Mem. Differential Equations Math. Phys. 32(2004), 151–153

T. Kiguradze

ON BOUNDED SOLUTIONS OF THIRD ORDER NONLINEAR HYPERBOLIC EQUATIONS

(Reported on June 7, 2004)

Let $I \subset \mathbb{R}$ be a compact interval containing zero. For the nonlinear hyperbolic equation

$$u^{(2,1)} = f(x,t,u,u^{(1,0)},u^{(2,0)},u^{(0,1)},u^{(1,1)}) \tag{1}$$

consider the following problems on bounded in the half strip $\mathbb{R}_+ \times I$ and in the strip $\mathbb{R} \times I$ solutions

$$\begin{gathered} u(x,0) = \varphi(x) \quad \text{for} \quad x \in \mathbb{R}_+, \\ u^{(0,1)}(0,t) = \psi(t), \quad \sup\{|u(x,t)| : x \in \mathbb{R}_+\} < +\infty \quad \text{for} \quad t \in I; \end{gathered} \tag{2_1}$$

$$\begin{gathered} u(x,0) = \varphi(x) \quad \text{for} \quad x \in \mathbb{R}_+, \\ u^{(1,0)}(0,t) = \psi(t), \quad \sup\{|u(x,t)| : x \in \mathbb{R}_+\} < +\infty \quad \text{for} \quad t \in I; \end{gathered} \tag{2_2}$$

$$u(x,0) = \varphi(x) \quad \text{for} \quad x \in \mathbb{R}, \qquad \sup\{|u(x,t)| : x \in \mathbb{R}\} < +\infty \quad \text{for} \quad t \in I. \tag{2_3}$$

Here

$$u^{(j,k)}(x,y) = \frac{\partial^{j+k}u(x,y)}{\partial x^j \partial y^k},$$

$f : \mathbb{R} \times I \times \mathbb{R}^5 \to \mathbb{R}$ and $\psi : I \to \mathbb{R}$ are continuous functions, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function such that

$$\sup\{|\varphi(x)| + |\varphi'(x)| + |\varphi''(x)| : x \in \mathbb{R}\} < +\infty.$$

By a solution of the equation (1) we understand a classical solution, i.e., a function u having the continuous partial derivatives $u^{(j,k)}$ $(j = 0,1,2;\ k = 0,1)$ and satisfying the equation (1) at every point of the domain under consideration.

Such problems arise in the theory of seepage of homogeneous fluids through fissured rocks [1]. In [2] problems $(1), (2_k)$ $(k = 1,2,3)$ are studied in the case where

$$f(x,t,u_0,u_1,u_2,v_0,v_1) \equiv f(x,t,u_0,u_1,u_2,v_0).$$

To our best knowledge in general case these problems were not studied.

We consider the case, where in the set $\mathbb{R} \times I \times \mathbb{R}^5$ the function f satisfies the following conditions:

(E_1) there exists a positive constant l such that

$$|f(x,t,u_0,u_1,u_2,v_0,v_1)| \le l(1 + |u_0| + |u_1| + |u_2| + |v_0| + |v_1|);$$

(E_2) there exists a positive constant δ such that

$$\big(f(x,t,u_0,u_1,u_2,v_0,v_1) - f(x,t,u_0,u_1,u_2,\overline{v}_0,v_1)\big)\operatorname{sgn}(v_0 - \overline{v}_0) \ge \delta|v_0 - \overline{v}_0|;$$

2000 *Mathematics Subject Classification.* 35L35, 35L55, 35L80.

Key words and phrases. Third order nonlinear hyperbolic equation, bounded solution.

(E_3) $f(x,t,u_0,u_1,u_2,v_0,v_1)$ is locally Lipschitz continuous with respect to u_2 and v_1, i.e., there exists a continuous function $\gamma : \mathbb{R}^7 \to \mathbb{R}_+$ such that

$$|f(x,t,u_0,u_1,u_2,v_0,v_1) - f(x,t,u_0,u_1,\overline{u}_2,v_0,\overline{v}_1)| \leq$$
$$\leq \gamma(u_0,u_1,u_2,\overline{u}_2,z_0,z_1,\overline{z}_1)\big(|u_2-\overline{u}_2| + |v_1-\overline{v}_1|\big);$$

(E_4) $f(x,t,u_0,u_1,u_2,v_0,v_1)$ is locally Lipschitz continuous with respect to u_0 and u_1, i.e., there exists a continuous function $\eta : \times\mathbb{R}^7 \to \mathbb{R}_+$ such that

$$|f(x,t,u_0,u_1,u_2,v_0,v_1) - f(x,t,\overline{u}_0,\overline{u}_1,u_2,v_0,v_1)| \leq$$
$$\leq \eta(u_0,\overline{u}_0,u_1,\overline{u}_1,u_2,v_0,v_1)\big(|u_0-\overline{u}_0| + |u_1-\overline{u}_1|\big).$$

Theorem 1. *Let the conditions* (E_1)*–*(E_3) *hold. Then for any* $k \in \{1,2,3\}$ *the problem* $(1),(2_k)$ *is solvable. Moreover, if in addition the condition* (E_4) *holds, then problem* $(1),(2_k)$ *is uniquely solvable.*

