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I. Kiguradze and N. Partsvania

ON VANISHING AT INFINITY SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Suppose $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$, and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is the function satisfying the local Carathéodory conditions. Boundary value problems on the interval $\mathbb{R}_+$ for the differential equation

$$u'' = f(t, u) \tag{1}$$

are applied to various fields of natural sciences and used as the subjects of numerous investigations (see, e.g., [1]–[7] and the references therein). In the present paper, for that equation we study the problem

$$u(0) = c, \quad \lim_{t \to +\infty} u(t) = 0, \quad \int_0^{+\infty} \varphi(s) u'^2(s) ds < +\infty, \tag{2}$$

where $c \in \mathbb{R}$, and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a twice continuously differentiable function such that

$$\int_0^{+\infty} \frac{ds}{\varphi(s)} < +\infty.$$

A function $u : \mathbb{R}_+ \to \mathbb{R}$ is said to be **a solution of Eq. (1) defined on** $\mathbb{R}_+$ if it is absolutely continuous together with its first derivative on every finite interval and satisfies Eq. (1) almost everywhere on $\mathbb{R}_+$. A solution of Eq. (1), defined on $\mathbb{R}_+$ and satisfying conditions (2), is said to be **a solution of problem (1), (2).**

Put

$$\Phi(t) = \int_t^{+\infty} \frac{ds}{\varphi(s)},$$

and along with (1) consider the perturbed differential equation

$$u'' = f(t, u) + h(t), \tag{3}$$

where $h : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function such that

$$\int_0^{+\infty} \varphi(s) \Phi^{1/2}(s) |h(s)| ds < +\infty. \tag{4}$$

We introduce the following definition.

Definition. Problem (1), (2) is said to be stable with respect to a small perturbation of the right-hand member of Eq. (1) if for an arbitrary locally integrable function h :

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$\mathbb{R}_+ \to \mathbb{R}$, satisfying condition (4), problem (3), (2) has a unique solution u_h and there exists a positive constant r, independent of the function h, such that

$$\Big[\int\limits_0^{+\infty} \varphi(s)|u_h'(s)-u_0(s)|^2 ds\Big]^{1/2} \le r \int\limits_0^{+\infty} \varphi(s)\Phi^{1/2}(s)|h(s)|ds,$$

where u_0 is a solution of problem (1), (2).

It is easy to see that if problem (1), (2) is solvable, then its arbitrary solution satisfies the conditions

$$u(t) = o\big(\Phi^{1/2}(t)\big) \quad \text{for} \quad t \to +\infty, \qquad \int\limits_0^{+\infty} \frac{u^2(s)}{\varphi(s)\Phi^2(s)} ds < +\infty.$$

If this problem is stable, then

$$|u_h(t) - u_0(t)| \le r\Phi^{1/2}(t) \int\limits_0^{+\infty} \varphi(s)\Phi^{1/2}(s)|h(s)|ds \quad \text{for} \quad t \in \mathbb{R}_+.$$

Below optimal in a certain sense conditions are given guaranteeing the solvability of problem (1), (2) and the stability of this problem with respect to a small perturbation of the right-hand member of Eq. (1).

For any $\lambda \in [0,1]$ and natural k, we consider the differential equation

$$u'' = (1-\lambda)p(t)u + \lambda f(t,u) \tag{5}$$

with the boundary conditions

$$u(0) = \lambda c, \quad u(k) = 0. \tag{6}$$

The following theorem is valid (a principle of a priori boundedness).

Theorem 1. *Let there exist a locally integrable function $p : \mathbb{R}_+ \to \mathbb{R}$ and a positive constant ρ such that for any $\lambda \in [0,1]$ and natural k, an arbitrary solution of problem* (5), (6) *satisfies the inequality*

$$\int\limits_0^k \varphi(s)u'^2(s)ds \le \rho.$$

Then problem (1), (2) *has at least one solution.*

The above-formulated principle gives us the possibility to obtain effective sufficient conditions for the solvability of problem (1), (2). In particular, on the basis of this principle the following theorem is proved.

