},\quad \omega_1,\omega_2\in[\gamma_i,\gamma_i+\delta)\times O^s.$$

Then there exist numbers $\varepsilon_2>0$, $\delta_2>0$ such that for an arbitrary $(t,\varepsilon,\mu)\in[\tilde t_1-\delta_2,\tilde t_1+\delta_2]\times[0,\varepsilon_2]\times V^{+}$, $V^{+}=\{\delta\mu\in V:\delta t_0\ge 0\}$ the formula (3) *is valid, where $\delta x(t;\delta\mu)$ has the form*

$$\delta x(t;\delta\mu)=\{Y(\tilde t_0+;t)[\dot{\tilde\varphi}(\tilde t_0)-\sum_{j=1}^{k}A_j(\tilde t_0)\dot{\tilde\varphi}(\eta_j(\tilde t_0))+\sum_{i=0}^{p}(\hat\gamma_{i+1}^{+}-\hat\gamma_i^{+})f_i^{+}]-$$

$$-\sum_{i=p+1}^{s}Y(\gamma_i+;t)f_i^{+}\dot\gamma_i^{+}\}\delta t_0+\beta(t;\delta\mu),$$

$$\hat\gamma_0^{+}=1,\quad \hat\gamma_i^{+}=\dot\gamma_i^{+},\ \ i=1,\dots,p,\quad \hat\gamma_{p+1}^{+}=0.$$

Theorem 3. *Let the assumptions of Theorems* 1, 2 *be fulfilled and*

$$\gamma_i,\tilde t_0\notin\{\eta_{k_1}(\eta_{k_2}(\dots(\eta_{k_e}(\tilde t_1)),\dots,))\in(a,\tilde t_1):$$
$$e=1,2,\dots,m=1,\dots,e,\ \ k_m=1,\dots,k\},\quad i=p+1,\dots,s;$$

$$\sum_{i=0}^{p}(\hat\gamma_{i+1}^{-}-\hat\gamma_i^{-})f_i^{-}=\sum_{i=0}^{p}(\hat\gamma_{i+1}^{+}-\hat\gamma_i^{+})f_i^{+}=f_0,\,f_i^{-}\dot\gamma_i=f_i^{+}\dot\gamma_i^{+}=f_i,\quad i=p+1,\dots,s.$$

Then there exist numbers $\varepsilon_2>0$, $\delta_2>0$ such that for an arbitrary $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_2,\tilde t_1+\delta_2]\times[0,\varepsilon_2]\times V$ the formula (3) *is valid, where $\delta x(t;\delta\mu)$ has the form*

$$\delta x(t;\delta\mu)=\{Y(\tilde t_0;t)[\dot{\tilde\varphi}(\tilde t_0)-\sum_{j=1}^{k}A_j(\tilde t_0)\dot{\tilde\varphi}(\eta_j(\tilde t_0))-f_0]+\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i\}\delta t_0+\beta(t;\delta\mu).$$

Finally we note that the formulas of variation of solution for various classes of delay and neutral differential equations are given in [1–6].

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