A particular case of the equation (1) is the linear equation

$$u^{(2,1)} = \sum_{j=0}^{2}\sum_{k=0}^{1} p_{jk}(x,t)u^{(j,k)} + q(x,t), \tag{3}$$

where $p_{jk} : \mathbb{R}\times I \to \mathbb{R}$ $(j=0,1,2;\ k=0,1)$ and $q : \mathbb{R}\times I \to \mathbb{R}$ are continuous functions. For this equation from Theorem 1 we get

Corollary 1. *Let there exist positive constants* l *and* δ *such that*

$$|p_{jk}(x,t)| \leq l \quad (j=0,1,2;\ k=0,1), \qquad |q(x,t)| \leq l \quad \textit{for} \quad (x,t)\in\mathbb{R}\times I,$$
$$p_{01}(x,t) \geq \delta \quad \textit{for} \quad (x,t)\in\mathbb{R}\times I.$$

Then for any $k \in \{1,2,3\}$ *the problem* $(3),(2_k)$ *is uniquely solvable.*

Now consider the case where $f(x,t,u_0,u_1,u_2,v_0,v_1)$ is independent of u_2, i.e., the equation (1) has the form

$$u^{(2,1)} = f(x,t,u,u^{(1,0)},u^{(0,1)},u^{(1,1)}), \tag{1'}$$

and the function f on the set $\mathbb{R}\times I\times\mathbb{R}^4$ satisfies the following conditions:
(E_1') there exist a positive constant l such that

$$|f(x,t,u_0,u_1,0,v_1)| \leq l(1+|u_0|+|u_1|+|v_1|);$$

(E_2') there exists a positive constant δ such that

$$\big(f(x,t,u_0,u_1,v_0,v_1) - f(x,t,u_0,u_1,\overline{v}_0,v_1)\big)\,\mathrm{sgn}(v_0-\overline{v}_0) \geq \delta|v_0-\overline{v}_0|;$$

(E_3') $f(x,t,u_0,u_1,v_0,v_1)$ is locally Lipschitz continuous with respect to v_1;
(E_4') $f(x,t,u_0,u_1,v_0,v_1)$ is locally Lipschitz continuous with respect to u_0 and u_1.

Theorem 2. *Let the conditions* (E_1')*–*(E_3') *hold. Then for any* $k \in \{1,2,3\}$ *the problem* $(1'),(2_k)$ *is solvable. Moreover, if in addition the condition* (E_4') *holds, then the problem* $(1'),(2_k)$ *is uniquely solvable.*

Unlike to Theorem, 1 Theorem 2 covers the case where $f(x,t,u_0,u_1,v_0,v_1)$ is a rapidly growing function with respect to v_0. For the equation

$$u^{(2,1)} = f_0(x,t,u) + f_1(x,t,u^{(1,0)}) + f_2(x,t,u^{(0,1)}) + f_3(x,t,u^{(1,1)}). \tag{4}$$

Theorem 2 implies

Corollary 2. *Let* $f_j : \mathbb{R}\times I\times\mathbb{R}\to\mathbb{R}$ $(j=0,1,2,3)$ *be continuous functions having continuous partial derivative with respect to the third argument. Moreover, let there exist positive constants* δ *and* l *such that on the set* $\mathbb{R}\times I\times\mathbb{R}$ *the inequalities*

$$\Big|\frac{\partial f_j(x,t,z)}{\partial z}\Big| \leq l \quad (j=0,1,3), \qquad \frac{\partial f_2(x,t,z)}{\partial z} \geq \delta,$$
$$|f_j(x,t,0)| \leq l \quad (j=0,1,2,3)$$

hold. Then for any $k \in \{1,2,3\}$ *the problem* $(4), (2_k)$ *is uniquely solvable.*

As an example consider the equation

$$u^{(2,1)} = p_0(x,t)\frac{u^{k_0}}{1+|u|^{m_0}} + p_1(x,t)\frac{\left(u^{(1,0)}\right)^{k_1}}{1+\left|u^{(1,0)}\right|^{m_1}} + p_2(x,t)\frac{\left(u^{(1,1)}\right)^{k_2}}{1+\left|u^{(1,1)}\right|^{m_2}} +$$
$$+ p(x,t)\sinh\left(u^{(0,1)}\right) + q(x,t). \qquad (5)$$

Here $p_i : \mathbb{R} \times I \to \mathbb{R}$ $(i = 0,1,2,3)$, p and $q : \mathbb{R} \times I \to \mathbb{R}$ are continuous functions, k_i $(i = 0,1,2)$ are natural numbers, m_i are nonnegative integers and $m_i \geq k_i - 1$ $(i = 0,1,2)$. Moreover there exist positive constants δ and l such that the inequalities

$$|p_i(x,t)| \leq l \ ; \ (i = 0,1,2), \quad |q(x,t)| \leq l,$$
$$\delta \leq p(x,t) \leq l$$

hold on the set $\mathbb{R} \times I$. Then by Corollary 2, for any $k \in \{1,2,3\}$ the problem $(5), (2_k)$ is uniquely solvable.

This work was supported by GRDF (Grant No. 3318).

References

1. G. Barenblat, I. Zheltov and I. Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks.(Russian) *J. Appl. Math. Mech.*, **24** (1960), 1286-1303.

2. T. Kiguradze and V. Lakshmikantham, On initial-boundary value problems in bounded and unbounded domains for a class of nonlinear hyperbolic equations of the third order (submitted).

Author's address:

Florida Institute of Technology
Department of Mathematical Sciences
150 W. University Blvd.
Melbourne, Fl 32901
USA
E-mail: tkigurad@fit.edu