Theorem 2. *Let there exist a positive constant δ and a locally integrable function $q : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\int\limits_0^{+\infty} \varphi(s)\Phi^{1/2}(s)q(s)ds < +\infty \tag{7}$$

and on $\mathbb{R}_+ \times \mathbb{R}$ the inequality

$$f(t,x)\operatorname{sgn} x \ge p(t)|x| - q(t)$$

holds, where

$$p(t) = \frac{\varphi''(t)}{2\varphi(t)} - \frac{1-\delta}{4\varphi^2(t)\Phi^2(t)}. \tag{8}$$

Then problem (1), (2) *has at least one solution.*

Theorem 3. *Let*

$$\int_0^{+\infty} \varphi(s)\Phi^{1/2}(s)|f(s,0)|ds < +\infty, \tag{9}$$

$$\liminf_{t\to+\infty} \Phi(t)\varphi'(t) > -\infty,$$

and let there exist a positive constant δ such that on the set $\mathbb{R}_+ \times \mathbb{R}$ the condition

$$\big[f(t,x) - f(t,y)\big]\operatorname{sgn}(x-y) \geq p(t)|x-y| \tag{10}$$

is satisfied, where p is the function defined by equality (8). *Then problem* (1), (2) *is uniquely solvable and stable with respect to a small perturbation of the right-hand member of Eq.* (1).

Theorem 3′. *Let conditions* (9) *and* (10) *be fulfilled, where p is the function defined by equality* (8)*, and δ is a positive constant. Let, moreover,*

$$p(t) \geq 0 \quad \textit{for} \quad t \in \mathbb{R}_+. \tag{11}$$

Then:

(*i*) *the differential equation* (1) *has a unique solution satisfying the boundary conditions*

$$u(0) = c, \quad \lim_{t\to+\infty} u(t) = 0; \tag{12}$$

(*ii*) *the solution of problem* (1), (12) *is a solution of problem* (1), (2) *as well;*

(*iii*) *problem* (1), (2) *is stable with respect to a small perturbation of the right-hand member of Eq.* (1).

A solution u of Eq. (1), defined on $\mathbb{R}_+$, is said to be of **Kneser type** if

$$u(t)u'(t) \geq 0 \quad \text{for} \quad t \in \mathbb{R}_+.$$

It is well-known (see, e.g., [3], [5]) that if $f(t,0) \equiv 0$ and the function f does not decrease in the second argument, then for any $c \in \mathbb{R}$ the differential equation (1) has a unique Kneser type solution satisfying the initial condition

$$u(0) = c.$$

Thus Theorem 3′ implies the following corollary.

Corollary 1. *Let $f(t,0) \equiv 0$ and there exist a positive constant δ such that conditions* (10) *and* (11) *are fulfilled, where p is the function defined by equality* (8). *Then Eq.* (1) *has a one-parametric set of Kneser type solutions, and every such solution satisfies the conditions*

$$u(t) = o\big(\Phi^{1/2}(t)\big) \quad \textit{for} \ t \to +\infty,$$

$$\int_0^{+\infty} \frac{1}{\varphi(s)\Phi^2(s)} u^2(s)ds < +\infty, \quad \int_0^{+\infty} \varphi(s)u'^2(s)ds < +\infty.$$

Finally, we give corollaries of the above-formulated theorems concerning the cases where $\varphi(t) = (1+t)^\alpha$ and $\varphi(t) = \exp(\beta t)$, i.e. the boundary conditions (2) have the forms

$$u(0) = c, \quad \lim_{t\to+\infty} u(t) = 0, \quad \int_0^{+\infty} (1+s)^\alpha u'^2(s)ds < +\infty \tag{13}$$

and

$$u(0) = c, \quad \lim_{t\to+\infty} u(t) = 0, \quad \int_0^{+\infty} \exp(\beta s) u'^2(s)ds < +\infty, \tag{14}$$

where $\alpha > 1$ and $\beta > 0$.

Corollary 2. *Let there exist a constant ℓ and a locally integrable function $q : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\ell > \frac{\alpha^2 - 1}{4}, \tag{15}$$

$$\int_0^{+\infty} (1+s)^{\frac{\alpha+1}{2}} q(s)ds < +\infty, \tag{16}$$

and on $\mathbb{R}_+ \times \mathbb{R}$ the inequality

$$f(t,x)\operatorname{sgn} x \geq \frac{\ell|x|}{(1+t)^2} - q(t) \tag{17}$$

holds. Then problem (1), (13) *has at least one solution.*

Corollary 3. *Let*

$$\int_0^{+\infty} (1+s)^{\frac{\alpha+1}{2}} |f(s,0)|ds < +\infty \tag{18}$$

and let there exist a constant ℓ, satisfying inequality (15), *such that on $\mathbb{R}_+ \times \mathbb{R}$ the condition*

$$\big[f(t,x) - f(t,y)\big]\operatorname{sgn}(x-y) \geq \frac{\ell|x-y|}{(1+t)^2} \tag{19}$$

holds. Then:

(*i*) *problem* (1), (12) *has a unique solution which is a solution of problem* (1), (13) *as well;*

(*ii*) *problem* (1), (13) *is stable with respect to a small perturbation of the right-hand member of Eq.* (1).

Remark 1. In Theorems 2 and 3, we cannot put $\delta = 0$, and in Corollaries 2 and 3, we cannot replace (15) by the condition

$$\ell \geq \frac{\alpha^2 - 1}{4}.$$

Indeed, every solution of the differential equation

$$u'' = \frac{\alpha^2 - 1}{4(1+t)^2} u + q(t), \tag{20}$$

where

$$q(t) = \frac{3\alpha^2 + 4\alpha + 1}{4}(1+t)^{-\alpha-2},$$

has the form

$$u(t) = c_1(1+t)^{\frac{1+\alpha}{2}} + c_2(1+t)^{\frac{1-\alpha}{2}} + (1+t)^{-\alpha}, \quad c_1, c_2 \in \mathbb{R}.$$

Thus for $c \neq 1$ problem (20), (13) does not have a solution. On the other hand, the function

$$f(t,x) = \frac{\alpha^2 - 1}{4(1+t)^2} x + q(t)$$

satisfies all the conditions of Corollary 3 except for (15). Instead of (15) we have

$$\ell = \frac{\alpha^2 - 1}{4}.$$

Remark 2. Condition (7) (condition (9)) in Theorem 2 (in Theorem 3) cannot be replaced by the condition

$$\int_0^{+\infty} \varphi(s)\Phi^{1/2+\varepsilon}(s)q(s)ds < +\infty \quad \left(\int_0^{+\infty} \varphi(s)\Phi^{1/2+\varepsilon}(s)|f(s,0)|ds < +\infty \right)$$

no matter how small $\varepsilon > 0$ would be. Analogously, condition (16) (condition (18)) in Corollary 2 (in Corollary 3) cannot be replaced by the condition

$$\int_0^{+\infty} (1+s)^{\frac{\alpha+1}{2}-\varepsilon} q(s)ds < +\infty \quad \left(\int_0^{+\infty} (1+s)^{\frac{\alpha+1}{2}-\varepsilon} |f(s,0)|ds < +\infty \right). \tag{21}$$

Indeed, an arbitrary solution of the equation

$$u'' = \frac{\gamma^2 - 1}{4(1+t)^2} u + h(t),$$

where

$$\gamma > \alpha, \quad h(t) = \frac{\alpha^2 - \gamma^2}{4} (1+t)^{-\frac{3+\alpha}{2}},$$

has the form

$$u(t) = c_1(1+t)^{\frac{1+\gamma}{2}} + c_2(1+t)^{\frac{1-\gamma}{2}} + (1+t)^{\frac{1-\alpha}{2}}, \quad c_1, c_2 \in \mathbb{R}.$$

Thus problem (1), (13) does not have a solution, although the function f satisfies all the conditions of Corollary 2 (Corollary 3) except for condition (16) (condition (18)), instead of which condition (21) holds for an arbitrary $\varepsilon > 0$.

Corollary 4. *Let $f(t,0) \equiv 0$ and there exist a constant ℓ, satisfying inequality* (15)*, such that on the set $\mathbb{R}_+ \times \mathbb{R}$ condition* (19) *holds. Then Eq.* (1) *has a one-parametric set of Kneser type solutions, and every such solution satisfies the conditions*

$$u(t) = o\Big(t^{\frac{1-\alpha}{2}}\Big) \quad \textit{for} \quad t \to +\infty,$$

$$\int_0^{+\infty} \frac{u^2(s)}{1+s} ds < +\infty, \qquad \int_0^{+\infty} (1+s)^\alpha u'^2(s)ds < +\infty.$$

Corollary 5. *Let there exist a constant ℓ and a locally integrable function $q : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\ell > \frac{\beta^2}{4}, \tag{22}$$

$$\int_0^{+\infty} \exp\Big(\frac{\beta s}{2}\Big) q(s)ds < +\infty, \tag{23}$$

and on $\mathbb{R}_+ \times \mathbb{R}$ the inequality

$$f(t,x)\operatorname{sgn} x \geq \ell|x| - q(t)$$

holds. Then problem (1), (14) *has at least one solution.*

Corollary 6. *Let*

$$\int_0^{+\infty} \exp\Big(\frac{\beta s}{2}\Big)|f(s,0)|ds < +\infty, \tag{24}$$

and let there exist a constant ℓ, satisfying inequality (22)*, such that on $\mathbb{R}_+ \times \mathbb{R}$ the condition*

$$\big[f(t,x) - f(t,y)\big] \operatorname{sgn}(x-y) \geq \ell|x-y| \tag{25}$$

holds. Then:

(i) *problem* (1), (12) *has a unique solution which is a solution of problem* (1), (14) *as well;*

(ii) *problem* (1), (14) *is stable with respect to a small perturbation of the right-hand member of Eq.* (1).

Remark 3. In Corollaries 5 and 6, condition (22) cannot be replaced by the condition

$$\ell \geq \frac{\beta^2}{4},$$

and condition (23) (condition (24)) cannot be replaced by the condition

$$\int_0^{+\infty} \exp\Big(\frac{(\beta-\varepsilon)s}{2}\Big)q(s)ds < +\infty \quad \bigg(\int_0^{+\infty} \exp\Big(\frac{(\beta-\varepsilon)s}{2}\Big)|f(s,0)|ds < +\infty\bigg)$$

no matter how small $\varepsilon > 0$ would be. To convince ourselves that the above-said is true, it suffices to consider the equations

$$u'' = \frac{\beta^2}{4}u + q(t)$$

and

$$u'' = \frac{\gamma^2}{4}u + h(t),$$

where

$$q(t) = \frac{3\beta^2}{4}\exp(-\beta t), \quad \gamma > \beta, \quad h(t) = \frac{\beta^2-\gamma^2}{4}\exp\Big(-\frac{\beta t}{2}\Big).$$

Corollary 7. *Let $f(t,0) \equiv 0$ and there exist a constant ℓ, satisfying inequality* (22)*, such that on the set $\mathbb{R}_+ \times \mathbb{R}$ condition* (25) *holds. Then Eq.* (1) *has a one-parametric set of Kneser type solutions, and every such solution satisfies the conditions*

$$u(t) = o\Big(\exp\big(-\frac{\beta t}{2}\big)\Big) \quad \textit{for} \quad t \to +\infty,$$

$$\int_0^{+\infty} \exp(\beta s)u^2(s)ds < +\infty, \quad \int_0^{+\infty} \exp(\beta s){u'}^2(s)ds < +\infty.$$

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Authors' address:

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 0193
Georgia